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ON THE CLASSIFICATION OF UNSTABLE $H^*V - A$-MODULES

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Abstract

In this work, we begin studying the classification, up to isomorphism, of unstable H^*V-A-modules E such that $\mathbb{F}_2\otimes_{\mathrm{H}^*V}E$ is isomorphic to a given unstable A-module M. In fact this classification depends on the structure of M as unstable A-module. In this paper, we are interested in the case M a nil-closed unstable A-module and the case M is isomorphic to $\sum^n \mathbb{F}_2$. We also study, for $V = \mathbb{Z}/2\mathbb{Z}$, the case M is the Brown-Gitler module J(2).

1. Introduction

Let V be an elementary abelian 2-group of rank d, that is a group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^d$, $d \in \mathbb{N}$, BV be a classifying space for the group V and $\mathrm{H}^*V = H^*(BV;\mathbb{F}_2)$. We recall that H^*V is an $\mathbb{F}_2$-polynomial algebra $\mathbb{F}_2[t_1,\ldots,t_d]$ on d generators t_i, $1 \leqslant i \leqslant d$, of degree one.

Let A be the mod.2 Steenrod algebra and $\mathcal{U}$ the category of unstable A-modules. We recall that $\mathrm{H}^*V-\mathcal{U}$ is the category whose objects are unstable H^*V-A-modules and morphisms are H^*V-linear and A-linear maps of degree zero. For example, the mod.2 equivariant cohomology of a V-CW-complex, which is the cohomology of the Borel construction, is an unstable $\mathrm{H}^*V - A$-module.

Let E be an unstable $\mathrm{H}^*V - A$-module, we denote by $\overline{E}$ the unstable A-module $\mathbb{F}_2 \otimes_{\mathrm{H}^*V} E = E/\widetilde{\mathrm{H}^*V}.E$, where $\widetilde{\mathrm{H}^*V}$ denotes the augmentation ideal of H^*V .

We have the following problem:

$(\mathcal{P})$: Let M be an unstable A-module.
Classify, up to isomorphism, unstable $\mathrm{H}^*V - A$-modules
such that $\overline{E} \cong M$ (as unstable A-modules).

It is clear that, for every subgroup W of V, the unstable $\mathrm{H}^*V - A$-module:

$$\mathrm{H}^*W \otimes M$$

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is a solution for the problem $(\mathcal{P})$.

For $W = 0$, a solution of $(\mathcal{P})$ is given by the unstable $\mathrm{H}^*V - A$-module M which is trivial as an H^*V-module.

For $W = V$, a solution of $(\mathcal{P})$ is given by the unstable $\mathrm{H}^*V - A$-module $\mathrm{H}^*V \otimes M$ which is free as an H^*V-module.

If $V = \mathbb{Z}/2\mathbb{Z}$ and $M = \Sigma N$ a suspension of an unstable A-module N, then we have, at least, the following two solutions of the problem $(\mathcal{P})$ which are free as $H^*(\mathbb{Z}/2\mathbb{Z})$-modules:

1. $\Sigma(H^*(\mathbb{Z}/2\mathbb{Z}) \otimes N)$.
2. $((H^*(\mathbb{Z}/2\mathbb{Z})^{\geqslant 1}) \otimes N$.

These two solutions are different as unstable A-modules (here $H^*(\mathbb{Z}/2\mathbb{Z})^{\geqslant 1}$ is the sub-algebra of $H^*(\mathbb{Z}/2\mathbb{Z})$ of elements of degree bigger than or equal to one). This shows that the solutions of the problem $(\mathcal{P})$ i.e. the classification, up to isomorphism, of unstable $\mathrm{H}^*V - A$-modules such that $\overline{E} \cong M$ (as unstable A-modules), depends on the structure of E as an H^*V-module and on the structure of M as unstable A-module.

In this paper we will discuss the solutions of $(\mathcal{P})$ if M is a nil-closed unstable A-module and E is free as an H^*V-module and the solutions of $(\mathcal{P})$ if M is isomorphic to $\sum^n \mathbb{F}_2$ or to $\mathrm{J}(2)$ and E is free as an H^*V-module .

We begin by proving the following result (which is solution of $(\mathcal{P})$ when M is a nil-closed unstable A-module).

Theorem 1.1. *Let E be unstable $\mathrm{H}^*V - A$-module which is free as an H^*V-module. If $\overline{E}$ is a nil-closed unstable A-module, then there exists two reduced $\mathcal{U}$-injectives I_0, I_1 and an $\mathrm{H}^*V - A$-linear map*
$\varphi : \mathrm{H}^*V \otimes I_0 \to \mathrm{H}^*V \otimes I_1$ such that:

1. *$E \cong ker\varphi$*
2. *$\overline{E} \cong ker\overline{\varphi}$*

The proof of this result is based on the classification of $\mathrm{H}^*V - \mathcal{U}$-injectives and on some properties of the injective hull in the category $\mathrm{H}^*V - \mathcal{U}$.

Our work is naturally motivated by topology as shown in the study of homotopy fixed points of a $\mathbb{Z}/2$-action (see [**L1**]). Let X be a space equipped with an action of $\mathbb{Z}/2$ and $X^{\mathrm{h}\mathbb{Z}/2}$ denote the space of homotopy fixed points of this action. The problem of determining the mod. 2 cohomology of $X^{\mathrm{h}\mathbb{Z}/2}$ (we ignore deliberately the questions of 2-completion) involves two steps:

– determining the mod. 2 equivariant cohomology $\mathrm{H}^*_{\mathbb{Z}/2}X$;
– determining $\mathrm{Fix}_{\mathbb{Z}/2}\, \mathrm{H}^*_{\mathbb{Z}/2}X$ (for the definition of the functor $\mathrm{Fix}_{\mathbb{Z}/2}$ see section 2).

