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COBORDISM CATEGORY OF PLUMBED 3-MANIFOLDS AND INTERSECTION PRODUCT STRUCTURES

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Abstract

In this paper, we introduce a category of graded commutative rings with certain algebraic morphisms, to investigate the cobordism category of plumbed 3-manifolds. In particular, we define a non-associative distributive algebra that gives necessary conditions for an abstract morphism between the homologies of two plumbed 3-manifolds to be realized geometrically by a cobordism. Here we also consider the homology cobordism monoid, and give a necessary condition using w-invariants for the homology 3-spheres to belong to the inertia group associated to some homology 3-spheres.

1. Introduction

In this paper, we introduce a category of graded commutative rings with certain algebraic morphisms in order to investigate the cobordism category of plumbed 3-manifolds. In fact, we define a non-associative distributive algebra, which we use to give necessary conditions for algebraic morphisms between the homologies of two plumbed 3-manifolds to be realized geometrically by cobordism. This paper is a generalization of the paper [2] on homology cobordisms, to general cobordisms between closed 3-manifolds. More precisely, we state the problem as follows. Let $\mathcal{C}_3$ be the cobordism category of 3-manifolds whose objects $M \in \mathrm{ob}\,(\mathcal{C}_3)$ are closed oriented 3-manifolds and whose morphisms $W \in \mathcal{C}_3\,(M, M')$ between two 3-manifolds M and M' are cobordisms $(W; M, M')$. On the other hand, let $\mathcal{L}_3$ be a category whose objects $(H_*, \bullet) \in \mathrm{ob}\,(\mathcal{L}_3)$ are graded commutative rings of dimension 3 and whose morphisms $(L_*; i, i', \bullet) \in \mathcal{L}_3\,((H_*, \bullet), (H'_*, \bullet))$ between two objects $(H_*, \bullet)$ and $(H'_*, \bullet)$ are composed of graded modules L_* of dimension 4 with certain product structures $\bullet$ and homomorphisms $H_* \xrightarrow{i} L_* \xleftarrow{i'} H'_*$ satisfying compatibility conditions on products. Note that there exists a functor $H_* : \mathcal{C}_3 \to \mathcal{L}_3$ given by $M \longmapsto (H_*\,(M; \mathbb{Z})\,, \bullet)$ and $(W; M, M') \longmapsto (L_*\,(W; \mathbb{Z})\,; i_*, i'_*, \bullet)$, where i_* and i'_*

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are the induced homomorphisms

$$H_*(M;\mathbb{Z}) \xrightarrow{i_*} L_*(W;\mathbb{Z}) \xleftarrow{i'_*} H_*(M';\mathbb{Z}),$$

to the $\mathbb{Z}$-module $L_*(W;\mathbb{Z})$ defined by

$$L_k(W;\mathbb{Z}) = \mathrm{Im}\left(H_k(M;\mathbb{Z}) \oplus H_k(M';\mathbb{Z}) \xrightarrow{i_*+i'_*} H_k(W;\mathbb{Z})\right).$$

Let $R_*(W)$ be a $\mathbb{Z}$-module defined by

$$R_k(W;\mathbb{Z}) = \mathrm{Ker}\left(H_{k-1}(M;\mathbb{Z}) \oplus H_{k-1}(M';\mathbb{Z}) \xrightarrow{i_*+i'_*} H_{k-1}(W;\mathbb{Z})\right)$$

and $\partial_* \oplus \partial'_* : R_k(W;\mathbb{Z}) \to H_{k-1}(M;\mathbb{Z}) \oplus H_{k-1}(M';\mathbb{Z})$ be the induced homomorphism. Then the problem can be stated as follows.

Problem 1. *Let $M, M' \in \mathrm{ob}(\mathcal{C}_3)$ be two closed oriented 3-manifolds. Let $\phi = (L_*; i, i', \bullet) \in \mathcal{L}_3((H_*(M;\mathbb{Z}), \bullet), (H_*(M';\mathbb{Z}), \bullet))$ be an algebraic morphism of homology rings. Then does there exist a cobordism $(W; M, M') \in \mathcal{C}_3(M, M')$ such that $(L_*(W;\mathbb{Z}); i_*, i'_*, \bullet) \cong \phi = (L_*; i, i', \bullet)$.*

Note that by the exact sequence of the pair $(W, M \sqcup M')$, the cobordism $(W; M, M')$ preserves the intersection product structures, that is, the induced morphism $(L_*(W;\mathbb{Z}); i_*, i'_*, \bullet) \in \mathcal{L}_3((H_*(M;\mathbb{Z}), \bullet), (H_*(M';\mathbb{Z}), \bullet))$ must be composed of ring homomorphisms i_*, i'_* with respect to the intersection pairings.

Lemma 2. *Let $(W; M, M')$ be a cobordism between two closed oriented 3-manifolds M and M'. Let $(L_*(W;\mathbb{Z}); i_*, i'_*, \bullet)$ be the induced homomorphism $H_*(M;\mathbb{Z}) \xrightarrow{i_*} L_*(W;\mathbb{Z}) \xleftarrow{i'_*} H_*(M';\mathbb{Z})$. If we denote the intersection parings on M and W by $\bullet : H_k(M;\mathbb{Z}) \otimes H_\ell(M;\mathbb{Z}) \to H_{k+\ell-3}(M;\mathbb{Z})$ and $\bullet : R_{k+1}(W;\mathbb{Z}) \otimes L_\ell(W;\mathbb{Z}) \to L_{k+\ell-3}(W;\mathbb{Z})$, respectively, for any non-negative integers k, ℓ with $k + \ell \geqslant 3$, then we have $i_*(\partial_*(\eta) \cdot \theta) = \eta \cdot i_*(\theta)$ for any $\eta \in R_{k+1}(W;\mathbb{Z})$, $\theta \in L_\ell(W;\mathbb{Z})$.*

In fact, this lemma provides a necessary condition for the existence of cobordisms described above. However, the following is an example to which Lemma 2 cannot be applied.

