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OCHA AND THE SWISS-CHEESE OPERAD

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(*communicated by James Stasheff*)

Abstract

In this paper we show that the relation between Kajiura-Stasheff's OCHA and A. Voronov's swiss-cheese operad is analogous to the relation between SH Lie algebras and the little discs operad. More precisely, we show that the OCHA operad is quasi-isomorphic to the operad generated by the top-dimensional homology classes of the swiss-cheese operad.

Introduction

OCHA refers to the homotopy algebra of open and closed strings introduced by Kajiura and Stasheff [15] inspired by the work of Zwiebach on string field theory [35]. In [15] the *A_∞-algebras over L_∞-algebras* are also introduced, they are the strong homotopy version of $\mathfrak{g}$-algebras (or Leibniz pairs, see [8]). An OCHA is a structure obtained by adding other operations to an A_∞-algebra over an L_∞-algebra. The physical meaning of those additional operations is given by the "opening of a closed string into an open one".

Considering that its relevance to Physics is well acknowledged (see also [16]), in the present paper we further explore the *mathematical significance* of a full OCHA, not restricted to an A_∞-algebra over an L_∞-algebra. In [13] we have proven that any degree one coderivation $D \in \mathrm{Coder}(S^cL \otimes T^cA)$ such that $D^2 = 0$ defines an OCHA structure on the pair (L, A). In this work we study the relation between OCHA's and A. Voronov's swiss-cheese operad and show that it is analogous to the relation between SH Lie algebras and the little discs operad. A graded Lie algebra is part of the structure of a Gerstenhaber Algebra, which in turn is equivalent to an algebra over the homology little discs operad. The Lie part of a Gerstenhaber algebra is given by the top-dimensional homology classes of the little discs operad.

We will study the suboperad of the homology swiss-cheese operad generated by top-dimensional homology classes and show that it is quasi-isomorphic to the operad

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whose algebras are OCHAs. The quasi-isomorphism, however, is not of operads but only a quasi-isomorphism of modules over the operad $\mathcal{L}_\infty$ of L_∞-algebras.

Let $\mathcal{OC}_\infty$ be the OCHA operad and let $\mathcal{OC}$ denote the suboperad of the homology swiss-cheese operad generated by top-dimensional homology classes. Our main result is the following.

Theorem. *There is a morphism of differential graded $\mathcal{L}_\infty$-modules $\mu : \mathcal{OC}_\infty \longrightarrow \mathcal{OC}$ which induces an isomorphism in cohomology.*

The paper is organized as follows. In section 1 we briefly review F. Cohen's theorem on the homology of the little discs operad and state it using trees. Section 2 reviews the analogous description of the homology swiss-cheese operad in terms of generators and relations given by trees. In section 3 we define OCHA in a grading and signs convention which is different from the original one in [15]. The definition given here is appropriate for studying its correspondence with the compactified configuration space of points on the closed upper half plane. We show that both definitions are equivalent through the (de)suspension operator. A definition of the OCHA operad $\mathcal{OC}_\infty$ is provided in section 4 using the partially planar trees, a type of tree which is defined in the same section. Section 5 reviews the construction of the compactified configuration spaces $\overline{C(p,q)}$ first introduced by Kontsevich in [19]. The combinatorial structure of its boundary strata is described in terms of partially planar trees and some examples are provided. The well known equivalence between $\overline{C(1,q)}$ and the cyclohedron W_{q+1} is explained in terms of those trees. In section 6 we prove the quasi-isomorphism between $\mathcal{OC}_\infty$ and $\mathcal{OC}$ viewed as modules over the operad $\mathcal{L}_\infty$ of L_∞ algebras. The main tool used in its proof is the spectral sequence of $\overline{C(p,q)}$ as a manifold with corners.

Notation and Conventions

Let us fix a field k of characteristic zero. In this paper, all vector spaces are over k and 'graded vector space' will always mean '$\mathbb{Z}$-graded vector space', unless otherwise stated. Let V be a graded vector space, we define a left action of the symmetric group S_n on $V^{\otimes n}$ in the following way: if $\tau \in S_2$ is a transposition, then the action is given by $x_1 \otimes x_2 \overset{\tau}{\mapsto} (-1)^{|x_1||x_2|} x_2 \otimes x_1$. Since any $\sigma \in S_n$ is a composition of transpositions, the sign of the action of σ on $V^{\otimes n}$ is well defined:

$$x_1 \otimes \cdots \otimes x_n \overset{\sigma}{\mapsto} \epsilon(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}. \tag{1}$$

We will refer to $\epsilon(\sigma)$ as the Koszul sign of the permutation. Let us define $\chi(\sigma) = (-1)^\sigma \epsilon(\sigma)$, where $(-1)^\sigma$ is the sign of the permutation.

Given two homogeneous maps $f, g : V \to W$ between graded vector spaces, according to the Koszul sign convention (which will be used throughout this work), we have:

$$(f \otimes g)(v_1 \otimes v_2) = (-1)^{|g||v_1|}(f(v_1) \otimes g(v_2)). \tag{2}$$

We will use the notation of Lada-Markl [20] for the suspension and desuspension operators: $\uparrow$ and $\downarrow$. Let $\uparrow V$ (resp. $\downarrow V$) denote the suspension (resp. desuspension) of the graded vector space V defined by: $(\uparrow V)^p = V^{p-1}$ (resp. $(\downarrow V)^p = V^{p+1}$). We thus have the natural maps $\uparrow : V \to \uparrow V$ of degree 1, and $\downarrow : V \to \downarrow V$ of degree -1. Let

$\uparrow^{\otimes n}$ denote $\bigotimes^n \uparrow\colon \bigotimes^n V \to \bigotimes^n \uparrow V$ and $\downarrow^{\otimes n}$ is defined analogously. The operators $\uparrow^{\otimes n}$ and $\downarrow^{\otimes n}$ transform symmetric operations into anti-symmetric ones. In fact, let E (resp. A) denote the symmetric (resp. anti-symmetric) left action of the group of permutations S_n on $V^{\otimes n}$:

$$E(\sigma)(x_1 \otimes \cdots \otimes x_n) = \epsilon(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \tag{3}$$

$$A(\sigma)(x_1 \otimes \cdots \otimes x_n) = \chi(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \tag{4}$$

Both actions are related by: $\uparrow^{\otimes n} E(\sigma) \downarrow^{\otimes n} = (-1)^{n(n-1)/2} A(\sigma)$, for any $\sigma \in S_n$. In particular, $\uparrow^{\otimes n} \circ \downarrow^{\otimes n} = (-1)^{n(n-1)/2} \cdot \mathbb{1}$. The sign $(-1)^{n(n-1)/2}$ is a consequence of the Koszul sign convention (2) defined above (see also [7]).

Let us now describe how the notation for operads and its related concepts (such as: representations, ideals and modules) will be used in this paper. Our description will not necessarily include precise definitions. Those can be found in [24]. An operad is any sequence $\mathcal{O} = \{\mathcal{O}(n)\}_{n \geqslant 1}$ of objects in a symmetric monoidal category (such as the category of topological spaces or the category of vector spaces) endowed with a right action of the symmetric group S_n on each $\mathcal{O}(n)$ and a composition law satisfying natural associative and equivariance conditions.

Given a graded vector space V, the endomorphism operad of V is defined as $\mathrm{End}_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$. The composition law $\circ_i$ in End_V is defined by the usual composition in the ith variable of multilinear maps and the right action of S_n on $\mathrm{Hom}(V^{\otimes n}, V)$ is the composition with the symmetric left action E defined by (3). In particular, this means that for graded vector spaces, according to our conventions, 'symmetric' always mean 'graded symmetric'.

Among the standard examples of operads are those defined in terms of *trees*. In this paper, in accordance with [10], trees are oriented and not necessarily compact: an edge may be terminated by a vertex at only one end (or none). Such an edge is called *external*. An external edge oriented toward its vertex is called a *leaf*, otherwise it is called *the root*. Trees are assumed to have only one root. The leaves of each tree are labeled by natural numbers. The action of S_n on trees with n leaves is defined by permuting the labels. The composition law $\circ_i$ on operads defined in terms of trees is given by the grafting operation, i.e., the identification of the root of one tree with the leaf labelled i of the other tree.

We also need to mention the *coloured operads*, a concept that goes back to Boardman and Vogt [3]. Following the notation of Berger and Moerdijk [2], given a set of colours C, a C-coloured operad $\mathcal{P}$ is defined by assigning to each $(n+1)$-tuple of colours $(c_1, \ldots, c_n; c)$ an object

$$\mathcal{P}(c_1, \ldots, c_n; c) \qquad \text{in some monoidal category}$$

endowed with a composition law and a symmetric group action. The defining conditions for coloured operads are analogous to those for ordinary operads.

Given a family $A = \{A_c\}_{c \in C}$ of vector spaces indexed by C, the C-coloured operad $\mathrm{End}(A)$ is defined by:

$$\mathrm{End}(A)(c_1, \ldots, c_n; c) = \mathrm{Hom}(A_{c_1} \otimes \cdots \otimes A_{c_n}, A_c). \tag{5}$$

For coloured operads, the composition law is only defined when the colour of the output coincides with the colour of the input. Another example of a coloured operad is given by trees with coloured edges, i.e., trees such that for each edge is assigned a element in some set C. For trees with coloured edges, the grafting operation is only defined when the colour of the root coincides with the colour of the corresponding leaf. Coloured trees will be used throughout the present paper. In fact, we will use 2-coloured trees where the colours of the edges are wiggly or straight, according to the notation used in [15].

Let $\mathcal{P}$ be an operad and let $M = \{M(n)\}_{n\geqslant 1}$ be a sequence of objects where each $M(n)$ has a right S_n-action. We say that M is a left $\mathcal{P}$-module if it is endowed with a left 'operadic action' $\circ_i^\lambda$:

$$\circ_i^\lambda : \mathcal{P}(n) \otimes M(m) \to M(m+n-1) \tag{6}$$

which is equivariant and satisfies associative conditions analogous to those in the definition of operads. The definition of right modules is similar.

An *ideal* in an operad $\mathcal{P}$ is a sequence of objects $\mathcal{I} = \{I(n)\}_{n\geqslant 1}$ with $I(n) \subseteq \mathcal{P}(n)$, where each $I(n)$ is invariant under the action of S_n and $\mathcal{I}$ is a left and right $\mathcal{P}$-module. We refer the reader to [24] for the precise definitions and further details about these concepts.

1. The homology little disks operad

We begin by recalling the description of the homology little discs operad $H_\bullet(\mathcal{D})$ in terms of generators and relations. To keep our notation in accordance with [15], we will represent classes in $H_\bullet(\mathcal{D})$ by trees with wiggly edges. As usual in operad theory, all trees are assumed to be *rooted* and *oriented* toward the root. A tree with only one vertex and n incoming edges is called an n-corolla.

The little disks operad (also called little 2-disks operad) is a sequence $\mathcal{D} = \{\mathcal{D}(n)\}_{n\geqslant 1}$ of topological spaces $\mathcal{D}(n)$ defined as the space of all maps

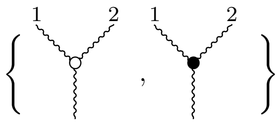
$$d : \coprod_{1\leqslant s\leqslant n} D_s \to D$$

from the disjoint union of n numbered standard two dimensional disks $D_1, \ldots, D_n$ to D, where

$$D := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leqslant 1\},$$

such that d, when restricted to each disk, is a composition of translation and multiplication by a positive real number and the images of the interiors of the disks are disjoint. The symmetric group acts by renumbering the disks. We may interpret $d \in \mathcal{D}(n)$ as the standard disk D with n numbered disjoint circular holes. The operad composition $\gamma_{\mathcal{D}}(d; d_1, \ldots, d_n)$ is, intuitively speaking, given by gluing n disks $d_1, \ldots, d_n$ in the holes of the disk with holes d, and erasing the seams (see [24, 25] for more precise definitions).