For the first step, see for example [**DL**], the main information one has about the $\mathbb{Z}/2$-space X is that the Serre spectral sequence, for mod . 2 cohomology, associated to the fibration

$$X \to X_{\mathrm{h}\mathbb{Z}/2} \to \mathrm{B}\mathbb{Z}/2$$

collapses ($X_{\mathrm{h}\mathbb{Z}/2}$ denotes the Borel construction $\mathrm{E}\mathbb{Z}/2 \times_{\mathbb{Z}/2} X$). This collapsing implies that $\mathrm{H}^*_{\mathbb{Z}/2}X$ is H-free and that $\overline{\mathrm{H}^*_{\mathbb{Z}/2}X}$ is canonically isomorphic to H^*X. This gives clearly a topological application of problem $(\mathcal{P})$.

We then prove the following results (related to the case $\overline{E}$ is $\sum^n \mathbb{F}_2$ and $\mathrm{J}(2)$).

Theorem 1.2. *Let E be unstable $\mathrm{H}^*V - A$-module which is free as an H^*V-module. If $\overline{E}$ is isomorphic to $\sum^n \mathbb{F}_2$, then there exists an element u in H^*V such that:*

1. $u = \prod_i \theta_i^{\alpha_i}$, *where* $\theta_i \in (\mathrm{H}^1V) \setminus \{0\}$ *and* $\alpha_i \in \mathbb{N}$
2. $E \cong \sum^d u\mathrm{H}^*V$ *with* $d + \sum_i \alpha_i = n$

Proposition 1.3. *Let E be an $\mathrm{H} - A$-module which is* H*-free and such that $\overline{E}$ is isomorphic to* $\mathrm{J}(2)$ *then:*
$E \cong \mathrm{H} \otimes \mathrm{J}(2)$
or
E is the sub-$\mathrm{H} - A$-module of $\mathrm{H} \oplus \sum \mathrm{H}$ generated by $(t, \Sigma 1)$ and $(t^2, 0)$.

The proofs of these two results are based on Smith theory, some properties of the functor Fix and on a result of J.P. Serre.

The paper is structured as follows. In section 2, we introduce the definitions of reduced and nil-closed unstable A-modules. We give the classification of injective modules in the category $\mathcal{U}$ and in the category $H^*V - \mathcal{U}$. We also recall the algebraic Smith theory. In section 3, we establish some properties of E when $\overline{E}$ is a reduced unstable A-module. The results will be useful in section 4, where we give the solutions of the problem $(\mathcal{P})$ when E is free as an H^*V-module and $\overline{E}$ is nil-closed. In section 5, we give some topological applications. In section 6, we give the solutions of the problem $(\mathcal{P})$ when E is free as an H^*V-module and $\overline{E}$ is isomorphic to $\sum^n \mathbb{F}_2$, we also give a topological application. In section 7, we solve the problem $(\mathcal{P})$ when $\overline{E}$ is the Brown-Gitler module $\mathrm{J}(2)$ and V is $\mathbb{Z}/2\mathbb{Z}$.

2. Preliminaries on the categories $\mathcal{U}$ and $\mathrm{H}^*V - \mathcal{U}$

In this section, we will fix some notations, recall some definitions and results about the categories $\mathcal{U}$ and $\mathrm{H}^*V - \mathcal{U}$.

2.1. Nilpotent unstable A-modules

Let N be an unstable A-module. We denote by Sq_0 the $\mathbb{Z}/2\mathbb{Z}$-linear map:

$$Sq_0 : N \to N,\ x \mapsto Sq_0(x) = Sq^{|x|}x.$$

An unstable A-module N is called nilpotent if:

$$\forall\, x \in N,\ \exists\, n \in \mathbb{N};\ Sq_0^n x = 0.$$

For example, finite unstable A-modules and suspension of unstable A-modules are nilpotent. Let $Tor_1^{\mathrm{H}^*V}(\mathbb{F}_2, N)$ be the first derived functor of the functor $\mathbb{F}_2 \otimes_{\mathrm{H}^*V} -\ :\ \mathrm{H}^*V - \mathcal{U} \to \mathcal{U}$, we have the following useful result.

Proposition 2.1.1. *([**S**] page 150) Let N be an unstable $\mathrm{H}^*V - A$-module, then the unstable A-module $Tor_1^{\mathrm{H}^*V}(\mathbb{F}_2, N)$ is nilpotent.*

2.2. Reduced unstable A-modules

An unstable A-module M is called reduced if the $\mathbb{Z}/2\mathbb{Z}$-linear map:

$$Sq_0 : M \to M,\ x \mapsto Sq_0(x) = Sq^{|x|}x,$$

is an injection.

Another characterization of reduced unstable A-module in terms of nilpotent modules is the following.

Lemma 2.2.1. *([**LZ1**]) An unstable A-module is reduced if it does not contain a non-trivial nilpotent module.*

In particular, any A-linear map from a nilpotent A-module to a reduced one is trivial.

2.3. Nil-closed unstable A-modules

Let M be an unstable A-module. We denote by Sq_1 the $\mathbb{Z}/2\mathbb{Z}$-linear map:

$$Sq_1 : N \to N,\ x \mapsto Sq_1(x) = Sq^{|x|-1}x.$$

Definition 2.3.1. *([**EP**]) An unstable A-module M is called nil-closed if:*

1. *M is reduced.*
2. *$Ker(Sq_1) = Im(Sq_0)$.*

We have the following two characterizations of unstable nil-closed A-modules.

Lemma 2.3.2. *([**LZ1**]) Let M be an unstable A-module and $\mathcal{E}(M)$ be its injective hull. The unstable A-module M is nil-closed if and only if M and the quotient $\mathcal{E}(M)/M$ are reduced.*

Let $Ext^s_{\mathcal{U}}(-, M)$ be the s-th derived functor of the functor $\mathrm{H}om_{\mathcal{U}}(-, M)$.

Lemma 2.3.3. *([**LZ1**]) An unstable A-module M is nil-closed if and only if $Ext^s_{\mathcal{U}}(N, M) = 0$ for any nilpotent unstable A-module N and $s = 0, 1$.*

2.4. Injectives in the category $\mathcal{U}$

Let I be an unstable A-module, I is called an injective in the category $\mathcal{U}$ or $\mathcal{U}$-injective for short, if the functor $\mathrm{H}om_{\mathcal{U}}(-, I)$ is exact.