Example 3. *Let (Γ, ω), (Γ', ω') be two Seifert graphs defined by*

$$\Gamma = (V, E), \; V = \{1, 2\}, \; E = \{(1,2), (2,1)\}$$
$$\begin{cases} \omega_1 = \{2; (5,3), (5,3), (5,4)\}, \\ \omega_2 = \{2; (9,1), (9,1), (9,4)\}. \end{cases}$$

$$\Gamma' = (V', E'), \; V' = \{1, 2\}, \; E' = \{(1,2), (2,1)\}$$
$$\begin{cases} \omega'_1 = \{2; (5,3), (5,3), (5,4)\}, \\ \omega'_2 = \{1; (9,2), (9,2), (9,2)\}. \end{cases}$$

Note that the associated plumbed 3-manifold $M(\Gamma)$ can be obtained by plumbing of S^1-V-bundles $E_v \to \Sigma_v$, $v \in V$ over the closed oriented V-surfaces Σ_v with Seifert *invariants $\omega(v)$ according to the graphs Γ, see Appendix 6 or [3]. This plumbed V-manifold can also be described by using the decorated plumbing graphs of N. Saveliev [12]. Then the homology groups of $M(\Gamma)$, $M(\Gamma')$ are calculated as follows.*

$$H_1(M(\Gamma);\mathbb{Z}) \cong \mathbb{Z}^8 \oplus \mathbb{Z}/45 \oplus \mathbb{Z}/675, \quad H_2(M(\Gamma);\mathbb{Z}) \cong \mathbb{Z}^8,$$
$$H_1(M(\Gamma');\mathbb{Z}) \cong \mathbb{Z}^6 \oplus \mathbb{Z}/45 \oplus \mathbb{Z}/675, \quad H_2(M(\Gamma');\mathbb{Z}) \cong \mathbb{Z}^6.$$

Let $\{\alpha_{vj}\}_{v\in\{1,2\},j\in\{1,2,\ldots,2g_v\}} \subset H_1\left(\bar{\Sigma}_v;\mathbb{Z}\right)$ be a system of α,β-cycles on the underlying topological space $\bar{\Sigma}_v$ of the V-surface Σ_v of genus g_v, so that $\alpha_{vj}\cdot\alpha_{v'j'} = \delta_{vv'}\varepsilon_{jj'}$ for $v,v'\in\{1,2\}$ and $j,j'\in\{1,2,\ldots,2g_v\}$, where

$$\varepsilon_{jj'} = \begin{cases} 1 & (j=2j''-1,\ j'=2j'',\ j''\in\{1,2,\ldots,g_v\}) \\ -1 & (j=2j'',\ j'=2j''-1,\ j''\in\{1,2,\ldots,g_v\}) \\ 0 & (\textit{otherwise}) \end{cases}.$$

We denote the same symbol $\{\alpha_{vj}\}_{v\in\{1,2\},j\in\{1,2,\ldots,2g_v\}} \subset H_1(M(\Gamma);\mathbb{Z})$ to be the corresponding homology classes in $M(\Gamma)$, and fix an isomorphism on the free part $H_1(M(\Gamma);\mathbb{Z})/\mathrm{Tor} \cong \mathbb{Z}^8$ by using this $\{\alpha_{vj}\}$. For each $\alpha \in H_1\left(\bar{\Sigma}_v;\mathbb{Z}\right)$, let $\theta_\alpha \in H_2(M(\Gamma);\mathbb{Z})$ be the corresponding 2-cycle obtained by using the natural homomorphism defined in Lemma 30.1. Then $\{\theta_{\alpha_{vj}}\}_{v\in\{1,2\},j\in\{1,2,\ldots,2g_v\}}$ $\subset H_2(M(\Gamma);\mathbb{Z})$ form a basis, and we can fix an isomorphism $H_2(M(\Gamma);\mathbb{Z}) \cong \mathbb{Z}^6$ by using $\{\theta_{\alpha_{vj}}\}$. Let $L_* = \bigoplus_{k=0}^4 L_k$ be the graded $\mathbb{Z}$-module defined by

$$L_0 = \mathbb{Z},\ L_1 = \mathbb{Z}^8 \oplus \mathbb{Z}/45 \oplus \mathbb{Z}/675,\ L_2 = \mathbb{Z}^6,\ L_3 = \mathbb{Z},\ L_4 = 0.$$