Since $\mathcal{D}(2)$ is homotopy equivalent to S^1, its homology has two generators. The zero dimensional generator will be denoted by a 2-corolla with a "white" vertex: $\circ$, while the one dimensional generator will be denoted by a 2-corolla with a "black" vertex: $\bullet$. Let

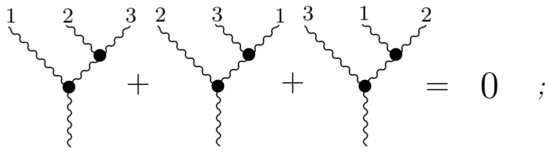


be a basis for $H_\bullet(\mathcal{D}(2))$, where the first generator has degree zero, the second has degree one and both are invariant under the action of the permutation group. An algebra over $H_\bullet(\mathcal{D})$ will thus have two graded symmetric operations. Notice however that the bracket defined below by (9) is *skew* graded symmetric, as required by the definition of a Gerstenhaber algebra. F. Cohen's famous result about the homology of $\mathcal{D}$ can be stated in the language of trees, as follows.

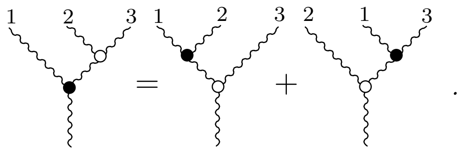
Theorem 1.1 (F. Cohen [5]). *The homology little disks operad $H_\bullet(\mathcal{D})$ is isomorphic, as an operad of $\mathbb{Z}$-graded vector spaces, to the operad generated by the above trees, subject to the following relations:*

i) Both generators are invariant under permutation of their labels;

ii) Jacobi identity:



iii) Leibniz rule:



Let us now describe algebras over $H_\bullet(\mathcal{D})$. It is well known that algebras over $H_\bullet(\mathcal{D})$ are equivalent to Gerstenhaber algebras. However, a brief description of that equivalence will help clarify our exposition involving the swiss-cheese operad and its relations to OCHA.

Remember that V is an algebra over $H_\bullet(\mathcal{D})$ if there is a morphism of operads $\Phi : H_\bullet(\mathcal{D}) \to \mathrm{End}_V$ where $\mathrm{End}_V(n) = \{\mathrm{Hom}(V^{\otimes n}, V)\}$, is the endomorphism operad. Consequently, V is a graded vector space endowed with two graded symmetric operations $m_2 : V \otimes V \to V$ of degree $|m_2| = 0$, corresponding to the first generating tree, and $l_2 : V \otimes V \to V$ of degree $|l_2| = 1$ corresponding to the second generating tree.

The two relations presented above in terms of trees correspond to the equalities:

$$\sum_\sigma l_2 \circ (\mathbb{1} \otimes l_2) \circ E(\sigma) = 0 \tag{7}$$

and

$$l_2 \circ (\mathbb{1} \otimes m_2) = m_2 \circ (l_2 \otimes \mathbb{1}) + m_2 \circ (\mathbb{1} \otimes l_2) \circ E(\tau_{1,2}). \tag{8}$$

In the first equality, σ runs over all cyclic permutations and $E(\sigma)$ is defined by (3). In the second equality $\tau_{1,2}$ denotes the transposition (1 2). Notice that $E(\sigma)$ appears in both formulas above because, by definition, the right action of S_n on $\mathrm{Hom}(V^{\otimes n}, V)$ is given by composition with $E(\sigma)$. Given homogeneous elements $x, y, z \in V$, identities (7) and (8) are expressed by:

$$(-1)^{|x||z|} l_2(l_2(x,y),z) + (-1)^{|x||y|} l_2(l_2(y,z),x) + (-1)^{|y||z|} l_2(l_2(z,x),y) = 0$$

$$l_2(x, y \cdot z) = l_2(x,y) \cdot z + (-1)^{(|x|-1)|y|} y \cdot l_2(x,z)$$

where the dot product denotes m_2. Notice that the sign $(-1)^{(|x|-1)|y|}$ occurs in the second expression as a consequence of the transposition $\tau_{1,2}$ and the Koszul sign convention.

To see that an algebra over $H_\bullet(\mathcal{D})$ is equivalent to a Gerstenhaber algebra, we just need to define the bracket:

$$[x,y] := (-1)^{|x|} l_2(x,y) \tag{9}$$

it is not dificult to see that

$$[x,y] = -(-1)^{(|x|-1)(|y|-1)}[y,x].$$

and

$$(-1)^{(|x|-1)(|z|-1)}[[x,y],z] + (-1)^{(|x|-1)(|y|-1)}[[y,z],x] + (-1)^{(|y|-1)(|z|-1)}[[z,x],y] = 0.$$

This shows that the structure of an algebra over $H_\bullet(\mathcal{D})$ is equivalent to the structure of an Gertenhaber algebra on V with bracket defined by (9).

2. The Homology swiss-cheese operad

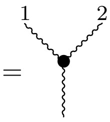
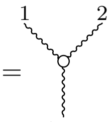
In this section we recall the definition of the swiss-cheese operad [32], denoted by $\mathcal{SC}$. We will present the homology swiss-cheese operad using 2-coloured trees. Harrelson [11] has presented similarly the homology of open-closed strings in the wider context of PROPs. The following presentation of the homology swiss-cheese operad is a particular case of Harrelson's open-closed homology PROP.

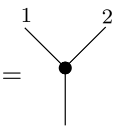
The swiss-cheese operad $\mathcal{SC}$ is a 2-coloured operad. We will use the initials of open and closed as our set of colours: $C = \{o, c\}$. For $m \geqslant 0, n \geqslant 0$ with $m + n \geqslant 1$, $\mathcal{SC}(m,n;o)$ is the configuration space of non-overlapping disks labeled 1 through m and upper semi-disks labeled 1 through n embedded by translations and dilations in the standard unit upper semi-disk so that the embedded semi-disks are all centered on the diameter of the big semi-disk.

For $m \geqslant 1$ and $n = 0$, $\mathcal{SC}(m,0;c) = \mathcal{D}(m)$ is just the usual component of the little disks operad, and $\mathcal{SC}(m,n;c)$ is the empty set for $n \geqslant 1$.

Observation 2.1. *The components of the form $\mathcal{SC}(m,0;o)$ were excluded in the original definition of the operad $\mathcal{SC}$, (see [32]), i.e., those components which have only discs as inputs and intervals as output were not to be considered in the original definition. Here, however, we shall keep the components $\mathcal{SC}(m,0;o)$, $m \geqslant 1$, since they are crucial for the OCHA structure. In fact, as we will see in this paper, a* *zero dimensonal generator of the homology of $\mathcal{SC}(1,0;o)$ corresponds to the map $n_{1,0} : L \to A$. The physical meaning of $n_{1,0}$ being given by the "opening of a closed string into an open one", see* [15, 16].

Let us now describe the homology swiss-cheese operad $H_\bullet(\mathcal{SC})$ in terms of generators and relations using trees. Since $H_\bullet(\mathcal{SC})$ is a 2-coloured operad, our trees must also be 2-coloured. The colours we use are wiggly and straight.

Let $l_2 =$  and $m_2 =$  denote the generators of $H_\bullet(\mathcal{SC}(2,0;c)) = H_\bullet(\mathcal{D}(2))$. Both spaces $\mathcal{SC}(1,0;o)$ and $\mathcal{SC}(0,2;o)$ are contractible because the elements of $\mathcal{SC}(1,0;o)$ have only one interior disc while $\mathcal{SC}(0,2;o)$ is homeomorphic to $\mathcal{C}_1(2)$ (where $\mathcal{C}_1$ is the little intervals operad) which is well known to be contractible. Their degree zero homology generators will be denoted respectively by

$n_{1,0} =$  and $n_{0,2} =$  .

So, the degrees of the above generators are: $|l_2| = 1$ and $|m_2| = |n_{1,0}| = |n_{0,2}| = 0$. The analog of Cohen's theorem can be stated as follows.

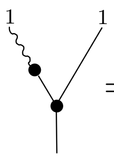
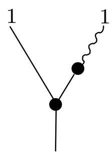
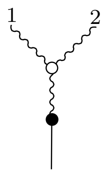
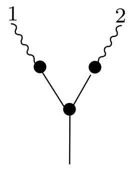
Theorem 2.2. *The homology swiss-cheese operad $H_\bullet(\mathcal{SC})$ is isomorphic, as a $\mathbb{Z}$-graded 2-coloured operad, to the 2-coloured operad generated by: l_2, m_2, $n_{1,0}$ and $n_{0,2}$, satisfying the following relations:*

a) l_2 is invariant under permutation and satisfies the Jacobi identity;

b) m_2 is invariant under permutation and associative;

c) l_2 and m_2 satisfy the Leibniz rule;

d) $n_{0,2}$ is associative;

e)  $=$  *and*  $=$  .

Observation 2.3. *Given a coloured tree, if it has k leaves of some colour c, then those leaves are labeled 1 to k. So, on the same tree we may have two leaves of different colours with the same label.*

Proof of Theorem 2.2. We first show that l_2, m_2, $n_{1,0}$ and $n_{0,2}$ generate the operad $H_\bullet(\mathcal{SC})$. In fact, $\mathcal{SC}(0,n;o)$ is homeomorphic to the little intervals operad $\mathcal{C}_1$ and hence $H_\bullet(\mathcal{SC}(0,n;o))$ is generated by $n_{0,2}$. The space $\mathcal{SC}(1,0;o)$ consists of configurations of one disk inside the standard semi disk and is also contractible, hence the need for the zero dimensional generator $n_{1,0}$. Finally observe that $\mathcal{SC}(m,n;o)$ is homotopy equivalent to $\mathcal{D}(m)\times S_n$, thus from Theorem 1.1 any class in $H_\bullet(\mathcal{SC}(m,n;o))$ is obtained from operadic composition of l_2, $n_{1,0}$ and $n_{0,2}$. In order to generate the full operad $H_\bullet(\mathcal{SC})$ we need to consider $\mathcal{SC}(m,0;c) = \mathcal{D}(m)$. But we already know that its homology is generated by l_2 and m_2 from Theorem 1.1.

We now show that the generators in fact satisfy the above relations. Relation *e)* involve only zero dimensional homology classes and one can easily check that the compositions indicated in *e)* belong to the same path component of the swiss-cheese. Item *d)* follows immediately from the fact that $\mathcal{SC}(0,n;o)$ is the little intervals operad $\mathcal{C}_1(n)$. Since $\mathcal{SC}(m,0;c) = \mathcal{D}(m)$, relations *a), b)* and *c)* are precisely the statement of Theorem 1.1. □

We will now study algebras over $H_\bullet(\mathcal{SC})$. Since our main interest is in OCHA and, as said in the introduction, OCHA is related to *part* of the structure of the homology swiss-cheese, let us define a suboperad of $H_\bullet(\mathcal{SC})$ containing the relevant structure.

Definition 2.4. *$\mathcal{OC}$ is the suboperad of $H_\bullet(\mathcal{SC})$ generated by l_2, $n_{1,0}$ and $n_{0,2}$. Algebras over $\mathcal{OC}$ will be called* open-closed algebras *or simply OC-*algebras.