The classification of $\mathcal{U}$-injectives (see [**LZ1**], [**LS**]) is the following.

Let $\mathrm{J}(n)$, $n \in \mathbb{N}$, be the n-th Brown- Gitler module, characterized up to isomorphism, by the functorial bijection on the unstable A-module M:

$$\mathrm{H}om_{\mathcal{U}}(M, \mathrm{J}(n)) \cong \mathrm{H}om_{\mathbb{F}_2}(M^n, \mathbb{F}_2)$$

Clearly $\mathrm{J}(n)$ is an $\mathcal{U}$-injective and it is a finite module.
Let $\mathcal{L}$ be a set of representatives for $\mathcal{U}$-isomorphism classes of indecomposable direct factors of $\mathrm{H}^*(\mathbb{Z}/2\mathbb{Z})^m$, $m \in \mathbb{N}$ (each class is represented in $\mathcal{L}$ only once).
We have:

Theorem 2.4.1. *Let I be an $\mathcal{U}$-injective module. Then there exists a set of cardinals $a_{L,n}$, $(L, n) \in \mathcal{L} \times \mathbb{N}$, such that $I \cong \bigoplus_{(L,n)} (L \otimes \mathrm{J}(n))^{\oplus a_{L,n}}$.*
Conversely, any unstable A-module of that form is $\mathcal{U}$-injective.

Let's remark that H^*V is an $\mathcal{U}$-injective.

2.5. The injectives of the category $\mathrm{H}^*V - \mathcal{U}$

The classification of injectives of the category $\mathrm{H}^*V - \mathcal{U}$ ($\mathrm{H}^*V - \mathcal{U}$-injectives for short) is given by Lannes-Zarati [**LZ2**] as follows.
Let $\mathrm{J}_V(n)$, $n \in \mathbb{N}$, be the unstable $\mathrm{H}^*V - A$-module characterized, up to isomorphism, by the functorial bijection on the unstable $\mathrm{H}^*V - A$-module M:

$$\mathrm{H}om_{\mathrm{H}^*V-\mathcal{U}}(M, \mathrm{J}_V(n)) \cong \mathrm{H}om_{\mathbb{F}_2}(M^n, \mathbb{F}_2)$$

Clearly $\mathrm{J}_V(n)$ is an $\mathrm{H}^*V - \mathcal{U}$-injective.
Let $\mathcal{W}$ be the set of subgroups of V and let $(W, n) \in \mathcal{W} \times \mathbb{N}$, we write

$$E(V, W, n) = \mathrm{H}^*V \otimes_{\mathrm{H}^*V/W} \mathrm{J}_{V/W}(n)$$

(in this formula H^*V is an H^*V/W-module via the map induced in mod.2 cohomology by the canonical projection $V \to V/W$).

Theorem 2.5.1. *([**LZ2**]) If I is an injective of the category of $\mathrm{H}^*V - \mathcal{U}$, then $I \cong \bigoplus_{(L,W,n)\in\mathcal{L}\times\mathcal{W}\times\mathbb{N}} (E(V, W, n) \otimes_{\mathbb{F}_2} L)^{\oplus a_{L,W,n}}$.*
Conversely, each $\mathrm{H}^*V - A$-module of this form is an $\mathrm{H}^*V - \mathcal{U}$-injective.

Clearly H^*V is an $\mathrm{H}^*V - \mathcal{U}$-injective.

2.6. Algebraic Smith theory

2.6.1. The functors $\mathrm{F}ix$

We introduce the functors $\mathrm{F}ix$ ([**L1**], [**LZ2**]). We denote by

$$\mathrm{F}ix_V : \mathrm{H}^*V - \mathcal{U} \to \mathcal{U}$$

the left adjoint of the functor

$$\mathrm{H}^*V \otimes - : \mathcal{U} \to \mathrm{H}^*V - \mathcal{U}$$

We have the functorial bijection:

$$\mathrm{H}om_{\mathrm{H}^*V-\mathcal{U}}(N,\ H^*V \otimes P) \cong \mathrm{H}om_{\mathcal{U}}(\mathrm{F}ix_V N,\ P)$$

for every unstable $\mathrm{H}^*V - A$-module N and every unstable A-module P.
The functor $\mathrm{F}ix_V$ has the following properties.
2.6.1.1. The functor $\mathrm{F}ix_V$ is an exact functor.
2.6.1.2. Let N be an unstable $\mathrm{H}^*V - A$-module and $\mathcal{E}(N)$ be its injective hull. Then, the module $\mathrm{F}ix_V\mathcal{E}(N)$ is the injective hull of $\mathrm{F}ix_V N$.

2.6.2.
Let N be an unstable $\mathrm{H}^*V - A$-module, we denote by

$$\eta_V : N \to \mathrm{H}^*V \otimes \mathrm{F}ix_V N$$

the adjoint of the identity of $Fix_V N$. We denote by $\mathrm{c}_V = \prod_{u \in \mathrm{H}^1V - \{0\}} u$ the top Dickson invariant, we have the following result (see [**LZ2**] corollary 2.3).

Proposition 2.6.1. *Let N be an unstable $\mathrm{H}^*V - A$-module. The localization of the map η_V*

$$\eta_V[\mathrm{c}_V^{-1}] : N[\mathrm{c}_V^{-1}] \to \mathrm{H}^*V[\mathrm{c}_v^{-1}] \otimes \mathrm{F}ix_V N$$

is an injection.

This shows in particular, that if N is torsion-free then the map η_V is an injection. The proposition 2.6.1 can be reformulated as follows.