Let $H_*(M(\Gamma)) \xrightarrow{i} L_* \xleftarrow{i'} H_*(M(\Gamma'))$ be two graded homomorphisms defined by

$$\begin{array}{ccccc} H_1(M(\Gamma);\mathbb{Z}) & \xrightarrow{i} & L_1 & \xleftarrow{i'} & H_1(M(\Gamma');\mathbb{Z}) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \mathbb{Z}^8 & \xrightarrow{\Phi_1} & \mathbb{Z}^8 & \xleftarrow{\Phi_1'} & \mathbb{Z}^6 \\ \oplus & & \oplus & & \oplus \\ \mathbb{Z}/45 & \xrightarrow{1} & \mathbb{Z}/45 & \xleftarrow{1} & \mathbb{Z}/45 \\ \oplus & & \oplus & & \oplus \\ \mathbb{Z}/675 & \xrightarrow{1} & \mathbb{Z}/675 & \xleftarrow{1} & \mathbb{Z}/675 \end{array},$$

where

$$\Phi_1 = \begin{pmatrix} I_2 & O & O & O \\ O & I_2 & O & O \\ O & O & O & O \\ O & O & O & I_2 \end{pmatrix}, \quad \Phi_1' = \begin{pmatrix} I_2 & O & O \\ O & O & I_2 \\ O & I_2 & O \\ O & O & O \end{pmatrix},$$
$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{array}{ccccc} H_2(M(\Gamma);\mathbb{Z}) & \xrightarrow{i} & L_2 & \xleftarrow{i'} & H_2(M(\Gamma');\mathbb{Z}) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \mathbb{Z}^8 & \xrightarrow{\Phi_2} & \mathbb{Z}^6 & \xleftarrow{\Phi_2'} & \mathbb{Z}^6 \end{array},$$

and where

$$\Phi_2 = \begin{pmatrix} I_2 & O & O & O \\ O & I_2 & O & O \\ O & O & I_2 & O \end{pmatrix}, \quad \Phi_2' = \begin{pmatrix} I_2 & O & O \\ O & O & I_2 \\ O & O & O \end{pmatrix},$$

and

$$\begin{array}{ccccc} H_3\left(M\left(\Gamma\right);\mathbb{Z}\right) & \overset{i}{\rightarrow} & L_3 & \overset{i'}{\leftarrow} & H_3\left(M\left(\Gamma'\right);\mathbb{Z}\right) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \mathbb{Z} & \overset{1}{\rightarrow} & \mathbb{Z} & \overset{1}{\leftarrow} & \mathbb{Z} \end{array}.$$

On the other hand, let $R_* = \bigoplus_{k=0}^{4} R_k$ *be the graded* $\mathbb{Z}$*-module defined by*

$$R_k = \mathrm{Ker}\left(H_{k-1} \oplus H'_{k-1} \overset{i-i'}{\rightarrow} L_{k-1}\right),$$

then we have

$$R_0 = 0,\ R_1 = \mathbb{Z},\ R_2 = \mathbb{Z}^6,\ R_3 = \mathbb{Z}^8,\ R_4 = \mathbb{Z}.$$

Let $\bullet : R_2 \otimes L_2 \to L_0 = \mathbb{Z}$ *and* $\bullet : R_3 \otimes L_2 \to L_1$ *be bilinear pairings defined by*

$$R_2\overset{L_2}{\begin{pmatrix} \varepsilon_2 & O & O \\ O & \varepsilon_2 & O \\ O & O & \varepsilon_2 \end{pmatrix}}, \quad R_3\overset{L_1}{\begin{pmatrix} -\varepsilon_2 & O & O & O \\ O & -\varepsilon_2 & O & O \\ O & O & O & -\varepsilon_2 \\ O & O & -\varepsilon_2 & O \end{pmatrix}}, \quad R_3\overset{L_2}{\begin{pmatrix} 45\varepsilon_2\delta & O & O \\ O & 45\varepsilon_2\delta & O \\ O & O & O \\ O & O & O \end{pmatrix}},$$

$$\varepsilon_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \delta = 1 \in \mathbb{Z}/675 \subset L_1.$$

We fix spin structures,

$$c = \left(\left(1,1,0\right),\left(1,1,0\right)\right),\ c' = \left(\left(1,1,0\right),\left(0,0,0\right)\right),$$

on $M\left(\Gamma\right)$, $M\left(\Gamma'\right)$*, respectively, where* $0, 1$ *are determined by the spin structures on punctured Riemann surfaces around* V*-singular points, see [2]. Then there exists no (spin) cobordism*

$$((W,\tilde{c});(M,c),(M',c'))$$

such that $\left(L_*\left(W;\mathbb{Z}\right);i_*,i'_*,\bullet\right) \cong \phi = \left(L_*;i,i',\bullet\right)$ *and* $L_*\left(W;\mathbb{Z}\right) = H_*\left(W;\mathbb{Z}\right)$*.*

Remark 4. *Note that the generators of* $R_2 = \mathbb{Z}^6$ *and* $R_3 = \mathbb{Z}^8$ *correspond to*

$$R_2 = \left\langle \overline{\alpha_{11}\alpha'_{11}}, \overline{\alpha_{12}\alpha'_{12}}, \overline{\alpha_{13}\alpha'_{21}}, \overline{\alpha_{14}\alpha'_{22}}, \overline{\alpha_{21}0}, \overline{\alpha_{22}0} \right\rangle,$$
$$R_3 = \left\langle \overline{\theta_{\alpha_{11}}\theta'_{\alpha'_{11}}}, \overline{\theta_{\alpha_{12}}\theta'_{\alpha'_{12}}}, \overline{\theta_{\alpha_{13}}\theta'_{\alpha'_{21}}}, \overline{\theta_{\alpha_{14}}\theta'_{\alpha'_{22}}}, \overline{\theta_{\alpha_{23}}0}, \overline{\theta_{\alpha_{24}}0}, \overline{0\theta'_{\alpha'_{13}}}, \overline{0\theta'_{\alpha'_{14}}} \right\rangle,$$

where we denote by $\overline{\alpha\alpha'}$ *(resp.* $\overline{\theta_\alpha\theta'_{\alpha'}}$*) the* 2*(resp.* 3*)-cycles corresponding to*