Observation 2.5. *Let $\mathcal{L}$ be the operad defined by $\mathcal{L}(n) = H_{n-1}(\mathcal{D}(n))$ for $n \geqslant 1$, i.e., $\mathcal{L}$ is the operad of top dimensional homology classes of the little discs operad. From Theorem 1.1, we see that the operad $\mathcal{L}$ is generated by an element $l_2 \in H_1(\mathcal{D}(2))$ which is invariant under the symmetric group action and satisfies the Jacobi identity. Consequently, l_2 corresponds to a degree one graded commutative bilinear operation satisfying the Jacobi identity. Under operadic desuspension, the new generator will have degree zero, will be graded anti-commutative and will satisfy the Jacobi identity. In other words, the operadic desuspension $\mathfrak{s}^{-1}$ transforms $\mathcal{L}$ into the Lie operad: $\mathcal{L}ie = \mathfrak{s}^{-1}\mathcal{L}$ (see [24] for the definition of operadic (de)suspension). In this paper we refer to $\mathcal{L}$ as the* Lie operad *by "abus de langage".*

There is an analogy between the operads $\mathcal{OC}$ and $\mathcal{L}$ which is sumarized bellow:

$$\mathcal{OC} \iff \text{top-dimensional generators of } H_\bullet(\mathcal{SC})$$
$$\mathcal{L} \iff \text{top-dimensional generators of } H_\bullet(\mathcal{D}).$$

An algebra over $\mathcal{OC}$ (or OC-algebra) consists of a pair of $\mathbb{Z}$-graded vector spaces L and A such that L is endowed with a degree one symmetric operation $l_2 : L \otimes L \to L$ satisfying the Jacobi identity, A has a degree zero operation $m_2 : A \otimes A \to A$ defining a structure of associative algebra and there is a degree zero linear map $n_{1,0} : L \to A$. From the first identity in item e) of Theorem 2.2, it follows that $n_{1,0}$ takes L into the center of A.

3. Open-closed homotopy algebras

OCHA's were originally defined in a particular grading and signs convention where all multilinear maps have degree one and, after being lifted as a coderivation $D \in \mathrm{Coder}(S^cL \otimes T^cA)$, the OCHA axioms are translated into the single condition: $D^2 = 0$.

In order to study the relation between OCHA and the swiss-cheese operad, we need a definition where grading and signs are given by the corresponding compactified configuration space. More specifically, a definition where the degrees are equal to minus the dimension of the configuration space and the signs in the axioms are chosen so as to make them compatible with the boundary operator in the first row of the E^1 term of the spectral sequence of the compactified configuration space. In this section we present the definition in this geometrical setting. It is proven in the Appendix that both definitions are equivalent.

Let us first recall the definition of SH Lie [21] algebras in a grading and signs convention compatible with its compactified configuration space description (see [18, 31]).

Definition 3.1 (Strong Homotopy Lie algebra). *A strong homotopy Lie algebra (or L_∞-algebra) is a $\mathbb{Z}$-graded vector space V endowed with a collection of graded symmetric n-ary brackets $l_n : V^{\otimes n} \to V$, of degree $3-2n$ such that $l_1^2 = 0$ and for $n \geqslant 2$:*

$$\partial\, l_n(v_1,\ldots,v_n) \;=\; \sum_{\substack{\sigma\in\Sigma_{k+l=n}\\ k\geqslant 2,l\geqslant 1}} \epsilon(\sigma)\; l_{1+l}(l_k(v_{\sigma(1)},\ldots,v_{\sigma(k)}),v_{\sigma(k+1)},\ldots,v_{\sigma(n)}) = 0 \tag{10}$$

where σ runs over all (k,l)-unshuffles, i.e., permutations $\sigma \in S_n$ such that $\sigma(i) < \sigma(j)$ for $1 \leqslant i < j \leqslant k$ and for $k+1 \leqslant i < j \leqslant k+l$.

Observation. *The operator ∂ in the above definition denotes the induced differential on the endomorphism complex, i.e.:*

$$\partial\, l_n = l_1 l_n + l_n(l_1 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \;\cdots\; + \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes l_1).$$

Definition 3.2 (Open-Closed Homotopy Algebra − OCHA). *An OCHA consists of a 4-tuple $(L, A, \mathfrak{l}, \mathfrak{n})$ where L and A are $\mathbb{Z}$-graded vector spaces, $\mathfrak{l} = \{l_n : L^{\otimes n} \to L\}_{n\geqslant 1}$ and $\mathfrak{n} = \{n_{p,q} : L^{\otimes p} \otimes A^{\otimes q} \to A\}_{p+q\geqslant 1}$ are two families of multilinear maps where l_n has degree $3-2n$ and $n_{p,q}$ has degree $2-2p-q$, such that $(L,\mathfrak{l})$ is an L_∞-algebra and the two families satisfy the following compatibility condition:*

$$\begin{aligned}
&\partial\, n_{n,m}(v_1,\ldots,v_n,a_1,\ldots,a_m) = \\
&= \sum_{\sigma\in\Sigma_{p+r=n},\; p\geqslant 2} (-1)^{\epsilon(\sigma)} n_{1+r,m}(l_p(v_{\sigma(1)},\ldots,v_{\sigma(p)}),v_{\sigma(p+1)},\ldots,v_{\sigma(n)},a_1,\ldots,a_m) + \\
&\quad + \sum_{\substack{\sigma\in\Sigma_{p+r=n},\; i+j=m-s\\ (r,s)\neq(0,1),(n,m)}} (-1)^{\mu_{p,i}(\sigma)} n_{p,i+1+j}(v_{\sigma(1)},..,v_{\sigma(p)},a_1,..,a_i,n_{r,s}(v_{\sigma(p+1)},..,v_{\sigma(n)},\\
&\qquad\qquad a_{i+1},..,a_{i+s}),a_{i+s+1},..,a_m).
\end{aligned}$$

where $\mu_{p,i}(\sigma) = s+i+si+ms+\epsilon(\sigma)+s(v_{\sigma(1)}+\cdots+v_{\sigma(p)}+a_1+\cdots+a_i)+ + (a_1+\cdots+a_i)(v_{\sigma(i+1)})+\cdots+v_{\sigma(n)}).$

Observation. *The operator ∂ in the above definition denotes the induced differential on the endomorphism complex, i.e.:*

$$\partial\, n_{n,m} = n_{0,1}n_{n,m} \;-(-1)^m\; n_{n,m}(d_{L^n} \otimes \mathbf{1}_A^{\otimes m} + \mathbf{1}_L^{\otimes n} \otimes d_{A^m})$$

where $d_{L^n} = l_1\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}+\;\cdots\;+\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes l_1$ *and* $d_{A^m} = n_{0,1}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}+\;\cdots\;+\mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes n_{0,1}$.

It is convenient to have a shorthand expression for the OCHA relations:

$$\partial\, n_{n,m} = \sum_{\sigma\in\Sigma_{p+r=n},\; p\geqslant 2} n_{1+r,m}(l_p\otimes \mathbf{1}_L^{\otimes r}\otimes \mathbf{1}_A^{\otimes m})(E(\sigma)\otimes \mathbf{1}_A^{\otimes m}) +$$

$$+ \sum_{\substack{\sigma\in\Sigma_{p+r=n},\; i+j=m-s \\ (r,s)\neq(0,1),(n,m)}} (-1)^{s+i+si+ms} n_{p,i+1+j}(\mathbf{1}_L^{\otimes p}\otimes \mathbf{1}_A^{\otimes i}\otimes n_{r,s}\otimes \mathbf{1}_A^{\otimes j})(E(\sigma)\otimes \mathbf{1}_A^{\otimes m}) \quad (11)$$

where $E(\sigma)$ was defined by formula (3) on page 125. The complicated sign of the definition is absorbed in the above expression if we assume the following standard convention: given two maps $h_1, h_2 : V\otimes W\to U$, the tensor product $h_1\otimes h_2$ defined on $V^{\otimes^2}\otimes W^{\otimes^2}$ is given by: $(h_1\otimes h_2)((v_1\otimes v_2)\otimes(w_1\otimes w_2)) = (-1)^{|v_2||w_1|}h_1(v_1,w_1)\otimes h_2(v_2,w_2)$.

Example 3.3. *Here is a list of the first few OCHA relations:*

$$\partial\, n_{0,1} = 2\,(n_{0,1})^2 = 0 \quad (12)$$

$$\partial\, n_{1,1} = n_{0,2}(n_{1,0}\otimes \mathbb{1}_A) - n_{0,2}(\mathbb{1}_A\otimes n_{1,0}) \quad (13)$$

$$\partial\, n_{2,0} = n_{1,0}l_2 + n_{1,1}(\mathbb{1}_L\otimes n_{1,0}) + n_{1,1}(\mathbb{1}_L\otimes n_{1,0})E(\tau_{1,2}) \quad (14)$$

$$\begin{aligned}\partial\, n_{1,2} &= n_{1,1}(\mathbf{1}_L\otimes n_{0,2}) - n_{0,2}(n_{1,1}\otimes \mathbf{1}_A) - n_{0,2}(\mathbf{1}_A\otimes n_{1,1}) + \\ &+ n_{0,3}(n_{1,0}\otimes \mathbf{1}_A\otimes \mathbf{1}_A) - n_{0,3}(\mathbf{1}_A\otimes n_{1,0}\otimes \mathbf{1}_A) + n_{0,3}(\mathbf{1}_A\otimes \mathbf{1}_A\otimes n_{1,0}) \quad (15)\end{aligned}$$

$$\begin{aligned}\partial\, n_{2,1} &= n_{1,1}(l_2\otimes \mathbf{1}_A) + n_{1,1}(\mathbf{1}_L\otimes n_{1,1}) + n_{1,1}(\mathbf{1}_L\otimes n_{1,1})(E(1\;2)\otimes \mathbf{1}_A) + \\ &+ n_{0,2}(n_{2,0}\otimes \mathbf{1}_A) - n_{0,2}(\mathbf{1}_A\otimes n_{2,0}) + n_{1,2}(\mathbf{1}_L\otimes n_{1,0}\otimes \mathbf{1}_A) - n_{1,2}(\mathbf{1}_L\otimes \mathbf{1}_A\otimes n_{1,0}) + \\ &+ n_{1,2}(\mathbf{1}_L\otimes n_{1,0}\otimes \mathbf{1}_A)(E(1\;2)\otimes \mathbf{1}_A) - n_{1,2}(\mathbf{1}_L\otimes \mathbf{1}_A\otimes n_{1,0})(E(1\;2)\otimes \mathbf{1}_A) \quad (16)\end{aligned}$$

Relation (12) simply says that $n_{0,1}$ is a differential operator. On the other hand, (13) means that $n_{1,0}$ takes L into the homotopy center of A where $n_{1,1}$ is the homotopy operator. The configuration space corresponding to $n_{1,1}$ is the cyclohedron W_2 (see example 5.3). The configuration space corresponding to relation (14) is “The Eye” (Figure 4 pg.148). Relation (15) corresponds to the configuration space W_3 (Figure 3 pg.148). Finally, relation (16) corresponds to the configuration space illustrated by Figure 5 pg.149. If we consider an OCHA structure where the maps $n_{1,0}$ and $n_{2,0}$ are set equal to zero, then relations (15) and (16) together say that $n_{1,1} : L\otimes A\to A$ is a Lie algebra action by derivations up to homotopy.

It is a well known fact that A_∞ and L_∞ algebras can be defined both in the *geometrical setting* (where the degrees of the multilinear maps are minus the dimension of the corresponding configuration space) and the *algebraic setting* where all the maps have degree 1. It is also well known that both definitions are equivalent through the (de)suspension operator. The same is true for OCHA.

Proposition 3.4. *An OCHA structure* $(L, A, \mathfrak{l}, \mathfrak{n})$*, in the grading and signs conventions of defintion 3.2, is equivalent to a degree one coderivation* $D \in \mathrm{Coder}(S^c(\downarrow L) \otimes T^c(\downarrow A))$ *such that* $D^2 = 0$.

The proof of this fact amounts to an appropriate use of the Lada-Markl notation for the suspension and desuspension operators $\uparrow$ and $\downarrow$ (see [20]) and is provided in the Appendix.

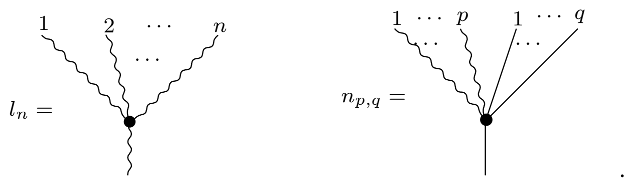
4. The OCHA operad $\mathcal{OC}_\infty$

In this section we study the operad $\mathcal{OC}_\infty$ whose algebras are precisely OCHA's as given in Definition 3.2. Our presentation of $\mathcal{OC}_\infty$ is slightly different from (but naturally equivalent to) the original definition in [15] because of the different conventions. We begin by defining the partially planar trees.