Proposition 2.6.2. *Let N be an unstable $\mathrm{H}^*V - A$-module. If N is torsion-free then its injective hull in $\mathrm{H}^*V - \mathcal{U}$ is free as an H^*V-module and is isomorphic to* $\bigoplus_{(L,n)\in\mathcal{L}\times\mathbb{N}} (\mathrm{H}^*V \otimes \mathrm{J}(n)) \otimes L$

Proof. Since the module is torsion-free then the map $\eta_V : N \to \mathrm{H}^*V \otimes \mathrm{F}ix_V N$ adjoint of the identity of $\mathrm{F}ix_V N$ is an injection. So N is a sub-$\mathrm{H}^*V - A$-module of $\mathrm{H}^*V \otimes \mathrm{F}ix_V N$. By 2.6.1.1 and 2.6.1.2, we have that the injective hull of N is isomorphic to $\mathrm{H}^*V \otimes I$, where I is an $\mathcal{U}$-injective. □

Remark 2.6.3. As a consequence of proposition 2.6.2, we have that if E is an unstable $\mathrm{H}^*V - A$-module which is free as an H^*V-module then its injective hull (in the category $\mathrm{H}^*V - \mathcal{U}$) is also free as an H^*V-module.

Proposition 2.6.4. *[**LZ2**]. Let N be an unstable $\mathrm{H}^*V - A$-module which is of finite type as an H^*V-module. The localization of the map η_V*

$$\eta_V[\mathrm{c}_V^{-1}] : N[\mathrm{c}_V^{-1}] \to \mathrm{H}^*V[\mathrm{c}_V^{-1}] \otimes \mathrm{F}ix_V N$$

is an isomorphism.

In particular, the previous result shows that:

1. If N is free as an H^*V-module, then the map η_V is an injection.
2. The isomorphism of the proposition proves that $dim\overline{E} = dim\mathrm{F}ix_V E$ where dim is the total dimension (see [**LZ2**]).

3. Some properties of E when $\overline{E}$ is reduced

In this section we will prove some algebraic results which will be useful for section 4. In fact, we will analyze the relation between an unstable $\mathrm{H}^*V - A$-module E and its (associated) unstable A-module $\overline{E}$. For this, we will begin by giving some technical results.

3.1. Technical results

Lemma 3.1.1. *Let P and Q be unstable $\mathrm{H}^*V - A$-modules, free as H^*V-modules and $f : P \to Q$ an $\mathrm{H}^*V - A$-linear map. If the induced map $\overline{f} : \overline{P} \to \overline{Q}$ is an injection then f is also an injection.*

Proof. Let's denote by Imf the image of f, by $\widetilde{f} : P \to Imf$ the natural surjection and by $i : Imf \hookrightarrow Q$ the inclusion of Imf in Q. Since $\overline{f}$ is an injection so the induced map $\overline{(\widetilde{f})}$ is an isomorphism of unstable A-modules and then the induced map $\overline{i}$ is an injection. This shows that $\overline{Imf}$ is the image of $\overline{f}$. Since the module Imf is a sub-H^*V-module of the H^*V-free module Q and $\overline{i} : \overline{Imf} \hookrightarrow \overline{Q}$ is an injection, so Imf is free as an H^*V-module. In particular, we have that $Tor_1^{\mathrm{H}^*V}(\mathbb{F}_2, Imf)=0$ (see for example [**R**]). Let's denote by N the kernel of the map $\widetilde{f}$, so we have the following short exact sequence in $\mathrm{H}^*V - \mathcal{U}$:

$$0 \longrightarrow N \longrightarrow P \xrightarrow{\widetilde{f}} Imf \longrightarrow 0 \ .$$

By applying the functor $(\mathbb{F}_2 \otimes_{\mathrm{H}^*V} -)$ to the previous sequence, we prove that $\overline{N}$ is trivial (since the map $\overline{(\widetilde{f})}$ is an isomorphism and Imf is free as an $\mathrm{H}^*V - A$-module). Hence the module N is trivial and the map f is an injection. □

The converse of this lemma is not true in general, but we have the following result:

Lemma 3.1.2. *Let P and Q be unstable $\mathrm{H}^*V - A$-modules, free as H^*V-modules and $f : P \to Q$ an $\mathrm{H}^*V - A$-linear map which is an injection. If $\overline{P}$ is a reduced unstable A-module, then the induced map $\overline{f} : \overline{P} \to \overline{Q}$ is an injection.*

Proof. We denote by C the quotient of Q by P, we have the following short exact sequence in $\mathrm{H}^*V - \mathcal{U}$:

$$0 \longrightarrow P \xrightarrow{f} Q \longrightarrow C \longrightarrow 0 \ .$$

By applying the functor $(\mathbb{F}_2 \otimes_{\mathrm{H}^*V} -)$ to the previous sequence, we obtain an exact sequence in $\mathcal{U}$:

$$0 \longrightarrow Tor_1^{\mathrm{H}^*V}(\mathbb{F}_2, C) \longrightarrow \overline{P} \xrightarrow{\overline{f}} \overline{Q} \longrightarrow \overline{C} \longrightarrow 0 \ .$$

Since $\overline{P}$ is reduced as unstable A-module and $Tor_1^{\mathrm{H}^*V}(\mathbb{F}_2, C)$ is nilpotent (see proposition 2.1.1), then the map $\overline{f}$ is an injection. □

3.2. Statement of some properties of E when $\overline{E}$ is reduced

The first result of this paragraph concerns the relation between the injective hull of E and the induced module $\overline{E}$.

Theorem 3.2.1. *Let E be an unstable $\mathrm{H}^*V - A$-module which is free as an H^*V-module and let $\mathcal{E}(E)$ be its injective hull (in the category $\mathrm{H}^*V - \mathcal{U}$). We suppose that $\overline{E}$ is reduced and let I be its injective hull in the category $\mathcal{U}$.*
Then $\mathcal{E}(E)$ is isomorphic, as an unstable $\mathrm{H}^*V - A$-module, to $\mathrm{H}^*V \otimes I$.