$$(-\alpha) \oplus \alpha' \in \mathrm{Ker}\left(H_1 \oplus H'_1 \overset{i+i'}{\rightarrow} L_1\right) \quad \textit{(resp. } (-\theta_\alpha) \oplus \theta'_{\alpha'} \in \mathrm{Ker}\left(H_1 \oplus H'_1 \overset{i+i'}{\rightarrow} L_1\right)\textit{)}$$

with $k=2$ (resp. $k=3$). Take a generator $\delta_1,\dots,\delta_8,\bar{\varepsilon},\bar{\bar{\delta}}$ of L_1,

$$\langle\delta_1,\dots,\delta_8\rangle\oplus\langle\bar{\varepsilon}\rangle\oplus\left\langle\bar{\bar{\delta}}\right\rangle\cong\mathbb{Z}^8\oplus\mathbb{Z}/45\oplus\mathbb{Z}/675\cong L_1,$$

and a generator $\rho_1,\dots,\rho_6$ of L_2,

$$\langle\rho_1,\dots,\rho_6\rangle\cong\mathbb{Z}^6\cong L_2.$$

Then the pairings $\bullet: R_2\otimes L_2\to L_0=\mathbb{Z}$ and $\bullet: R_3\otimes L_2\to L_1$ are described as follows.

$$\bullet: R_2\otimes L_2\to L_0=\mathbb{Z}$$

$$\begin{aligned}
\overline{\alpha_{11}\alpha'_{11}}\cdot\rho_2&=1,\\
\overline{\alpha_{12}\alpha'_{12}}\cdot\rho_1&=-1,\\
\overline{\alpha_{13}\alpha'_{21}}\cdot\rho_4&=1,\\
\overline{\alpha_{14}\alpha'_{22}}\cdot\rho_3&=-1,\\
\overline{\alpha_{21}0}\cdot\rho_6&=1,\\
\overline{\alpha_{22}0}\cdot\rho_5&=-1\\
x\cdot y&=0\quad\textit{(otherwise)}.
\end{aligned}$$

$$\bullet: R_3\otimes L_1\to L_0=\mathbb{Z}$$

$$\begin{aligned}
\overline{\theta_{\alpha_{11}}\theta_{\alpha'_{11}}}\cdot\delta_2&=-1,\\
\overline{\theta_{\alpha_{12}}\theta_{\alpha'_{12}}}\cdot\delta_1&=1,\\
\overline{\theta_{\alpha_{13}}\theta_{\alpha'_{21}}}\cdot\delta_4&=-1,\\
\overline{\theta_{\alpha_{14}}\theta_{\alpha'_{22}}}\cdot\delta_3&=1,\\
\overline{\theta_{\alpha_{23}}0}\cdot\delta_8&=-1,\\
\overline{\theta_{\alpha_{24}}0}\cdot\delta_7&=1,\\
\overline{0\theta_{\alpha'_{13}}}\cdot\delta_6&=-1,\\
\overline{0\theta_{\alpha'_{14}}}\cdot\delta_5&=1,\\
x\cdot y&=0\quad\textit{(otherwise)}.
\end{aligned}$$

$$\bullet: R_3\otimes L_2\to L_1$$

$$\begin{aligned}
\overline{\theta_{\alpha_{11}}\theta_{\alpha'_{11}}}\cdot\rho_2&=-\bar{s},\quad\bar{s}=45\bar{\bar{\delta}},\\
\overline{\theta_{\alpha_{12}}\theta_{\alpha'_{12}}}\cdot\rho_1&=\bar{s},\\
\overline{\theta_{\alpha_{13}}\theta_{\alpha'_{21}}}\cdot\rho_4&=-\bar{s},\\
\overline{\theta_{\alpha_{14}}\theta_{\alpha'_{22}}}\cdot\rho_3&=\bar{s},\\
x\cdot y&=0\quad\textit{(otherwise)}.
\end{aligned}$$

As in the paper [2] on the homology cobordisms of plumbed 3-manifolds, we show the above statements by two approaches. Let $M(\Gamma)$ (resp. $M(\Gamma')$) be a plumbed

3-manifold associated to the tree Seifert graph Γ (resp. Γ') satisfying a certain condition (Ndeg) (Definition 6), and let $\phi = (L_*; i, i', \bullet)$ be an algebraic morphism between $H_*(M(\Gamma); \mathbb{Z})$ and $H_*(M(\Gamma'); \mathbb{Z})$. Then we construct a distributive algebra $\mathcal{R}_*(\Gamma, \Gamma', \phi)$ over $\mathbb{Q}$ by using the data (Γ, Γ', ϕ) as in Definition 6.

1. The 10/8-inequality.
   We apply a V-manifold version of the extended Furuta-Kametani-10/8-inequality for closed spin 4-manifolds with $b_1 > 0$. The 10/8-inequality contains terms depending on the quadruple cup product of the first cohomology on closed 4-V-manifolds. The quadruple products are calculated by using the algebra $\mathcal{R}_*(\Gamma, \Gamma', \phi)$ as a map $q(\Gamma, \Gamma', \phi) : \mathcal{R}_*(\Gamma, \Gamma', \phi)^{\otimes 4} \to \mathbb{Q}$.

2. The associativity of cup products.
   The distributive algebra $\mathcal{R}_*(\Gamma, \Gamma', \phi)$ is not necessarily associative by definition. However, if there exists a cobordism W between $M(\Gamma)$ and $M(\Gamma')$ realizing the algebraic morphism ϕ, then we see that there exists an injective ring homomorphism from $\mathcal{R}_*(\Gamma, \Gamma', \phi)$ to the homology ring $H_*(Z; \mathbb{Q})$ of a closed 4-V-manifold Z obtained by gluing $P(\Gamma)$, $P(\Gamma')$ and W along the boundaries $M(\Gamma)$ and $M(\Gamma')$, and hence that $\mathcal{R}_*(\Gamma, \Gamma', \phi)$ must be associative. Therefore, the associativity of the distributive algebra $\mathcal{R}_*(\Gamma, \Gamma', \phi)$ gives an obstruction to the existence of cobordisms W realizing ϕ.