Definition 4.1. *A* partially planar tree *is an isotopy class of oriented rooted trees embedded in the euclidean 3 dimensional space* $\mathbb{R}^3$ *such that a fixed subset of edges is contained in the xy-plane. Planar edges will be denoted by straight lines, while spatial edges will be denoted by wiggly lines.*

Observation 4.2. *Partially planar trees have appeared in the work of Merkulov [26]. Merkulov, however, uses wiggly lines for planar edges and straight lines for spatial edges.*

The partially planar trees relevant for the definition of $\mathcal{OC}_\infty$ have a specific form we now begin to describe. We define l_n as the corolla which has n leaves and only spatial edges and $n_{p,q}$ as the corolla with planar root, p spatial leaves and q planar leaves. Leaves of different colours are labelled by different sets:



(17)

As mentioned in the introduction, the grafting operation of a tree T_2 on some leaf of a tree T_1 is only defined when the colour of the root of T_2 is equal to the colour of the corresponding leaf of T_1. The grafting of a tree T_2 with spatial root on the ith spatial leaf of some tree T_1 will be denoted by:

$$T_1 \circ_i T_2 \tag{18}$$

On the other hand, the grafting of a tree with planar root T_4 on the ith planar leaf of some tree T_3 will be denoted by:

$$T_3 \bullet_i T_4 \tag{19}$$

Consider the set of all corollas $n_{p,q}$ and l_n with $2p + q \geqslant 2$ and $n \geqslant 2$. Let $\mathcal{T}(n)$ denote the set of all partially planar trees T with n leaves which can be obtained by grafting a finite number of corollas in the above set. Let $\mathcal{T}_o(p,q) \subseteq \mathcal{T}(p+q)$ denote the subset of trees with planar root having p spatial leaves and q planar leaves. Let $\mathcal{T}_c(n) \subseteq \mathcal{T}(n)$ be the subset of trees with spatial root.

Definition 4.3. *For $2p+q \geqslant 2$, we define $\mathcal{N}_\infty(p,q)$ as the vector space spanned by $\mathcal{T}_o(p,q)$ and for $n \geqslant 2$, $\mathcal{L}_\infty(n)$ is defined as the vector space spanned by $\mathcal{T}_c(n)$. The space $\mathcal{N}_\infty(0,1)$ is defined as the vector space spanned by the tree with only one planar edge and no vertices, while $\mathcal{L}_\infty(1)$ is defined similarly as the vector space spanned by the tree with only one spatial edge and no vertices.*

Observation 4.4. *Notice that if a tree in $\mathcal{T}(n)$ has a spatial root, then all of its edges must also be spatial because of the corollas we have chosen as generators.*

Let $|i(T)|$ be the number of internal edges of T, we define the degree of $T \in \mathcal{T}(n)$ as follows:

$$|T| = \begin{cases} |i(T)| + 2 - 2p - q, & \text{if } T \in \mathcal{T}_o(p,q) \\ |i(T)| + 3 - 2n, & \text{if } T \in \mathcal{T}_c(n) \end{cases} \tag{20}$$

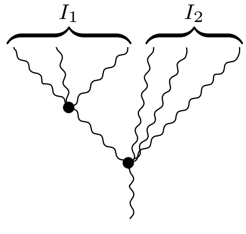
in particular, $|n_{p,q}| = 2 - 2p - q$ and $|l_n| = 3 - 2n$. Now we can define the spaces: $\mathcal{N}_\infty = \bigoplus_{k+l\geqslant 1} \mathcal{N}(k,l)$ and $\mathcal{L}_\infty = \bigoplus_{n\geqslant 1} \mathcal{L}_\infty(n)$, and finally define:

$$\mathcal{OC}_\infty = \mathcal{L}_\infty \oplus \mathcal{N}_\infty. \tag{21}$$

There is a symmetric group action on spatial leaves by permuting the labels of the spatial leaves, and there is no symmetric group action on planar leaves. In other words, given a tree $T \in \mathcal{OC}_\infty$ with p spatial leaves and q planar leaves, the group S_p acts on T by permuting the labels of the spatial leaves, while the planar leaves remain fixed.

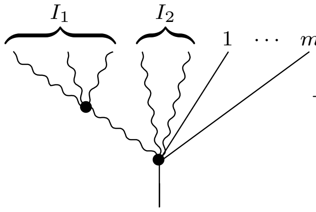
The space $\mathcal{OC}_\infty$ we have just defined has the structure of a 2-coloured operad of graded vector spaces defined by the grafting operations $\circ_i$ and $\bullet_i$ and by the symmetric group action on spatial leaves. Let us now define a differential operator $d : \mathcal{OC}_\infty \to \mathcal{OC}_\infty$. We proceed analogously to the definition given in [12] (see also [18]).

Let us first define the action of d on corollas l_n and $n_{p,q}$:

$$d\ l_n = \sum_{\substack{k+l=n+1 \\ k,l\geqslant 2}} \sum_{\substack{\text{unshuffles } \sigma: \\ \{1,2,\ldots,n\}=I_1\cup I_2 \\ \#I_1=k,\ \#I_2=l-1}}$$  (22)

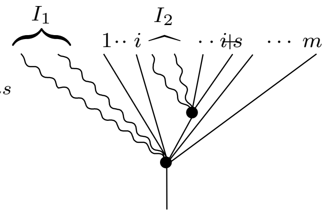
observing that an unshuffle σ is equivalent to a partition $(1,2,\ldots,n) = I_1 \cup I_2$ into two ordered subsets I_1 and I_2. On the other hand: $d\, n_{n,m} =$

$$= \sum_{\substack{k+l=n+1\\ k,l\geqslant 2}} \quad \sum_{\substack{\text{unshuffles } \sigma:\\ \{1,2,\dots,n\}=I_1\cup I_2\\ \#I_1=k,\ \#I_2=l-1}} \Bigg($$



$$+$$

$$\sum_{0\leqslant i,s\leqslant m} (-1)^{s+i+si+ms}$$



$$\Bigg). \qquad (23)$$

Once d is defined on the generators of $\mathcal{OC}_\infty$, it is extended to the whole operad by the Leibniz rule:

$$d\,(T \circ_i T_1) = d\,T \circ_i T_1 + (-1)^{|T|} T \circ_i d\,T_1 \qquad d\,(T \bullet_i T_2) = d\,T \bullet_i T_2 + (-1)^{|T|} T \;\bullet_i d\,T_2$$

where: T_1 is a tree with spatial root and T_2 is a tree with planar root. With the operator d, $\mathcal{OC}_\infty$ becomes a differential graded 2-coloured operad.

Observation 4.5. *For trees in $\mathcal{OC}_\infty$, let $T' \to T$ indicate that T is obtained from T' by contracting a spatial or planar internal edge. The above defined differential operator $d : \mathcal{OC}_\infty \to \mathcal{OC}_\infty$ is, up to sign, simply given by:*

$$d(T) = \sum_{T' \to T} \pm T',$$

the only difference between the above definition and the original one in [15] is the sign.

According to the grading defined by (20), the operator d has degree 1. So $\mathcal{OC}_\infty$ is an operad of cochain complexes.

Given two differential graded vector spaces L and A, we say that (L, A) is an algebra over $\mathcal{OC}_\infty$ if there is a morphism of differential graded 2-coloured operads:

$$\Psi : \mathcal{OC}_\infty \to \mathrm{End}_{L,A}$$

where $\mathrm{End}_{L,A}$ is the 2-coloured endomorphism operad of the pair (L, A), as described by (5). Since Ψ is a chain map and the differential operator ∂ on $\mathrm{End}_{L,A}$ is precisely the one used in formulas (10) and (11), it follows that (L, A) is an algebra over $\mathcal{OC}_\infty$ if, and only if, it admits the structure of an OCHA.

Observation 4.6. *By definition, $\mathcal{OC}_\infty = \mathcal{L}_\infty \oplus \mathcal{N}_\infty$, where $\mathcal{N}_\infty$ is spanned by trees with planar root. For trees in $\mathcal{OC}_\infty$, grafting two trees T_1 and T_2 where at least one of them has a planar root always results in a tree with a planar root. So, $\mathcal{N}_\infty$ is an ideal in $\mathcal{OC}_\infty$. On the other hand, $\mathcal{L}_\infty$ is a suboperad of $\mathcal{OC}_\infty$, since trees* *with spatial root can only have spatial edges. Finally, we observe that $\mathcal{OC}_\infty$ has a strucutre of module over $\mathcal{L}_\infty$ given by the grafting operation in $\mathcal{OC}_\infty$.*

5. The compactification $\overline{C(p,q)}$

In this section we recall the construction of the space $\overline{C(p,q)}$, first introduced by Kontsevich in [19]. We use the fact that $\overline{C(p,q)}$ is a manifold with corners and study the combinatorics of its boundary strata to show that the first row of the E^1 term of the spectral sequence determined by $\overline{C(p,q)}$ is isomorphic, as a differential complex, to $\mathcal{N}_\infty(p,q)$.

Let p,q be non-negative integers satisfying the inequality $2p+q \geqslant 2$. We denote by $\mathrm{Conf}(p,q)$ the configuration space of marked points on the upper half plane $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geqslant 0\}$ with p points in the interior and q points on the boundary (real line):

$$\mathrm{Conf}(p,q) = \{(z_1,\dots,z_p,x_1,\dots,x_q) \in H^{p+q} \mid z_{i_1} \neq z_{i_2},\, x_{j_1} \neq x_{j_2}\ \forall i_1 \neq i_2, j_1 \neq j_2 \\ \mathrm{Im}(z_i) > 0,\, \mathrm{Im}(x_j) = 0\ \forall i,j\}$$

The above configuration space $\mathrm{Conf}(p,q)$ is the cartesian product of an open subset of H^p and an open subset of $\mathbb{R}^q$ and, consequently, is a $2p+q$ dimensional smooth manifold. Let $C(p,q)$ be the quotient of $\mathrm{Conf}(p,q)$ by the action of the group of orientation preserving affine transformations that leaves the real line fixed:

$$C(p,q) = \mathrm{Conf}(p,q)\Big/(z \mapsto az+b) \quad a,b \in \mathbb{R},\ a>0.$$

The condition $2p+q \geqslant 2$ ensures that the action is free and thus $C(p,q)$ is a $2p+q-2$ dimensional smooth manifold.

Let $\mathrm{Conf}_n(\mathbb{C})$ be the configuration space of n points in the complex plane. We take the quotient by affine transformations $z \mapsto az+b$ where $a \in \mathbb{R}$, $a>0$ and $b \in \mathbb{C}$ and define $C(n) := \mathrm{Conf}_n(\mathbb{C})/(z \mapsto az+b)$. Again $C(n)$ is a smooth manifold. The real version of the Fulton-MacPherson compactification $\overline{C(n)}$ is defined in the usual way, see [1, 9, 27]. Let ϕ be the embedding:

$$\phi : C(p,q) \longrightarrow C(2p+q) \tag{24}$$

defined by $\phi(z_1,\dots,z_p,x_1,\dots,x_q) = (z_1,\bar{z}_1,\dots,z_p,\bar{z}_p,x_1,\dots,x_q)$, where $\bar{z}$ denotes complex conjugation.