Proof. Since E is free as an H^*V-module, then $\mathcal{E}(E)$ is isomorphic, in the category $\mathrm{H}^*V - \mathcal{U}$, to $\mathrm{H}^*V \otimes J$, where J is an $\mathcal{U}$-injective (see proposition 2.6.2).
Let's denote by i the inclusion of E in $\mathcal{E}(E)$, we have, by lemma 3.1.2, that the induced map $\bar{i}$ is an injection. We will prove, by using the definition, that J is the injective hull of $\overline{E}$, in the category $\mathcal{U}$. Let P be a sub-A-module of J such that the A-module $(\bar{i})^{-1}(P)$ is trivial, we have to show that the unstable A-module P is trivial.
Since $(\bar{i})^{-1}(P)$ is trivial then the composition: $\pi \circ \bar{i} : \ \overline{E} \xrightarrow{\bar{i}} J \xrightarrow{\pi} J/P$ is an injection. By lemma 3.1.1, the following composition
$E \xrightarrow{i} \mathrm{H}^*V \otimes J \longrightarrow \mathrm{H}^*V \otimes (J/P)$ is an injection, which proves that the unstable $\mathrm{H}^*V - A$-module $i^{-1}(\mathrm{H}^*V \otimes P)$ is trivial. Since $\mathrm{H}^*V \otimes J$ is the injective hull of E so the unstable $\mathrm{H}^*V - A$-module $\mathrm{H}^*V \otimes P$ is trivial. □

Corollary 3.2.2. *Let E be an unstable $\mathrm{H}^*V - A$-module such that:*

1. *E is free as an H^*V-module.*
2. *$\overline{E}$ is reduced as unstable A-module.*

Then E is reduced as unstable A-module.

Proof. We have, by theorem 3.2.1, that the injective hull of E is $\mathrm{H}^*V \otimes I$, where I is the injective hull of $\overline{E}$ in $\mathcal{U}$. Since $\overline{E}$ is reduced, then I is a reduced $\mathcal{U}$-injective. This shows that E is reduced as an unstable A-module because its injective hull (in the category $\mathrm{H}^*V - \mathcal{U}$) is $\mathrm{H}^*V \otimes I$ which is reduced as unstable A-module. □

Remark 3.2.3. In the previous result the condition (1): E is free as an H^*V-module is necessary. In fact, the finite $\mathrm{H} - A$-module $\mathrm{J}_{\mathbb{Z}/2\mathbb{Z}}(1)$ is not free as an H-module and not reduced as an unstable A-module, however $\overline{\mathrm{J}_{\mathbb{Z}/2\mathbb{Z}}(1)} = \mathbb{F}_2$ is a reduced unstable A-module. Observe that $\mathrm{J}_{\mathbb{Z}/2\mathbb{Z}}(1)$ is isomorphic, as unstable A-module, to $\mathbb{F}_2 \oplus \sum \mathbb{F}_2$, the structure of H-module is given by: $t.\iota = \Sigma\iota$, where ι is the generator of $\mathbb{F}_2$ and t the generator of H.
Observe that the converse of corollary 3.2.2 is false. In fact, the $\mathrm{H} - A$-module $E = \mathrm{H}^{\geqslant 1}$ is reduced as unstable A-module however the unstable A-module $\overline{E} \cong \sum \mathbb{F}_2$ is not reduced.

4. Description of E when $\overline{E}$ is nil-closed

The main result of this paragraph concerns the relation between the two first terms of a (minimal) injective resolution of E and $\overline{E}$.

Theorem 4.1. *Let E be an unstable $\mathrm{H}^*V - A$-module which is free as an H^*V-module. We suppose that:*

1. *$\overline{E}$ is nil-closed.*
2. *$0 \longrightarrow \overline{E} \longrightarrow I_0 \xrightarrow{i_1} I_1 \longrightarrow \cdots$ is the beginning of a (minimal) $\mathcal{U}$- injective resolution of $\overline{E}$.*

Then there exists an $\mathrm{H}^*V - A$-linear map $\varphi : \mathrm{H}^*V \otimes I_0 \to \mathrm{H}^*V \otimes I_1$ such that:

1. *$0 \longrightarrow E \longrightarrow \mathrm{H}^*V \otimes I_0 \xrightarrow{\varphi} \mathrm{H}^*V \otimes I_1 \longrightarrow \cdots$ is the beginning of a (minimal) injective resolution of E (in the category $\mathrm{H}^*V - \mathcal{U}$).*
2. *$\overline{\varphi} = i_1$*

Proof. The unstable A-module $\overline{E}$ is nil-closed so is reduced, we have then, by theorem 3.2.1, that the injective hull of E is $\mathrm{H}^*V \otimes I_0$. We denote by C_0 the quotient of $\mathrm{H}^*V \otimes I_0$ by E. We have the following short exact sequence in $\mathrm{H}^*V - \mathcal{U}$:

$$0 \longrightarrow E \xrightarrow{i_0} \mathrm{H}^*V \otimes I_0 \longrightarrow C_0 \longrightarrow 0 \ .$$

Since the induced map $\overline{i_0}$ is an injection (see lemma 3.1.2), then the unstable A-module $Tor_1^{\mathrm{H}^*V}(\mathbb{F}_2, C_0)$ is trivial; this shows that the module C_0 is free as an H^*V-module (see for example [**NS**], proposition A.1.5).
We verify that the $\mathcal{U}$-injective hull of $\overline{C_0}$ is I_1 and that C_0 is reduced since $\overline{C_0}$ is reduced (see corollary 3.2.2). This implies, by theorem 3.2.1, that the $\mathrm{H}^*V - \mathcal{U}$-injective hull of C_0 is isomorphic to $\mathrm{H}^*V \otimes I_1$. □

Remark 4.2. let M be a nil-closed unstable A-module and
$0 \longrightarrow M \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \longrightarrow \cdots$ be the beginning of a (minimal) $\mathcal{U}$-injective resolution of M. We denote by

$$(\mathrm{H}om_{\mathrm{H}^*V-\mathcal{U}}(\mathrm{H}^*V \otimes I_0,\ \mathrm{H}^*V \otimes I_1))_{i_1}$$

the set of $\mathrm{H}^*V - A$-linear map $\varphi : \mathrm{H}^*V \otimes I_0 \to \mathrm{H}^*V \otimes I_1$ such that $\overline{\varphi} = i_1$.
Using Lannes T-functor (see [**L1**]) we have:

$$(\mathrm{H}om_{\mathrm{H}^*V-\mathcal{U}}(\mathrm{H}^*V \otimes I_0,\ \mathrm{H}^*V \otimes I_1))_{i_1} \cong (\mathrm{H}om_{\mathcal{U}}(T_V I_0,\ I_1))_{i_1}$$

where $(\mathrm{H}om_{\mathcal{U}}(T_V I_0,\ I_1))_{i_1}$ is the set of A-linear map $\psi : T_V I_0 \to I_1$ such that $\psi \circ i = i_1$, where $i : I_0 \hookrightarrow T_V I_0$ denotes the natural inclusion.
The kernel of any element $\psi \in (\mathrm{H}om_{\mathcal{U}}(T_V I_0,\ I_1))_{i_1}$, which is free as an H^*V-module, is an unstable $\mathrm{H}^*V - A$-module such that $\overline{ker \psi} \cong M$.