Remark 5. *As in paper [2], the author does not know examples that can be detected by using the gauge theory in Approach 1 but cannot be detected by using the associativity of cup products in Approach 2.*

The homology cobordism category is defined to be the category whose objects are closed 3-manifolds and whose morphisms are homology cobordisms. If we take the quotient of the set of objects by the homology cobordism relation, then we obtain the homology cobordism monoid. In particular, we give a necessary condition, using w-invariants, for the homology 3-spheres to belong to the inertia group associated to some homology 3-spheres.

The organization of this paper is as follows. In Section 2, we introduce a distributive algebra $\mathcal{R}_*(\Gamma, \Gamma', \phi)$ and state the main theorems and their applications in Approach 1 using the 10/8-inequality, and in Approach 2 using the associativity of cup products. In Section 3, we recall the definition of cobordism the category of 3-manifolds and introduce a category of graded commutative rings with certain algebraic morphisms, which models that of the homology rings. In Section 5, we recall the definition of the w-invariants, which are integral lifts of the Rochlin invariants, and give several properties of the invariants under cobordisms and connected sum operations. Here we also give a necessary condition for the homology 3-spheres to belong to the inertia group associated to some 3-manifolds in the homology cobordism monoid. Finally in Section 6, to prove Lemma 32, we consider a cobordism of two plumbed 3-manifolds and calculate the intersection pairings of 3-cycles on closed 4-V-manifolds obtained by gluing 4-V-manifolds along the boundaries.

2. Main theorems and applications

2.1. A distributive algebra constructed from algebraic morphisms

Motivated by the above Lemma 32 below in Section 6, we introduce a distributive algebra $\mathcal{R}_*(\Gamma,\Gamma',\phi)$ using the data (Γ,Γ',ϕ).

Definition 6. *Let $\Gamma=(V,E,\omega)$, $\Gamma'=(V',E',\omega')$ be two tree Seifert graphs satisfying the condition (Ndeg) :*

1. *the intersection matrices $A(\Gamma)$, $A(\Gamma')$ of the plumbed V-manifold $P(\Gamma)$, $P(\Gamma')$ with respect to the standard basis are non-singular,*
2. *all the Euler numbers are non-zero $e(\omega_v)\neq 0$, $e(\omega'_{v'})\neq 0$ for all $v\in V$ and $v'\in V'$.*

Let $\phi=(L_*;i,i',\bullet)$ be an algebraic morphism between $(H_*(M(\Gamma);\mathbb{Z}),\bullet)$ and $(H_*(M(\Gamma');\mathbb{Z}),\bullet)$ in the $\mathcal{L}_3$ category . Then we define a graded commutative distributive algebra $\mathcal{R}_*(\Gamma,\Gamma',\phi)$ over $\mathbb{Q}$ to be

$$\begin{cases} \mathcal{R}_4(\Gamma,\Gamma',\phi)=\mathbb{Q}\ Z \\ \mathcal{R}_3(\Gamma,\Gamma',\phi)=R(\Gamma,\Gamma',\phi)\otimes\mathbb{Q} \\ \mathcal{R}_2(\Gamma,\Gamma',\phi)=\bigoplus_{v\in V}\mathbb{Q}\ \Sigma_v\ \oplus\bigoplus_{v'\in V'}\mathbb{Q}\ \Sigma'_{v'} \\ \mathcal{R}_1(\Gamma,\Gamma',\phi)=L(\Gamma,\Gamma',\phi)\otimes\mathbb{Q} \\ \mathcal{R}_0(\Gamma,\Gamma',\phi)=\mathbb{Q}\ pt \end{cases},$$

where

$$R(\Gamma,\Gamma',\phi)=\operatorname{Ker}\left(H(\Gamma)\oplus H(\Gamma')\overset{\theta\oplus\theta'}{\to}H_2(M(\Gamma);\mathbb{Z})\oplus H_2(M(\Gamma');\mathbb{Z})\overset{i+i'}{\to}L_2\right),$$

$$L(\Gamma,\Gamma',\phi)=\operatorname{Im}\left(H(\Gamma)\oplus H(\Gamma')\overset{\lambda\oplus\lambda'}{\to}H_1(M(\Gamma);\mathbb{Z})\oplus H_1(M(\Gamma');\mathbb{Z})\overset{i+i'}{\to}L_1\right),$$

as in Lemma 30 in Section 6. If we denote the elements corresponding to $(-\alpha)\oplus\alpha'\in R(\Gamma,\Gamma',\phi)$ formally by $\widetilde{\theta_\alpha\theta'_{\alpha'}}$ then the product structure on $\mathcal{R}_*(\Gamma,\Gamma',\phi)$ is given by