Definition 5.1. *The compactification of the configuration space $C(p,q)$ is defined as the closure in $\overline{C(2p+q)}$ of the image of ϕ. It will be denoted by $\overline{C(p,q)}$.*

The compactification $\overline{C(n)}$ of points in the plane can be intuitivelly described through "*bubbling offs*" on the sphere (the one point compactification of the plane). In the case of $\overline{C(p,q)}$, one can think of the closed disc as the one point compactification of the upper half plane and think of the embedding ϕ as taking the closed disc to the upper hemisphere of the above sphere. Punctures in the bulk of the disc are reflected through the equator. Points in $\overline{C(p,q)}$ can be intuitivelly described through "*bubbling offs*" on the disc. Those bubbling offs are pictured in the next figures.

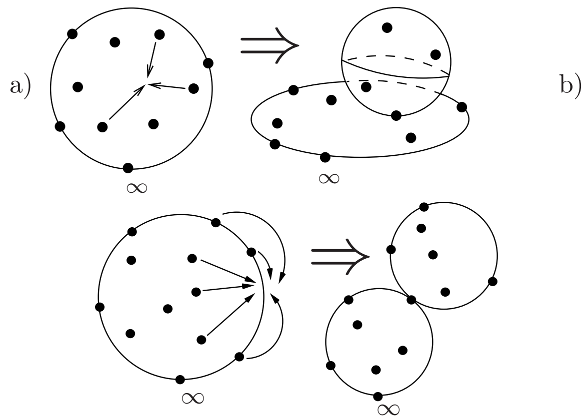


Figure 1: The two possible types of bubbling off on the closed disc.

5.1. The Stratification of $\overline{C(p,q)}$

The combinatorics of the compactification $\overline{C(n)}$ of the configuration space of points in the complex plane is well known to be described in terms of trees. In other words, its boundary strata can be labeled by trees (see: [1, 9, 18, 27, 33, 34]). Since $\overline{C(p,q)}$ was defined through the embedding $\phi : C(p,q) \to C(2p+q)$, it naturally inherits its combinatorics from that of $\overline{C(2p+q)}$. Leaves corresponding to the p points in the bulk of the upper half plane are spatial, while leaves corresponding to the q points on the boundary (real line) are planar. The combinatorics of $\overline{C(p,q)}$ is thus described by partially planar trees. We follow the notation of [18] and state this fact in the following theorem.

Theorem 5.2. *There is a stratification of $\overline{C(p,q)}$ such that:*

(1) $\overline{C(p,q)} = \coprod_{T \in \mathcal{T}_o(p,q)} S_T$. *Each stratum S_T is a smooth submanifold and* $\mathrm{codim}_{\mathbb{R}} S_T = |i(T)| =$ *number of internal edges of T ;*

(2) *there is a unique open stratum $S_{n_{p,q}} = C(p,q) \quad 2p+q \geqslant 2$;*

(3) *for each tree $T \in \mathcal{T}_o(p,q)$ we have the identity*

$$S_T = S_{n_{p_1,q_1}} \times S_{\delta_1} \times \cdots \times S_{\delta_n}$$

where each δ_i is a corolla of the form $n_{k,l}$ or l_k, and T is obtained by grafting the corollas $\delta_1, \ldots, \delta_n$ to n_{p_1,q_1}.

(4) *The boundary of the closure $\overline{S_T}$ of each stratum is given by $\partial\overline{S_T} = \bigcup_{T' \to T} \overline{S_{T'}}$, where $T' \to T$ means that T is obtained from T' by contracting a internal edge.*

In case $p = 0$, the space $\overline{C(0,q)}$ is the associahedron K_q [28, 29]. The labeling of the boundary strata of $\overline{C(0,q)}$, in this case, reduces to the well known labeling of the facets of K_q by planar trees.

Example 5.3 ($\overline{C(1,q)}$ is the cyclohedron W_{q+1})**.** *The cyclohedron was introduced by Bott and Taubes [4] and received its name from Stasheff [30]. It is defined as* *the Fulton-MacPherson compactification of the configuration space of points on the circle S^1 modded out by the group of rotations $SO(2) = S^1$. The equivalence between $\overline{C(1,q)}$ and W_{q+1} will be described below in terms of partially planar trees using the above theorem.*

By fixing the interior point of $C(1,q)$ to be equal to $i \in \mathbb{C}$, the remaining points are on the real line. It is not difficult to see that $C(1,q)$ is an open simplex homeomorphic to the configuration space of points on the circle modded out by $SO(2)$. The compactification $\overline{C(1,q)}$ is obtained from that open simplex by performing iterated blow ups. Hence $\overline{C(1,q)}$ is a polytope. In order to show that $\overline{C(1,q)}$ and W_{q+1} are equivalent polytopes, we just need to establish a one-to-one correspondence between bracketings around the $q+1$ marked points on the circle (q points on the real line plus one point marked ∞) and the partially planar trees in $\mathcal{T}_o(1,q)$, showing also that the correspondence respects the incidence relations.

In fact, for any $q \geqslant 0$ the facets of the cyclohedron W_{q+1} are labeled by (i.e. are in one-to-one correspondence with) all the meaningful ways of inserting brackets in an expression of $q+1$ letters disposed on a circle. The codimension of the facet corresponding to a given bracketing is equal to the number of brackets inserted, as illustrated by Figure 3 *(see also Devadoss' paper* [6] *on the cyclohedra.).*

First recall that $\overline{C(1,q)}$ can be described as the compactified configuration space of points on the closed disc (the one point compactification of the upper half plane) with 1 point in the interior of the disc, q points on the boundary of the disc plus one boundary point marked as ∞. From the "bubbling off" description of points in the compactification, we know that each facet of codimension k in $\overline{C(1,q)}$ corresponds to $k+1$ discs joined at points in the boundary such that exactly one of those discs contains one point in the bulk while the others contain only points in the boundary.

Let us exhibit the correspondence between 'circular bracketings' and 'bubbling offs'. Consider a point in $\overline{C(1,q)}$ in the bubbling off description, as a number of discs joined at "double points". There is only one of these discs which contains the interior point, the remaining ones only contain points in the boundary. The correspondence goes as follows (see Figure 2):

1. *the disc containing the interior point corresponds to the circle;*
2. *points on the boundary of the disc containing the interior point correspond to points on the circle which are not inside any bracket;*
3. *a disc joined to the disc containing the interior point corresponds to a bracketing; two discs joined correspond to a bracketing inside another bracketing or to two disjoint bracketings, and so on.*

In order to get a tree from the bubbling off, we associate to the discs their dual graphs. According to the usual procedure, each disc correspond to a vertex; the point marked ∞ corresponds to the root; the double points correspond to the edges and the remaining marked points correspond to the leaves. Since the correspondence between bracketings on S^1 and joining discs is established, there follows the correspondence between bracketings on S^1 and trees in $\mathcal{T}_o(1,q)$ (see Figure 2). Since the facets of both polytopes are in a one-to-one correspondence compatible with their corresponding boundaries, it follows that $W_q = \overline{C(1,q)}$.

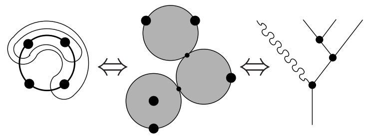


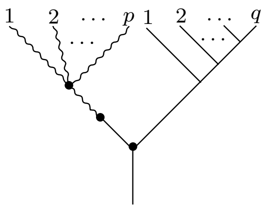
Figure 2: Example of the correspondence between circular bracketings and trees.

Figures 3 and 4 on page 148 illustrate the spaces $\overline{C(1,2)}$ and $\overline{C(2,0)}$. Figure 5 on page 149 is a portrait of $\overline{C(2,1)}$ due to S. Devadoss where we have included the partially planar trees corresponding to its codimension one boundary strata (see also [17]). The OCHA relations corresponding to the spaces $\overline{C(2,0)}$, $\overline{C(1,2)}$ and $\overline{C(2,1)}$ are given in formulas (14), (15) and (16).

5.2. The space $\overline{C(p)}$ as a deformation retract of $\overline{C(p,q)}$

We close the present section by pointing out a fact that will play a crucial role in the proof of our main theorem (Theorem 6.6). There is a stratum S_T in $\overline{C(p,q)}$ which is homeomorphic to $C(p)$, where T is the following tree:

$$T \quad = \qquad , \tag{25}$$



moreover, S_T is a deformation retract of $\overline{C(p,q)}$.

In fact, by putting a collar neighborhood along the boundary, we see that there is a deformation retraction of $\overline{C(p,q)}$ onto $C(p,q) = \overline{C(p,q)} \setminus \partial\,\overline{C(p,q)}$. This last space was defined as the quotient of $\mathrm{Conf}(p,q)$ by the group of affine transformations $z \mapsto az+b$, where $a, b \in \mathbb{R}$. Since that group is contractible, $C(p,q)$ is homotopy equivalent to $\mathrm{Conf}(p,q)$. Now, $\mathrm{Conf}(p,q)$ is homeomorphic to $\mathrm{Conf}_{\mathbb{C}}(p) \times \mathrm{Conf}_{\mathbb{R}}(q)$ and $\mathrm{Conf}_{\mathbb{R}}(q)$ is well known to be contractible. Thus, by composing all those contractions and homotopy equivalences, we get the claimed deformation retraction of $\overline{C(p,q)}$ onto S_T. To see that S_T is in fact homeomorphic to $C(p)$, just notice that T is obtained by grafting the trees l_p, $n_{1,0}$ and a binary planar tree. The configuration space corresponding to $n_{1,0}$ and to any binary planar tree is just one point, the homeomorphism thus follows from Theorem 5.2. Notice that the retraction is essentialy determined by the contraction of $\mathrm{Conf}_{\mathbb{R}}(q)$ to a single point. This means that the configuration of the interior points are unaffected during the contraction of $\overline{C(p,q)}$ onto $\overline{S_T} = \overline{C(p)}$.

Those facts will be used in the next section along with the fact, proven by P. May in [25], that $C(p)$ is S_p-equivariantly homotopy equivalent to $\mathcal{D}(p)$, where $\mathcal{D}$ denotes the little discs operad.

6. OCHA and the spectral sequence of $\overline{C(p,q)}$

In this section we show that the first row of the E^1 term of the spectral sequence of $\overline{C(p,q)}$ is isomorphic, as a chain complex, to $\mathcal{N}_\infty(p,q)$. The isomorphism is natural with respect to the operad composition. This fact depends crucially on the study of the stratification of $\overline{C(p,q)}$ as a manifold with corners.

Every compact manifold with corners induces a spectral sequence converging to its homology. In fact, the boundary strata of the manifold induces a natural filtration on its singular chain complex which ensures the existence of the spectral sequence. Since the boundary filtration is finite, the spectral sequence is convergent.

Let us study the spectral sequence in the case of the manifold $\overline{C(p,q)}$. Consider the topological filtration of $\overline{C(p,q)}$:

$$F^i\overline{C(p,q)} = \{\text{closure of the union } \coprod_T S_T \text{ of strata of dimension } i\} =$$

$$\bigcup \{\overline{S_T} \mid \dim S_T = i\}$$

We will denote $F^i\overline{C(p,q)}$ more simply by F^i, with $2p+q-2 \geqslant i \geqslant 0$, remembering that the dimension of $C(p,q)$ is $2p+q-2$. The topological filtration induces a filtration on the singular chain complex of $\overline{C(p,q)}$ and we have the spectral sequence.

Theorem 6.1. *There is a spectral sequence $E^r_{m,n}$ converging to $H_*(C(p,q))$. Its E^1 term has the form $E^1_{m,n} = H_{m+n}(F^m, F^{m-1})$ and, for $n=0$, the complex*

$$0 \to E^1_{2p+q-2,0} \to \cdots \to E^1_{m,0} \to E^1_{m-1,0} \to \cdots \to E^1_{0,0} \to 0$$

is isomorphic to the p,q component $\mathcal{N}_\infty(p,q)$ of the ideal $\mathcal{N}_\infty \lhd \mathcal{OC}_\infty$.