Remark 4.3. If $\overline{E}$ is an $\mathcal{U}$-injective then the only unstable free $\mathrm{H}^*V - A$-module, up to isomorphism, solution of the problem $(\mathcal{P})$ is $\mathrm{H}^*V \otimes \overline{E}$.
Let n be an even integer. The unstable free $\mathrm{H}-A$-modules, up to isomorphism, solution of the problem $(\mathcal{P})$ when M is $\mathrm{H}^*BSO(n)$ are $\mathrm{H}^*BO(n)$ and $\mathrm{H}\otimes\mathrm{H}^*BSO(n)$. We verify that these two $\mathrm{H}-A$-modules are not isomorphic in the category $\mathrm{H}-\mathcal{U}$ (since it does not exist an A-linear section of the projection $\mathrm{H}^*BO(n) \to \mathrm{H}^*BSO(n)$).

5. Applications

5.1.

Our first application concerns the determination of the mod. 2 cohomology of the mapping space $\mathbf{hom}(\mathrm{B}(\mathbb{Z}/2^n), Y)$ whose domain is a classifying space for the group $\mathbb{Z}/2^n$ and whose range is a space Y such that H^*Y is concentrated in even degrees.

We will just recall some facts, ignoring the p-completion problems. For further details see [**DL**].

One proceeds by induction on the integer n. Let us set

$$X = \mathbf{hom}(\mathrm{E}(\mathbb{Z}/2^n)/(\mathbb{Z}/2^{n-1}), Y) \qquad .$$

The space X has the homotopy type of $\mathbf{hom}(\mathrm{B}(\mathbb{Z}/2^{n-1}), Y)$ and is equipped of an action $\mathbb{Z}/2$ such that one has a homotopy equivalence

$$\mathbf{hom}(\mathrm{B}(\mathbb{Z}/2^n), Y) \cong X^{\mathrm{h}\,\mathbb{Z}/2} \qquad ,$$

$X^{\mathrm{h}\,\mathbb{Z}/2}$ denoting the homotopy fixed point space: $\mathbf{hom}_{\mathbb{Z}/2}(\mathrm{E}\mathbb{Z}/2, X)$. Using $\mathrm{Fix}_{\mathbb{Z}/2}$-theory [**L1**], one gets:

$$\mathrm{H}^*\mathbf{hom}(\mathrm{B}(\mathbb{Z}/2^n), Y) \cong \mathrm{Fix}_{\mathbb{Z}/2}\,\mathrm{H}^*_{\mathbb{Z}/2}X \qquad .$$

Since the computation of the functor $\mathrm{Fix}_{\mathbb{Z}/2}$ on an unstable $\mathrm{H}-\mathrm{A}$-module is not difficult in general, the determination of the mod. 2 cohomology of the mapping space $\mathbf{hom}(\mathrm{B}(\mathbb{Z}/2^n), Y)$ is reduced to the determination of the unstable $\mathrm{H}-\mathrm{A}$-module $\mathrm{H}^*_{\mathbb{Z}/2}X$. As we are going to explain, this last point is closely related to problem $(\mathcal{P})$.

One knows by induction on n that the mod . 2 cohomology of the space X as the one of the space Y is concentrated in even degrees and one checks that the action of $\mathbb{Z}/2$ on $\mathrm{H}^*(Y;\mathbb{Z})$ is trivial. These two facts imply that the Serre spectral sequence, for mod. 2 cohomology, associated to the fibration

$$X \to X_{\mathrm{h}\mathbb{Z}/2} \to \mathrm{B}\mathbb{Z}/2$$

collapses ($X_{\mathrm{h}\mathbb{Z}/2}$ denotes the Borel construction $\mathrm{E}\mathbb{Z}/2 \times_{\mathbb{Z}/2} X$). This collapsing implies in turn that $\mathrm{H}^*_{\mathbb{Z}/2}X$ is H-free and that $\overline{\mathrm{H}^*_{\mathbb{Z}/2}X}$ is isomorphic to H^*X. So the determination of $\mathrm{H}^*\mathbf{hom}(\mathrm{B}(\mathbb{Z}/2^n), Y)$ is indeed reduced to the resolution of a problem $(\mathcal{P})$.

We conclude this subsection by a concrete example (we follow [**De**], section 6); we take $n = 2$ and $Y = \mathrm{BSU}(2)$. Using $\mathrm{T}_{\mathbb{Z}/2}$-computations one sees that X has the homotopy type of $\mathrm{BSU}(2)\coprod\mathrm{BSU}(2)$; one checks also that the $\mathbb{Z}/2$-action preserves the connected components. The $(\mathcal{P})$-problem asociated to the determination of the unstable $\mathrm{H}-\mathrm{A}$-module $\mathrm{H}^*_{\mathbb{Z}/2}X$ is the following one:

Find the unstable $\mathrm{H}-\mathrm{A}$-modules E such that

- E is H-free;
- the unstable A-module $\overline{E}$ is isomorphic to $\mathrm{H}^*\mathrm{BSU}(2)$.

Using the fact that the injective hull, in the category $\mathrm{H}-\mathcal{U}$, of E is $\mathrm{H}\otimes\mathrm{H}$ (see theorem 3.2), one checks that one has two possibilities:

- $E \cong \mathrm{H}\otimes \mathrm{H}^*\mathrm{BSU}(2)$;
- $E \cong \mathrm{H}\otimes_{\mathrm{H}^*\mathrm{BU}(1)} \mathrm{H}^*\mathrm{BU}(2)$ (the structures of unstable $\mathrm{H}^*\mathrm{BU}(1) - \mathrm{A}$-modules on $\mathrm{H} = \mathrm{H}^*\mathrm{BO}(1)$ and $\mathrm{H}^*\mathrm{BU}(2)$ are respectively induced by the inclusion of $\mathrm{O}(1)$ in $\mathrm{U}(1)$ and the determinant homomorphism from $\mathrm{U}(2)$ to $\mathrm{U}(1)$).