$$\begin{cases} Z\cdot x=x\quad(x\in\mathcal{R}_*(\Gamma,\Gamma',\phi)) \\ \widetilde{\theta_\alpha\theta'_{\alpha'}}\cdot\widetilde{\theta_\beta\theta'_{\beta'}}=-\sum_{v,v'\in V}A(\Gamma)^{vv'}(\alpha_{v'}\cdot\beta_{v'})\Sigma_v+\sum_{v,v'\in V'}A(\Gamma')^{vv'}(\alpha'_{v'}\cdot\beta'_{v'})\Sigma'_v, \\ \widetilde{\theta_\alpha\theta'_{\alpha'}}\cdot\Sigma_v=\Sigma_v\cdot\widetilde{\theta_\alpha\theta'_{\alpha'}}=-i(\alpha_v),\ \widetilde{\theta_\alpha\theta'_{\alpha'}}\cdot\Sigma'_v=\Sigma'_v\cdot\widetilde{\theta_\alpha\theta'_{\alpha'}}=i'(\alpha'_v), \\ i(\alpha_v)\cdot\widetilde{\theta_\beta\theta'_{\beta'}}=-\widetilde{\theta_\beta\theta'_{\beta'}}\cdot i(\alpha_v)=-\alpha_v\cdot\beta_v,\ i'(\alpha'_v)\cdot\widetilde{\theta_\beta\theta'_{\beta'}}=-\widetilde{\theta_\beta\theta'_{\beta'}}\cdot i'(\alpha'_v)=\alpha'_v\cdot\beta'_v, \\ \qquad\text{for any } \alpha,\alpha'\in H(\Gamma),\ \beta,\beta'\in H(\Gamma'),\ \widetilde{\theta_\alpha\theta'_{\alpha'}},\widetilde{\theta_\beta\theta'_{\beta'}}\in R(\Gamma,\Gamma',\phi) \\ \Sigma_v\cdot\Sigma_{v'}=-A(\Gamma)_{vv'}\,pt,\ \Sigma'_v\cdot\Sigma'_{v'}=A(\Gamma')_{vv'}\,pt, \\ x\cdot y=0\quad(x,y:\ \text{otherwise}). \end{cases}$$

Note that $\mathcal{R}_*(\Gamma,\Gamma',\phi)$ is not necessarily associative in general.

By Lemma 29, 30 in Section 6 below, we have the following

Theorem 7. *Let $\Gamma=(V,E,\omega)$, $\Gamma'=(V',E',\omega')$ be two tree Seifert graphs satisfying the condition (Ndeg), and let $\phi=(L_*;i,i',\bullet)$ be a morphism between homology rings $(H_*(M(\Gamma);\mathbb{Z}),\bullet)$, $(H_*(M(\Gamma');\mathbb{Z}),\bullet)$ in the $\mathcal{L}_3$ category. If there exists a cobordism $(W;M(\Gamma),M(\Gamma'))$ realizing $\phi=(L_*;i,i',\bullet)$, then there exists an injective ring homomorphism $\mathcal{R}_*(\Gamma,\Gamma',\phi)\to H_*(Z;\mathbb{Q})$, and hence $\mathcal{R}_*(\Gamma,\Gamma',\phi)$ must be an associative ring.*

Remark 8. *The V-manifold Z can be regarded as a rational homology manifold, and hence the homology ring $H_*(Z;\mathbb{Q})$ can be defined over the rationals $\mathbb{Q}$.*

2.2. The 10/8-inequality and the quadruple cup products

The 11/8-conjecture, due to Y. Matsumoto, states that for any closed spin 4-manifold the second Betti number is greater than or equal to 11/8 times the absolute value of the signature. A weaker inequality, called the 10/8-inequality, was first proved by M. Furuta using a technique based on the finite-dimensional approximation of the Seiberg-Witten equation. This inequality was proved under the assumption that the first Betti number is zero, but this condition can always be realized by surgeries. However, by dealing with the first cohomology of closed 4-manifolds, M. Furuta and Y. Kametani improved the 10/8-inequality for closed spin 4-manifolds with positive first Betti numbers by considering $Pin(2)$-equivariant maps between sphere bundles over the Jacobi tori of 4-manifolds constructed from the finite-dimensional approximation of the Seiberg-Witten equation [9]. Their result is based on the joint work of by M. Furuta, Y. Kametani, H. Matsue, and N. Minami [8] on the stable homotopy version [7] of the Seiberg-Witten invariants. The improved inequality contains terms that come from the quadruple cup product structures on closed spin 4-manifolds.

In the paper [2], we extended the Furuta-Kametani-10/8-inequality to the case of V-manifolds. For various definitions concerning V-manifolds, see [11].

Let $((M,c),(X,\hat{c}))$ be a pair consisting of a closed 3-manifold M with spin structure c and a closed spin 4-V-manifold X with V-spin structure $\hat{c}$ satisfying $\partial(X,\hat{c}) = (M,c)$. Then we can define an integral lift of the Rochlin invariant, which we call the w-invariant (Definition 19),

$$w((M,c),(X,\hat{c})) \equiv -\mu(M,c) \mod 16.$$

The w-invariant was defined in joint work with M. Furuta on applications of the 10/8-inequality [4]. In fact, we proved the vanishing of the V-indices of the Dirac operators on closed 4-V-manifolds X with $b_2^{\pm}(X) \leqslant 2$, which implies the homology cobordism invariance of w in a certain class of homology 3-spheres. In joint work with M. Furuta and M. Ue [5], and in the extensive work of N. Saveliev [12], it is shown that the Neumann-Siebenmann invariant for plumbed homology 3-spheres is equal to the w-invariant for some auxiliary plumbed V-manifold and its V-spin structure. Recently, M. Ue proved that the Ozsváth-Szabó correction term is equal to the Neumann-Siebenmann invariant (and hence the w-invariant) for a large class of plumbed rational homology 3-spheres [15].

We introduce the following quadruple product structure.