Proof. From Theorem 5.2, we have: $F^m \setminus F^{m-1} = \coprod_{|T|=-m} S_T$, where $|T|$ is the degree of T defined by (20). The equality $|T| = -m$ means that the number of internal edges of T is equal to the codimension of its corresponding submanifold S_T. Using the Lefshetz duality theorem:

$$E^1_{m,0} = H_m(F^m, F^{m-1}) = H^0(F^m \setminus F^{m-1}) = H^0(\coprod_{|T|=-m} S_T) = \bigoplus_{|T|=-m} k. \tag{26}$$

As a vector space, $E^1_{m,0}$ is thus exactly the vector space generated by trees of degree $-m$ in $\mathcal{N}_\infty(p,q)$. The spectral sequence is a homology spectral sequence, hence $|d^1| = -1$. On the other hand, $\mathcal{N}_\infty$ is an ideal in $\mathcal{OC}_\infty$ which is an operad of cochain complex. So the diferential d in the ideal $\mathcal{N}_\infty$ has degree 1. This is consistent with the fact, used in (26), that the degree of a tree is minus the dimension of its corresponding relative homology class.

Now we need to check that the differential d^1 on the first row of the spectral sequence coincides with d defined by formulas (22) and (4).

In fact, by Theorem 5.2 and the Lefshetz duality, each relative class in $H_m(F^m, F^{m-1})$ is given by the closure $\overline{S_T}$ of a submanifold S_T. To see that d^1 coincides with d, recall that the operator d^1 is given by the relativization of the boundary operator ∂:

$$H_{m-1}(F^{m-1}) \xrightarrow{j_{m-1}}$$
$$H_m(F^m, F^{m-1}) \xrightarrow{\partial_m} \xrightarrow{d^1_m} H_{m-1}(F^{m-1}, F^{m-2}).$$

Item 4 of Thm. 5.2, says that: $\partial\overline{S_T} = \bigcup_{T' \to T} \overline{S_{T'}}$. Observation 4.5 (pg. 135) implies that $d^1 = d$. □

Observation 6.2. *It is still necessary to check that the signs given by the coboundary operator in the spectral sequence coincide with the signs given in formulas (22) and (4). That is a somewhat tedious exercise which consists of comparing the orientation induced on the product of two oriented manifolds with the orientation induced by the operadic embedding of that product manifold into the boundary strata of other oriented manifold.*

6.1. The Quasi-isomorphism of $\mathcal{L}_\infty$-modules

According to Definition 2.4, the open-closed operad $\mathcal{OC}$ is the operad generated by top-dimensional homology classes of the swiss-cheese operad, i.e., $\mathcal{OC}$ is the suboperad of $H_\bullet(\mathcal{SC})$ generated by $n_{1,0}$, l_2 and $n_{0,2}$ (see Definition 2.4). Recall $\mathcal{L}$ is the suboperad of $H_\bullet(\mathcal{D})$ defined by $\mathcal{L}(n) = H_{n-1}(\mathcal{D}(n))$ for $n \geqslant 1$. We refer to $\mathcal{L}$ as the Lie operad, since algebras over it are equivalent to Lie algebras (see Observation 2.5 pg. 130). From the tree description of $H_\bullet(\mathcal{SC})$ given in section 2, we see that $\mathcal{OC}$ is a suboperad of $\mathcal{OC}_\infty$.

We know that $\mathcal{OC}_\infty$ is an $\mathcal{L}_\infty$-module (see Observation 4.6) and that $\mathcal{OC}$ is an $\mathcal{L}$-module, since $\mathcal{L}$ is an suboperad of $\mathcal{OC}$. Consequently, $\mathcal{OC}$ has a natural structure of $\mathcal{L}_\infty$-module induced by the well known quasi-isomorphism of operads: $\mu : \mathcal{L}_\infty \to \mathcal{L}$ defined by $\mu(l_2) = l_2$ and $\mu(l_n) = 0$ for $n \geqslant 3$ (see [22–24] for details).

Proposition AppendixB.1 in the Appendix says that there is a morphism of differential graded $\mathcal{L}_\infty$-modules extending the identity on $\mathcal{OC}$:

$$\mu : \mathcal{OC}_\infty \to \mathcal{OC}. \tag{27}$$

The $\mathcal{L}_\infty$-morphism μ vanishes on the corollae that are not in $\mathcal{OC}$. The restriction of μ to $\mathcal{L}_\infty$ coincides with the above mentioned quasi-isomorphism between $\mathcal{L}_\infty$ and $\mathcal{L}$. Since $\mathcal{OC}_\infty = \mathcal{L}_\infty \oplus \mathcal{N}_\infty$, in order to prove that μ is a quasi-isomorphism of $\mathcal{L}_\infty$-modules, we need to study the cohomology of the ideal $\mathcal{N}_\infty$(see corollary 6.5).

Let us begin by showing that, for any p, q such that $2p + q \geqslant 2$, the cohomology of $\mathcal{N}_\infty(p, q)$ is isomorphic, as $\mathbb{Z}$-graded vector spaces, to $H_\bullet(\mathcal{D}(p))$ (where $\mathcal{D}$ is the little discs operad). From Theorem 6.1, $\mathcal{N}_\infty(p, q)$ is isomorphic to the complex given by:

$$0 \longrightarrow H_{2p+q-2}(F^{2p+q-2}, F^{2p+q-3}) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_m(F^m, F^{m-1}) \longrightarrow \cdots \longrightarrow H_0(F^0) \longrightarrow 0$$

where each F^i is the closure of the disjoint union of the i-dimensional strata of $\overline{C(p, q)}$.

Notice that $E^r_{m,n}$ is a homology spectral sequence converging to $H_*(C(p, q))$. In the proof of Theorem 6.1, we have seen that the above complex is isomorphic to a complex generated by trees whose degree is minus the degree of their corresponding relative homology classes. So, it is a cochain complex generated by trees, precisely: $\mathcal{N}_\infty(p,q)$. In what follows, we use this to establish a relation between the *cohomology* of $\mathcal{N}_\infty$ and the *homology* of the little disks operad.

Since each stratum is a smooth submanifold, it follows that F^i has the homotopy type of a CW complex. The manifold $\overline{C(p,q)}$ has thus the homotopy type of a CW complex X such that each skeleton X^i is homotopy equivalent to F^i. It is well known that for any CW complex, the map $H_n(X^n) \to H_n(X^{n+1}) \simeq H_n(X)$, induced by the inclusion $X^n \hookrightarrow X^{n+1}$, is surjective for all n. Consequently, the map

$$H_n(F^n) \to H_n(F^{n+1}) \simeq H_n(\overline{C(p,q)})$$

is also surjective. For the same reason, we have: $H_n(F^{n-1}) \simeq H_n(X^{n-1}) = 0$. Now, consider the usual commutative diagram:

$$0 \longrightarrow H_n(F^{n+1}) \simeq H_n(\overline{C(p,q)})$$
$$H_n(F^n) \xrightarrow{j_n}$$
$$\cdots \longrightarrow H_{n+1}(F^{n+1},F^n) \xrightarrow{\partial_{n+1}} \xrightarrow{d_{n+1}} H_n(F^n,F^{n-1}) \xrightarrow{\partial_n} \xrightarrow{d_n} H_{n-1}(F^{n-1},F^{n-2}) \longrightarrow \cdots,$$
$$H_{n-1}(F^{n-1}) \xrightarrow{j_{n-1}}$$
$$0 \longrightarrow$$

since $H_n(F^n) \to H_n(F^{n+1}) \simeq H_n(\overline{C(p,q)})$ is surjective and $H_n(F^n) \xrightarrow{j_n} H_n(F^n,F^{n-1})$ is injective, from the exactness of the sequence of the pair (F^n,F^{n-1}) one can see that the nth cohomology group of the complex $\mathcal{N}_\infty(p,q)$ is isomorphic to $H_n(\overline{C(p,q)})$. As observed before, $\overline{C(p,q)}$ is homotopy equivalent to $\mathcal{D}(p)$, so the following lemma is proved.

Lemma 6.3. *$H^k(\mathcal{N}_\infty(p,q)) \simeq H_k(\mathcal{D}(p))$ for every $k \geqslant 0$ and p,q such that $2p+q \geqslant 2$.*

For any $q \geqslant 0$, consider the following sequence of vector spaces:

$$H^\bullet(\mathcal{N}_\infty(\_,q)) := \{H^\bullet(\mathcal{N}_\infty(p,q))\}_{p\geqslant 1}.$$

Since $\mathcal{L}$ is just the operad generated by a binary tree l_2 of degree 1 which is invariant under the action of the symmetric group S_2 and satisfies the Jacobi identity, there is a natural injection $\mathcal{L} \hookrightarrow H^\bullet(\mathcal{OC}_\infty)$. Since $\mathcal{N}_\infty$ is an ideal in $\mathcal{OC}_\infty$, it follows that $H^\bullet(\mathcal{N}_\infty)$ is an ideal in $H^\bullet(\mathcal{OC}_\infty)$. Consequently, for any $q \geqslant 0$ we have a structure of $\mathcal{L}$-module on $H^\bullet(\mathcal{N}_\infty(\_,q))$. Since $\mathcal{L} = \{H_{n-1}(\mathcal{D}(n))\}_{n\geqslant 1}$ is a suboperad of $H_\bullet(\mathcal{D})$, we also have a natural structure of $\mathcal{L}$-module on $H_\bullet(\mathcal{D})$. The next proposition is a stronger version of the above lemma.

Proposition 6.4. *For any $q \geqslant 0$, $H^\bullet(\mathcal{N}_\infty(\_,q))$ and $H_\bullet(\mathcal{D})$ are isomorphic as $\mathcal{L}$-modules.*

Proof. At the end of section 5 we observed that $\overline{C(p,q)}$ deformation retracts to a stratum S_T which is homeomorphic to $C(p)$. That deformation retract takes each stratum of dimension m (represented by a partially planar tree of degree $-m$) in $\overline{C(p,q)}$ to an m-dimensional singular chain in $\overline{S_T} = \overline{C(p)}$.

In fact, following the notation of Theorem 5.2, let S_U be a stratum of dimension m corresponding to a tree U of degree $|U| = -m$. Its closure $\overline{S_U}$ is a connected smooth oriented manifold with corners (topologically it is a manifold with boundary). Let $[\overline{S_U}]$ be the relative fundamental class in $H_m(\overline{S_U}, \partial\overline{S_U})$. For each stratum S_U, take a singular chain in $\overline{C(p,q)}$ representing the fundamental class $[\overline{S_U}]$. By composing with the contraction $\overline{C(p,q)} \to \overline{C(p)}$, we see that those singular chains are taken to singular chains in $\overline{C(p)}$. Hence we have a chain map:

$$\psi_p : \mathcal{N}_\infty(p,q) \longrightarrow C_*(\overline{C(p)}) \tag{28}$$

and an induced map in homology

$$\Psi_p : H^\bullet(\mathcal{N}_\infty(p,q)) \longrightarrow H_\bullet(C(p)) \simeq H_\bullet(\mathcal{D}(p)), \qquad \text{for each } p \geqslant 1. \tag{29}$$

Since the contraction leaves the configuration of the interior points unaffected (see subsection 5.2), classes representred by trees with only spatial edges will also be unaffected. It follows that the class $[S_{\delta_k \circ_i U}] = [S_{\delta_k}] \times [S_U]$ will be taken to the class $[S_{\delta_k}] \times \psi_p([S_U])$ for any spatial corolla $\delta_k \in \mathcal{L}_\infty$. Consequently, the sequence of maps $\{\Psi_p\}$ define a morphism of $\mathcal{L}$-modules:

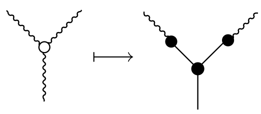
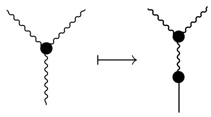
$$\Psi : H^\bullet(\mathcal{N}_\infty(\_, q)) \longrightarrow H_\bullet(\mathcal{D}).$$

In order to show that Ψ is an isomorphism, let us now construct a map from $H_\bullet(\mathcal{D}(p))$ to $H^\bullet(\mathcal{N}_\infty(p, 0))$, for each $p \geqslant 1$. In case $p = 1$, define the map:

$$\Phi_1 : H_\bullet(\mathcal{D}(1)) \longrightarrow H^\bullet(\mathcal{N}_\infty(1, 0))$$

by taking the identity in $e \in H_\bullet(\mathcal{D}(1))$ into $n_{1,0} =$ .