5.2.

The theorem 4.1 can be illustrated, topologically, as follows:

Proposition 5.2.1. *Let X be a CW-complex on which acts an elementary abelian group 2-group V. Suppose that:*

1. H^*X *is nil-closed*
2. $0 \longrightarrow \mathrm{H}^*X \longrightarrow I_0 \xrightarrow{\alpha} I_1 \longrightarrow \cdots$ *is the beginning of a (minimal) $\mathcal{U}$-injective resolution of* H^*X
3. $\mathrm{H}^*_V X$ *is free as an* H^*V*-module.*

Then there exists an $\mathrm{H}^*V - A$-linear map $\varphi:\ \mathrm{H}^*V\otimes I_0 \to \mathrm{H}^*V\otimes I_1$ such that:

1. $\mathrm{H}^*_V X \cong Ker(\varphi)$.
2. $0 \longrightarrow \mathrm{H}^*_V X \longrightarrow \mathrm{H}^*V\otimes I_0 \xrightarrow{\varphi} \mathrm{H}^*V\otimes I_1 \longrightarrow \cdots$ *is the beginning of a (minimal) injective resolution of $\mathrm{H}^*_V X$ (in the category $\mathrm{H}^*V - \mathcal{U}$).*
3. $\overline{\varphi} = \alpha : I_0 \to I_1$.

In particular, we have:

Corollary 5.2.2. *Let X be a CW-complex on which acts an elementary abelian group 2-group V. Suppose that:*

1. H^*X *is a reduced $\mathcal{U}$-injective,*
2. $\mathrm{H}^*_V X$ *is free as an* H^*V*-module.*

Then $\mathrm{H}^*_V X \cong \mathrm{H}^*V\otimes H^*X$.

6. Description of E when $\overline{E}$ is isomorphic to $\sum^n \mathbb{F}_2$

In this section, we prove the following result.

Theorem 6.1. *Let E be unstable $\mathrm{H}^*V - A$-module which is free as an H^*V-module. If $\overline{E}$ is isomorphic to $\sum^n \mathbb{F}_2$, then there exists an element u in H^*V such that:*

1. $u = \prod_i \theta_i^{\alpha_i}$, *where $\theta_i \in (\mathrm{H}^1V)\setminus\{0\}$ and $\alpha_i \in \mathbb{N}$*
2. $E \cong \sum^d u\mathrm{H}^*V$ *with* $d + \sum_i \alpha_i = n$.

Proof. Let N be an unstable A-module, we denote by $\mathrm{d}imN$ the total dimension of N that is $\mathrm{d}im\ N = \sum_i \mathrm{d}im\ N^i$. We have the equality $\mathrm{d}im\ \overline{E} = 1 = \mathrm{d}im\ \mathrm{F}ix_V E$

(see [**LZ3**]), so we deduce that $\mathrm{F}ix_V E = \sum^l \mathbb{F}_2$, where $l \in \mathbb{N}$. Let $\eta_V : E \to \mathrm{H}^*V \otimes \mathrm{F}ix_V E$ be the adjoint of the identity of $\mathrm{F}ix_V E$ (see [**LZ2**]). Since the map η_V is an injection, then the module E is a sub-$\mathrm{H}^*V - A$-module of $\sum^l \mathrm{H}^*V$. Let's write $E = \sum^l E'$, where E' is sub-$\mathrm{H}^*V - A$-module of H^*V . By a result of J-P. Serre (see [**Se**]), there exists N such that: $\mathrm{c}_V^N \mathrm{H}^*V \subset E' \subset \mathrm{H}^*V$. Since E' is free as an H^*V-module and of dimension one, then there exists $u \in \widetilde{\mathrm{H}}^*V$ such that $E' = u\mathrm{H}^*V$. The inclusion $\mathrm{c}_V^N \mathrm{H}^*V \subset u\mathrm{H}^*V$ proves that $u = \prod_i \theta_i^{\alpha_i}$, where $\theta_i \in (\mathrm{H}^1V) \setminus \{0\}$ and $\alpha_i \in \mathbb{N}$. □

Remark 6.2. We remark that by the previous result, we can determinate E when $\overline{E}$ is isomorphic to $\mathbb{F}_2 \oplus \sum^n \mathbb{F}_2$. In this case, we verify that $E \cong \mathrm{H}^*V \oplus \sum^d u\mathrm{H}^*V$, where $u = \prod_i \theta_i^{\alpha_i}$, $\theta_i \in \mathrm{H}^*V \setminus \{0\}$, $\alpha_i \in \mathbb{N}$ and $d + \sum_i \alpha_i = n$. In fact, since the $\mathrm{H}^*V - \mathcal{U}$-injective module H^*V is a sub-H^*V-module of E, then $E \cong \mathrm{H}^*V \oplus E'$, where E' is an unstable $\mathrm{H}^*V - A$-module, free as an H^*V-module and such that $\overline{E'} \cong \sum^n \mathbb{F}_2$. The result holds from theorem 6.1.

6.3 Example

We give an example showing how to realize topologically the cases of theorem 6.1 and remark 6.2.

Let $\rho : V \to \mathrm{O}(d)$ be a group homomorphism. ρ gives both an action of V on D^d, S^{d-1} and a d-dimensional orthogonal bundle whose mod.2 Euler class is denoted by $e(\rho)$.