Lemma 9. *Let $\Gamma = (V,E,\omega)$, $\Gamma' = (V',E',\omega)$ be two tree Seifert graphs and let $\phi = (L_*; i, i', \bullet)$ be a morphism between the homology rings $(H_*(M(\Gamma)),\bullet)$, $(H_*(M(\Gamma')),\bullet)$ in the $\mathcal{L}_3$ category. Then a quadruple product $q(\Gamma,\Gamma',\phi)$:*

$\mathcal{R}_3(\Gamma,\Gamma',\phi)^{\otimes 4} \to \mathbb{Q}$ *is calculated to be*

$$\begin{aligned}&\widetilde{\theta_\alpha\theta'_{\alpha'}}\cdot\widetilde{\theta_\beta\theta'_{\beta'}}\cdot\widetilde{\theta_\gamma\theta'_{\gamma'}}\cdot\widetilde{\theta_\delta\theta'_{\delta'}}\\ &= -\sum_{v,v'\in V} A(\Gamma)^{vv'}(\alpha_v\cdot\beta_v)(\gamma_{v'}\cdot\delta_{v'}) + \sum_{v,v'\in V'} A(\Gamma')^{vv'}(\alpha'_v\cdot\beta'_v)({\gamma'}_{v'}\cdot\delta'_{v'})\end{aligned}$$

By Lemma 32, we generalize the Theorem in [2] to obtain

Theorem 10. *Let* $\Gamma = (V, E, \omega)$, $\Gamma' = (V', E', \omega')$ *be two tree Seifert graphs such that*

1. $b^{\pm}(\Gamma) + b^{\mp}(\Gamma') \leqslant 2m + 2$,
2. *the intersection matrices* $A(\Gamma)$, $A(\Gamma')$ *of the plumbed* V*-manifold* $P(\Gamma)$, $P(\Gamma')$ *with respect to the standard basis are non-singular, and*
3. *the Euler numbers are non-zero* $e(\omega_v) \neq 0$, $e(\omega'_{v'}) \neq 0$ *for all* $v \in V$ *and* $v' \in V'$.

Let $\phi = (L_*; i, i', \bullet)$ *be a morphism between the homology rings* $(H_*(M(\Gamma);\mathbb{Z}),\bullet)$, $(H_*(M(\Gamma');\mathbb{Z}),\bullet)$ *in the* $\mathcal{L}_3$ *category. Suppose that the associated plumbed* 3*-manifolds with spin structures* $(M(\Gamma), c)$, $(M(\Gamma'), c')$ *are spin cobordant* $(M(\Gamma), c) \simeq^{\phi}_{(W,\tilde{c})} (M(\Gamma'), c')$ *for some compact spin* 4*-manifold* $(W, \tilde{c})$ *inducing an algebraic morphism* $\phi = (L_*; i, i', \bullet) \cong (L_*(W;\mathbb{Z}); i_*, i'_*, \bullet)$ *such that* $L_*(W;\mathbb{Z}) = H_*(W;\mathbb{Z})$. *If there exists an injective homomorphism*

$$h : H^1\left(\sharp_{i=1}^m T_i^4;\mathbb{Z}\right) \to R(\Gamma,\Gamma',\phi) \subset \mathcal{R}_3(\Gamma,\Gamma',\phi)$$

such that

$$h(x)\cdot h(y)\cdot h(z)\cdot h(w) \equiv \left\langle x\cup y\cup z\cup w, \left[\sharp_{i=1}^m T_i^4\right]\right\rangle \bmod 2,$$

for any $x, y, z, w \in H^1\left(\sharp_{i=1}^m T_i^4;\mathbb{Z}\right)$, *where the* T_i^4*'s are* m*-copies of the* 4*-torus* T^4. *Then we have*

$$w((M(\Gamma), c), (P(\Gamma), \hat{c})) = w((M(\Gamma'), c'), (P(\Gamma), \hat{c}')).$$

Example 11. *Let* Γ *and* Γ' *be two Seifert graphs in the above Example 3. Let* $\tilde{\Gamma} = \Gamma \sharp s\cdot(\{0\},\emptyset)$ *be the Seifert graph consisting of the disjoint union of* Γ *and* s*-vertices for* $s \leqslant 2$, *with no edges and with Seifert invariants*

$$\begin{aligned}&\tilde{\omega}(v) = \omega(v),\ v\in V,\\ &\tilde{\omega}_\ell(0) = \omega = \{0; (a_1,b_1),\ldots,(a_n,b_n)\},\ 1\leqslant r\leqslant s.\end{aligned}$$

We also define $\tilde{\Gamma}' = \Gamma'\sharp s\cdot(\{0\},\emptyset)$ *and the Seifert invariants* $\tilde{\omega}'$ *by* $\tilde{\omega}'(v) = \omega'(v)$ *for* $v \in V'$ *and* $\tilde{\omega}'(0) = \omega$. *Suppose that one or more of the* a_i*'s are even for* $\tilde{\omega}(0)$, *so that the associated disk* V*-bundle admits a* V*-spin structure. Then the plumbed* 3*-manifold* $M(\tilde{\Gamma})$ *is the connected sum* $(M(\Gamma), c)\sharp s\cdot(\Sigma, c_\Sigma)$ *of the plumbed* 3*-manifold* $(M(\Gamma), c)$ *and* s*-copies of the Seifert rational homology* 3*-sphere* (Σ, c_Σ) *of Seifert invariant* $\tilde{\omega}(0)$. *The plumbed* 4*-*V*-manifold* $P(\tilde{\Gamma})$ *is the boundary connected sum* $(P(\Gamma), \hat{c}) \natural s\cdot(E, c_E)$, *where* E *is the associated disk* V*-bundle of the* S^1*-fibration* Σ. *Suppose the Euler number* $e(\omega)$ *is positive then we have* $b_2^{\pm}(P(\Gamma)\natural s\cdot E) +$