When $p = 2$, the map $\Phi_2 : H_\bullet(\mathcal{D}(2)) \longrightarrow H^\bullet(\mathcal{N}_\infty(2, 0))$ is defined by:

 and  .

Now that we have defined our maps on the operad generators of $H_\bullet(\mathcal{D})$, we define the map $\Phi_p : H_\bullet(\mathcal{D}(p)) \to H^\bullet(\mathcal{N}_\infty(p, 0))$, for any p, in the following way:

i) *if $T \in H_\bullet(\mathcal{D}(p))$ has only white vertices (i.e., corresponds to a zero dimensional homology class), then $\Phi_p(T)$ is defined by grafting $n_{1,0}$ to all the leaves of the tree obtained from T by making all vertices black and all edges straight;*

ii) *extend Φ_p to the whole $H_\bullet(\mathcal{D}(p))$ so that the resulting map*

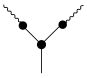
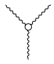
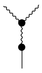
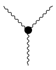
$$\Phi : H_\bullet(\mathcal{D}) \to H^\bullet(\mathcal{N}_\infty(\_, 0)) \tag{30}$$

becomes a morphism of left modules over $\mathcal{L} = \{H_{n-1}(\mathcal{D}(n))\}_{n \geqslant 1}$, i.e., such that

$$\Phi(T \circ_i l_2) = \Phi(T) \circ_i l_2, \text{ for any } T \in H_\bullet(\mathcal{D}).$$

In conclusion, we have defined another morphism of $\mathcal{L}$-modules

$$\Phi : H_\bullet(\mathcal{D}) \to H^\bullet(\mathcal{N}(\_,0)).$$

To see that $\Phi : H_\bullet(\mathcal{D}) \to H^\bullet(\mathcal{N}_\infty(\_,0))$ is an isomorphism, let us show that the composition $\Psi \circ \Phi$ is the identity in $H_\bullet(\mathcal{D})$. In fact: since both Ψ and Φ are morphisms of $\mathcal{L}$-modules, we need only to check that on generators. Observe that  correspond to a zero dimensional component of the boundary strata of $\overline{C(2,0)}$ and is taken to the zero dimensional generator  $\in H_0(\mathcal{D}(2))$ under the deformation retraction $\overline{C(2,0)} \to \overline{C(2)} \cong \mathcal{D}(2)$ used to define Ψ. On the other hand,  is homemorphic to S^1 (see Figure 4 pg. 148) and is naturally taken to  $\in H_1(\mathcal{D}(2))$ under the same deformation retraction, so $\Psi \circ \Phi = Id$. From lemma 6.3, we know that the vector spaces $H_\bullet(\mathcal{D}(p))$ and $H^\bullet(\mathcal{N}_\infty(p,0))$ have the same dimension for each $p \geqslant 1$. It follows that Φ is in fact a bijection.

Finally we just need to observe that $H^\bullet(\mathcal{N}_\infty(\_,0))$ is naturally isomorphic as an $\mathcal{L}$-module to $H^\bullet(\mathcal{N}_\infty(\_,q))$ for any $q \geqslant 0$. The isomorphism being induced by the grafting operation with some fixed binary planar tree T with $q+1$ leaves. $\square$

Corollary 6.5. *The cohomology $H^\bullet(\mathcal{N}_\infty)$ is the ideal of $H^\bullet(\mathcal{OC}_\infty)$ generated by $n_{1,0}$ and $n_{0,2}$.*

Proof. It is immediate from the explicit definition of the $\mathcal{L}$-isomorphism Φ that any class in $H^\bullet(\mathcal{N}_\infty(p,q))$ can be obtained by grafting a finite number of trees of the form $n_{1,0}$ and $n_{0,2}$ followed by grafting a finite number of the form l_2, i.e., by the action of $\mathcal{L}$ on $H^\bullet(\mathcal{N}(\_,q))$. $\square$

We can now prove our main result.

Theorem 6.6. *The morphism of differential graded $\mathcal{L}_\infty$-modules $\mu : \mathcal{OC}_\infty \longrightarrow \mathcal{OC}$ induces an isomorphism in cohomology.*

Proof. It is sufficient to show that the cohomology OCHA operad $H^\bullet(\mathcal{OC}_\infty)$ and $\mathcal{OC}$ are isomorphic as operads of graded vector spaces. Let us first recall that the operad $\mathcal{OC}_\infty$ is decomposed as a direct sum: $\mathcal{OC}_\infty = \mathcal{L}_\infty \oplus \mathcal{N}_\infty$, where $\mathcal{L}_\infty$ is the operad of L_∞-algebras and $\mathcal{N}_\infty$ is the is the ideal of partially planar trees with planar root. Since the differential operator d respects the direct sum decomposition, the homology of $\mathcal{OC}_\infty$ is a direct sum: $H^\bullet(\mathcal{OC}_\infty) = \mathcal{L} \oplus H^\bullet(\mathcal{N}_\infty)$. Now we just observe that $\mathcal{L}$ is the operad generated by l_2 and, from Corollary 6.5, $H^\bullet(\mathcal{N}_\infty)$ is generated by $n_{1,0}$ and $n_{0,2}$. The relations listed in the statement of Theorem 2.2 are naturally satisfied in $H^\bullet(\mathcal{OC}_\infty)$ since they are just the homology version of the OCHA axioms. $\square$

Considering that $\mathcal{OC}$ is a suboperad of $H_\bullet(\mathcal{SC})$, an interesting problem that might be pursued in a sequel to the present paper is to extend our results to the whole operad $H_\bullet(\mathcal{SC})$. That would involve the entire spectral sequence of $\overline{C(p,q)}$ (see also the comments at the end of [32]).

AppendixA. OCHA as a Coderivation Differential

We say that a coderivation $\phi \in \mathrm{Coder}(S^c(U) \otimes T^c(V))$ is in **OCHA form** if it can be written as a sumation

$$\phi = \sum_{n\geqslant 1} \tilde{g}_n + \sum_{p+q\geqslant 1} \tilde{f}_{p,q},$$

where $\tilde{g}_n$ and $\tilde{f}_{p,q}$ denote the lifting as a coderivation of some maps: $g_n : U^{\wedge n} \to U$ and $f_{p,q} : U^{\wedge p} \otimes V^{\otimes q} \to V$. In [13] we have proven that all coderivations in $\mathrm{Coder}(S^c(U) \otimes T^c(V))$ are in OCHA form for any vector spaces U and V over a field k of characteristic zero.

Proposition 3.4. *An OCHA structure $(L, A, \mathfrak{l}, \mathfrak{n})$, in the grading and signs conventions of defintion 3.2, is equivalent to a degree one coderivation $D \in \mathrm{Coder}(S^c(\downdownarrows L) \otimes T^c(\downarrow A))$ such that $D^2 = 0$.*

Proof. Let us begin by defining: $\tilde{l}_1 = -l_1$ and $\tilde{n}_{0,1} = -n_{0,1}$ as the differential operators respectively on $\downdownarrows L$ and on $\downarrow A$. Let $D \in \mathrm{Coder}(S^c(\downdownarrows L) \otimes T^c(\downarrow A))$ be any degree one coderivation such that $D^2 = 0$. Since any coderivation in $\mathrm{Coder}(S^c(\downdownarrows L) \otimes T^c(\downarrow A))$ is in OCHA form, D is obtained by lifting maps $\tilde{l}_n : (\downdownarrows L)^{\otimes n} \to \downdownarrows L$ for $n \geqslant 1$ and $\tilde{n}_{p,q} : (\downdownarrows L)^{\otimes p} \otimes (\downarrow A)^{\otimes q} \to \downarrow A$ for $p+q \geqslant 1$, where all the maps $\tilde{l}_n$ and $\tilde{n}_{p,q}$ have degree one.

Equation $D^2 = 0$ holds if and only if $\{\tilde{l}_n\}_{n\geqslant 1}$ satisfies the conditions of an L_∞ algebra and $\{\tilde{n}_{p,q}\}_{p+q\geqslant 1}$ satisfies the conditions of an OCHA as originally defined in [15]:

$$0 = \sum_{\sigma \in \Sigma_{p+r=n}} \Big(\tilde{n}_{1+r,m}(\tilde{l}_p \otimes \mathbf{1}_L^{\otimes r} \otimes \mathbf{1}_A^{\otimes m}) \; +$$

$$\sum_{i+j+s=m} \tilde{n}_{p,i+1+j}(\mathbf{1}_L^{\otimes p} \otimes \mathbf{1}_A^{\otimes i} \otimes \tilde{n}_{r,s} \otimes \mathbf{1}_A^{\otimes j}) \Big)(E(\sigma) \otimes \mathbf{1}_A^{\otimes m}). \tag{31}$$

Now define maps $l_n : L^{\otimes n} \to L$ and $n_{p,q} : L^{\otimes p} \otimes A^{\otimes q} \to A$, with $\deg(l_n) = 3 - 2n$ and $\deg(n_{p,q}) = 2 - 2p - q$ such that: $\tilde{l}_p = \downdownarrows l_p(\upuparrows)^{\otimes p}$ and $\tilde{n}_{p,q} = \downarrow n_{p,q}(\upuparrows^{\otimes p} \otimes \uparrow^{\otimes q})$.

Thus:

$$\tilde{n}_{1+r,m}(\tilde{l}_p \otimes \mathbf{1}_L^{\otimes r} \otimes \mathbf{1}_A^{\otimes m}) + \sum_{i+j+s=m} \tilde{n}_{p,i+1+j}(\mathbf{1}_L^{\otimes p} \otimes \mathbf{1}_A^{\otimes i} \otimes \tilde{n}_{r,s} \otimes \mathbf{1}_A^{\otimes j}) =$$

$$= \downarrow n_{1+r,m}(\upuparrows^{\otimes 1+r} \otimes \uparrow^{\otimes m})(\downdownarrows l_p(\upuparrows)^{\otimes p} \otimes \mathbf{1}_L^{\otimes r} \otimes \mathbf{1}_A^{\otimes m}) +$$

$$+ \sum_{i+j+s=m} \downarrow n_{p,i+1+j}(\upuparrows^{\otimes p} \otimes \uparrow^{\otimes i+1+j})(\mathbf{1}_L^{\otimes p} \otimes \mathbf{1}_A^{\otimes i} \otimes \downarrow n_{r,s}(\upuparrows^{\otimes r} \otimes \uparrow^{\otimes s}) \otimes \mathbf{1}_A^{\otimes j}) =$$

$$= \downarrow n_{1+r,m}(l_p \otimes \mathbf{1}_L^{\otimes r} \otimes \mathbf{1}_A^{\otimes m})(\upuparrows^{\otimes n} \otimes \uparrow^{\otimes m}) +$$

$$+ \sum_{i+j+s=m} (-1)^{js+i} \downarrow n_{p,i+1+j}(\mathbf{1}_L^{\otimes p} \otimes \mathbf{1}_A^{\otimes i} \otimes n_{r,s} \otimes \mathbf{1}_A^{\otimes j})(\upuparrows^{\otimes n} \otimes \uparrow^{\otimes m}) =$$

$$= \downarrow \Big(n_{1+r,m}(l_p \otimes \mathbf{1}_L^{\otimes r} \otimes \mathbf{1}_A^{\otimes m})$$

$$+ \sum_{i+j+s=m} (-1)^{js+i} n_{p,i+1+j}(\mathbf{1}_L^{\otimes p} \otimes \mathbf{1}_A^{\otimes i} \otimes n_{r,s} \otimes \mathbf{1}_A^{\otimes j}) \Big)(\upuparrows^{\otimes n} \otimes \uparrow^{\otimes m}),$$

where the sign $(-1)^{js+i}$ comes from the Koszul sign convention. Observing that $\tilde{l}_1 = -l_1$, $\tilde{n}_{0,1} = -n_{0,1}$ and $(-1)^{js+i} = (-1)^{s+i+si+ms}$, we obtain formula (11) from formula (AppendixA). □

AppendixB. Existence of the DG $\mathcal{L}_\infty$-module morphism

Proposition AppendixB.1. *There is a morphism of differential graded $\mathcal{L}_\infty$-modules $\mu : \mathcal{OC}_\infty \to \mathcal{OC}$ extending the identity on $\mathcal{OC}$, i.e., such that the following diagram is commutative:*

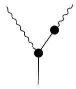
$$\begin{array}{ccc} \mathcal{OC}_\infty & \xrightarrow{\mu} & \\ \uparrow & & \\ \mathcal{OC} & \xrightarrow{id} & \mathcal{OC}\ . \end{array}$$

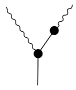
Proof. In this proof we shall omit the labels on trees because they are not crucial in the argument.