The long exact sequence of the pair $(\mathrm{D}^d,\ \mathrm{S}^{d-1})$ and the Thom isomorphism give the long (Gysin) exact sequence (see for example [**Hu**]):

$$\cdots \longrightarrow \mathrm{H}^{*-1}V \longrightarrow \mathrm{H}_V^{*-1}\mathrm{S}^{d-1} \longrightarrow \Sigma^{-d}\mathrm{H}^*V \xrightarrow{\smile e(\rho)} \mathrm{H}^*V \longrightarrow \mathrm{H}_V^*\mathrm{S}^{d-1} \longrightarrow \cdots$$

The decomposition $\rho \cong \oplus_{i=1}^d \rho_i$ of the representation ρ into orthogonal representations of dimension 1 gives $e(\rho) = \prod_i e(\rho_i)$. We have now two cases.

- If none of the representations ρ_i is trivial then $e(\rho)$ is non zero and $\mathrm{H}_V^*(\mathrm{D}^d, \mathrm{S}^{d-1})$ is isomorphic to $e(\rho)\mathrm{H}^*V$ as an $\mathrm{H}^*V - \mathrm{A}$-module. This illustrates theorem 6.1.
- Otherwise, let's write $\rho = \sigma \oplus \tau$, σ (resp. τ) being the direct sum of the non trivial (resp. trivial) representations ρ_i. Then $\mathrm{H}_V^*\mathrm{S}^{d-1} \cong \mathrm{H}^*V \oplus \Sigma^{\mathrm{d}im\tau}\ e(\sigma)\mathrm{H}^*V$ and $\mathrm{H}_V^*(\mathrm{S}^{d-1})$ is an illustration of the remark 6.2.

7. Determination of E when V is $\mathbb{Z}/2\mathbb{Z}$ and $\overline{E}$ is $\mathrm{J}(2)$

In this section, we assume that V is $\mathbb{Z}/2\mathbb{Z}$ and $\overline{E}$ is the Brown-Gitler module $\mathrm{J}(2)$. We denote by $\mathrm{H} = \mathbb{F}_2[t]$ the cohomology of $\mathbb{Z}/2\mathbb{Z}$, where t is an element of H of degree one. We have the following result.

Proposition 7.1. *Let E be an $\mathrm{H} - A$-module which is H-free and such that $\overline{E}$ is isomorphic to $\mathrm{J}(2)$ then:*
$E \cong \mathrm{H} \otimes \mathrm{J}(2)$
or
E is the sub-$\mathrm{H} - A$-module of $\mathrm{H} \oplus \sum \mathrm{H}$ generated by $(t, \Sigma 1)$ and $(t^2, 0)$.

Proof. This proof uses the Smith theory (see [**DW**], [**LZ2**] theorem 2.1) which gives us an exact sequence (*) in $\mathrm{H}-\mathcal{U}$:

$$(*) \qquad 0 \longrightarrow E \xrightarrow{\eta} \mathrm{H}\otimes \mathrm{F}ixE \longrightarrow C \longrightarrow 0$$

where C the quotient of $\mathrm{H}\otimes \mathrm{F}ixE$ is finite and also $\mathrm{F}ixE$ is finite.
If the module C is trivial then E is isomorphic to $\mathrm{H}\otimes \mathrm{J}(2)$.
When C is a non trivial module. By applying the functor $\mathbb{F}_2\otimes_{\mathrm{H}} -$ to the exact sequence (*), we obtain:

$$0 \longrightarrow \sum \tau C \longrightarrow \overline{E} = \mathrm{J}(2) \longrightarrow \mathrm{F}ixE \longrightarrow \overline{C} \longrightarrow 0$$

where τC is the trivial part of C (see [**BHZ**]).
Let's denote by Q the quotient of $\overline{E}$ by $\sum \tau C$. By properties of the module $\mathrm{J}(2)$, we have that $\sum \tau C = \sum^2 \mathbb{F}_2$ and $Q = \sum \mathbb{F}_2$. The exact sequence:

$$0 \longrightarrow \sum \mathbb{F}_2 \longrightarrow FixE \longrightarrow \overline{C} \longrightarrow 0$$

gives that $FixE \cong \sum \mathbb{F}_2 \oplus \overline{C}$. One checks that the module $\overline{C}$ is either isomorphic to $\mathbb{F}_2$ or $\sum \mathbb{F}_2$. If $\overline{C} = \sum \mathbb{F}_2$ then $\mathrm{F}ixE \cong \sum \mathbb{F}_2 \oplus \sum \mathbb{F}_2$ as an unstable A-module, which implies that the module E is a suspension which is impossible because $\overline{E} = \mathrm{J}(2)$ is not a suspension. We conclude that $\overline{C} = \mathbb{F}_2$. Since $\tau C = \sum \mathbb{F}_2$ then we get C is isomorphic to $\mathrm{H}^{\leqslant 1}$, where $\mathrm{H}^{\leqslant 1}$ denotes the sub-$\mathrm{H}-A$-module of H consisting of elements of degree less or equal than 1. We have the following exact sequence in $\mathrm{H}-\mathcal{U}$:

$$0 \longrightarrow E \longrightarrow \mathrm{H}\oplus \sum \mathrm{H} \xrightarrow{\varphi} \mathrm{H}^{\leqslant 1} \longrightarrow 0\ .$$

The module E, we are searching for, is the kernel of φ and we check that it is the sub-H − A-module of $\mathrm{H}\oplus\sum \mathrm{H}$ generated by the elements $(t, \Sigma 1)$ and $(t^2, 0)$. □

Remark 7.2. Let be $\mathbb{Z}/2\mathbb{Z}$ act on a real projective space $\mathbb{R}\mathrm{P}^2$; let x_0 be a fixed point of this action (the set of fixed point is not empty for example by an argument of Lefschetz number). We have:
- The Serre spectral sequence collapses to give that: $\mathrm{H}^*_V(\mathbb{R}\mathrm{P}^2, x_0)$ is H-free and $\overline{\mathrm{H}^*_V(\mathbb{R}\mathrm{P}^2, x_0)}$ is isomorphic to $\mathrm{J}(2)$.
- In [**DW**], Dwyer and Wilkerson have shown that $\mathrm{H}^*_V\mathbb{R}\mathrm{P}^2 = \mathbb{F}_2[t, y]/(f)$ where y restricts to x and $f = y^i(y + t)^j$ for $i + j = 3$. It is easy to check that this computation agrees with theorem 7.1.

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