The open-closed operad $\mathcal{OC}$ is a differential graded operad where the differential operator δ is trivial: $\delta \equiv 0$. On the other hand, the differential operator d of the OCHA operad $\mathcal{OC}_\infty$ is defined by formulas (22) and (4). We will exhibit a chain map $\mu : \mathcal{OC}_\infty \to \mathcal{OC}$ which is also a morphism of $\mathcal{L}_\infty$-modules. In other words, μ must satisfy two conditions:

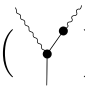
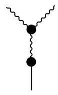
$$\mu(dT) = 0, \quad \forall T \in \mathcal{OC}_\infty$$

$$\mu(l \circ_i T) = l \circ_i \mu(T), \quad \forall T \in \mathcal{OC}_\infty \text{ and } \forall l \in \mathcal{L}_\infty.$$

Let $\mathcal{E}$ be the $\mathcal{L}_\infty$-submodule of $\mathcal{OC}_\infty$ generated by $\mathcal{OC}$ and by  :

$$\mathcal{E} = \Big\langle \mathcal{OC}\,,$$  $$\Big\rangle$$

On the generators of the submodule $\mathcal{E}$, the map μ is defined in the following way:

$$\mu(T) = T \quad \forall\, T \in \mathcal{OC} \quad \text{and} \quad \mu\Big($$  $$\Big) = -\frac{1}{2}$$ 

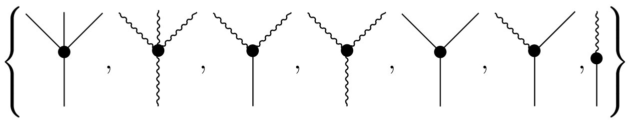
and it is extended to $\mathcal{E}$ as an $\mathcal{L}_\infty$-morphism. Finally, for any tree $T \in \mathcal{OC}_\infty$ such that $T \notin \mathcal{E}$, we define $\mu(T) = 0$. We thus have an $\mathcal{L}_\infty$-morphism:

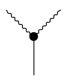
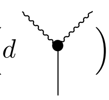
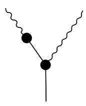
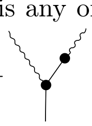
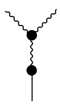
$$\mu : \mathcal{OC}_\infty \to \mathcal{OC}.$$

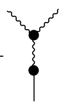
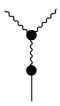
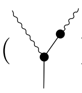
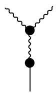
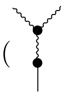
It remains to show that μ is a chain map, i.e., that $\mu(dT) = 0$ for any tree $T \in \mathcal{OC}_\infty$. Given any tree $T \in \mathcal{OC}_\infty$, dT is a summation of trees. By the definition of μ, if T is such that dT has no components in $\mathcal{E}$, then $\mu(dT) = 0$. Hence, we just need to consider those elements $T \in \mathcal{OC}_\infty$ such that dT has some component in $\mathcal{E}$. Such elements form an $\mathcal{L}_\infty$-submodule of $\mathcal{OC}_\infty$ which will be denoted by $\mathcal{E}'$. More precisely:

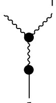
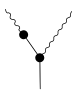
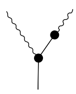
$$\mathcal{E}' := \{T \in \mathcal{OC}_\infty \;:\; dT = T_1 + T_2, \quad T_1 \in \mathcal{E}, T_1 \neq 0\}.$$

Any tree T is obtained by grafting a finite number of corollae which we call the *irreducible components* of T. Recall that, for $n \geqslant 3$, the $\mathcal{L}_\infty$-module action of $l_n \in \mathcal{L}_\infty$ on any element of $\mathcal{OC}$ is zero since that action is defined through the quasi-ismorphism $\mu : \mathcal{L}_\infty \to \mathcal{L}$, and $\mu(l_n) = 0$ for $n \geqslant 3$. From the definition of $\mu : \mathcal{OC}_\infty \to \mathcal{OC}$ and the definition of the $\mathcal{L}_\infty$-module structure on $\mathcal{OC}$, one can see that the irreducible components of any tree $T \in \mathcal{E}'$ such that $\mu(dT) \neq 0$ could only be one of the following corollae:



Consequently, we just need to check that $\mu(dT) = 0$ where T is any of the above corollae. In the case of $T =$  : $\mu\Big(d$  $\Big) = \mu\Big($  $+$  $+$  $\Big) =$

$-$  $+$  $= 0$, since by definition we have: $\mu($  $) = -\frac{1}{2}$  , $\mu($  $) =$

 and because the wiggly edges are spatial, we also have:  $=$  . The other corollae can be handled similarly. □

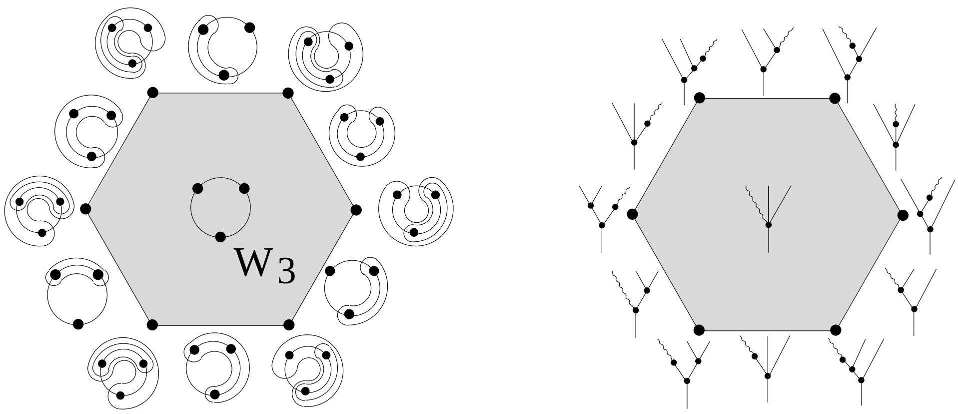


Figure 3: Cyclohedron $\overline{C(1,2)}$ and its cells labelled by circular bracketings and by trees.

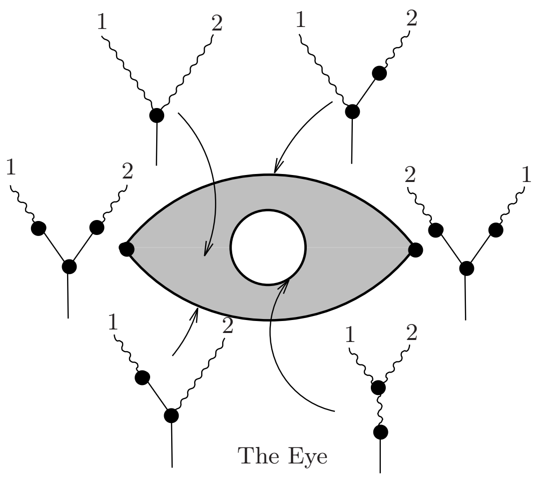


Figure 4: The space $\overline{C(2,0)}$ = "The Eye" and its boundary strata labelled by trees.

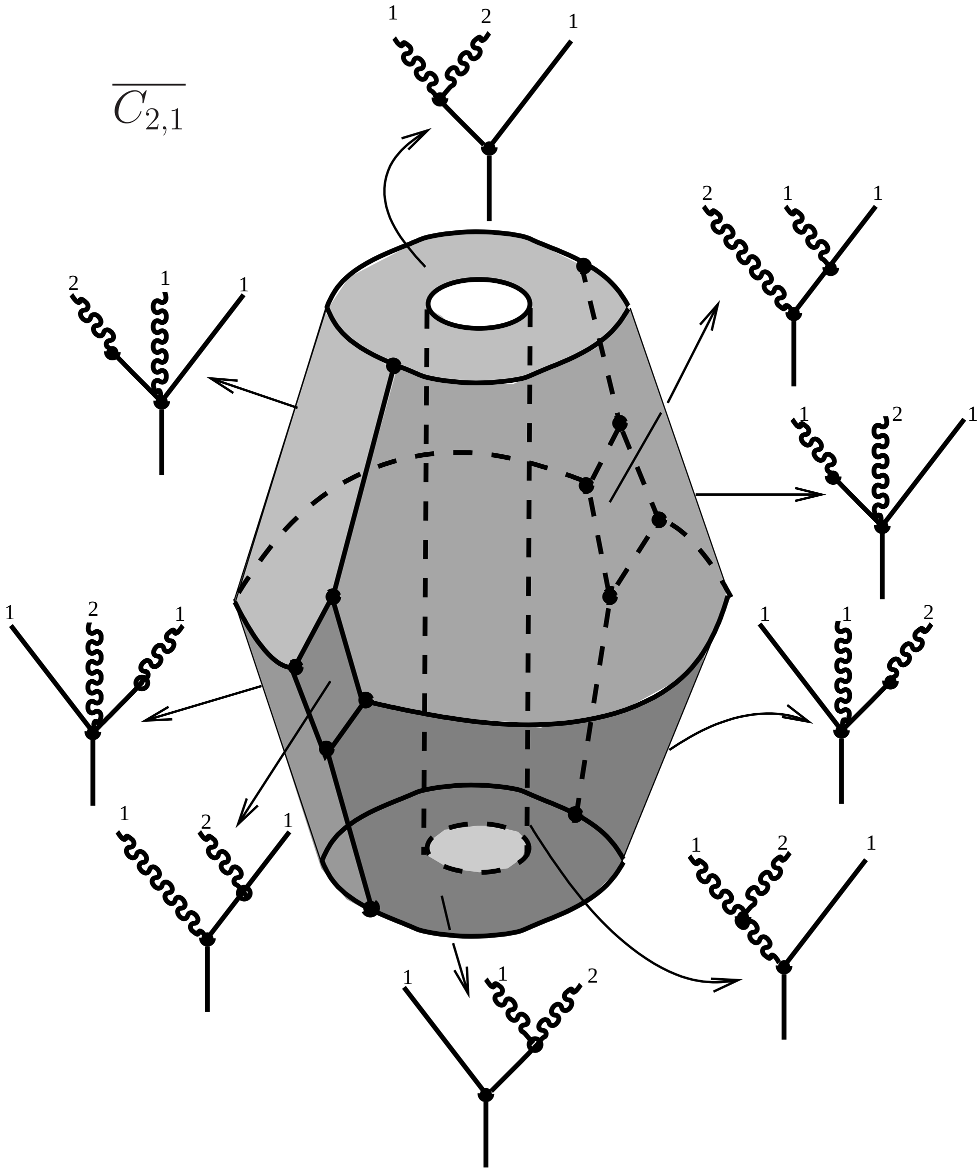


Figure 5: The space $\overline{C(2,1)}$, which is topologically equivalent to a solid torus, and its codimension 1 boundary components labelled by partially planar trees.

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