$b_2^{\mp}(P(\Gamma') \natural s \cdot E) \leqslant 2 + s$. Now the quadruple product associated to the 3-cocycles $\widetilde{\theta_{\alpha_{11}}\theta'_{\alpha'_{11}}}, \widetilde{\theta_{\alpha_{12}}\theta'_{\alpha'_{12}}}, \widetilde{\theta_{\alpha_{13}}\theta'_{\alpha'_{21}}}, \widetilde{\theta_{\alpha_{14}}\theta'_{\alpha'_{22}}} \in R(\Gamma, \Gamma', \phi)$ satisfies

$$\widetilde{\theta_{\alpha_{11}}\theta'_{\alpha'_{11}}} \cdot \widetilde{\theta_{\alpha_{12}}\theta'_{\alpha'_{12}}} \cdot \widetilde{\theta_{\alpha_{13}}\theta'_{\alpha'_{21}}} \cdot \widetilde{\theta_{\alpha_{14}}\theta'_{\alpha'_{22}}} = -5 \equiv 1 \bmod 2$$

In fact, by noting that

$$i(\theta_{\alpha_{11}}) = i'(\theta'_{\alpha'_{11}}),\ i(\theta_{\alpha_{12}}) = i'(\theta'_{\alpha'_{12}}),\ i(\theta_{\alpha_{13}}) = i'(\theta'_{\alpha'_{21}}),\ i(\theta_{\alpha_{14}}) = i'(\theta'_{\alpha'_{22}}),$$

and

$$A(\Gamma)^{-1} = A(\Gamma')^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix},$$

and applying Lemma 32, we obtain

$$\begin{aligned} & q(\Gamma, \Gamma', \phi)\left(\widetilde{\theta_{\alpha_{11}}\theta'_{\alpha'_{11}}} \otimes \widetilde{\theta_{\alpha_{12}}\theta'_{\alpha'_{12}}} \otimes \widetilde{\theta_{\alpha_{13}}\theta'_{\alpha'_{21}}} \otimes \widetilde{\theta_{\alpha_{14}}\theta'_{\alpha'_{22}}}\right) \\ &= -A(\Gamma)^{11}(\alpha_{11} \cdot \alpha_{12})(\alpha_{13} \cdot \alpha_{14}) + A(\Gamma')^{12}(\alpha'_{11} \cdot \alpha'_{12})(\alpha'_{21} \cdot \alpha'_{22}) \\ &= -2 \cdot 1 \cdot 1 + (-3) \cdot 1 \cdot 1 \\ &= -5 \equiv 1 \bmod 2. \end{aligned}$$

On the other hand, the w-invariants are calculated to be

$$\begin{aligned} & w((M(\Gamma), c) \sharp s \cdot (\Sigma, c_\Sigma), (P(\Gamma), \hat{c}) \natural s \cdot (E, c_E)) \\ & - w((M(\Gamma'), c') \sharp s \cdot (\Sigma, c_\Sigma), (P(\Gamma'), \hat{c}') \natural s \cdot (E, c_E)) \\ &= w((M(\Gamma), c), (P(\Gamma), \hat{c})) - w((M(\Gamma'), c'), (P(\Gamma'), \hat{c}')) \\ &= 12 - (-4) = 16 \neq 0 \end{aligned}$$

and $2+s \leqslant 2+2$. Hence by Theorem 10, we see that there exists no cobordism $(W, \tilde{c})$ between $(M(\Gamma), c) \sharp s \cdot (\Sigma, c_\Sigma)$ and $(M(\Gamma'), c') \sharp s \cdot (\Sigma, c_\Sigma)$ inducing $\phi = (L_*; i, i', \bullet)$ for $s \leqslant 2$ such that $L_*(W; \mathbb{Z}) = H_*(W; \mathbb{Z})$. Note that the difference is divisible by 16 and hence this cannot be detected by using the Rohlin invariant.

2.3. Associativity of intersection products on homology

On the other hand, we can prove the above statement in Approach 2. Motivated by Lemma 32, we introduce the following triple product.

Definition 12.
Let $\Gamma = (V, E)$, $\Gamma' = (V', E')$ be two Seifert graphs and let $\phi = (K_*; i, i', \bullet)$ be a morphism between homology rings $(H_*(M(\Gamma); \mathbb{Z}), \bullet)$, $(H_*(M(\Gamma'); \mathbb{Z}), \bullet)$ in the category $\mathcal{L}_3$. Then a triple product $t(\Gamma, \Gamma', \phi) : \mathcal{R}_3(\Gamma, \Gamma', \phi)^{\otimes 3} \to \mathcal{R}_1(\Gamma, \Gamma', \phi)$ is

defined and calculated to be

$$\begin{aligned}
& t\left(\Gamma,\Gamma',\phi\right)\left(\widetilde{\theta_\alpha\theta'_{\alpha'}}\otimes\widetilde{\theta_\beta\theta'_{\beta'}}\otimes\widetilde{\theta_\gamma\theta'_{\gamma'}}\right) \\
&\equiv \left(\widetilde{\theta_\alpha\theta'_{\alpha'}}\cdot\widetilde{\theta_\beta\theta'_{\beta'}}\right)\cdot\widetilde{\theta_\gamma\theta'_{\gamma'}} - \widetilde{\theta_\alpha\theta'_{\alpha'}}\cdot\left(\widetilde{\theta_\beta\theta'_{\beta'}}\cdot\widetilde{\theta_\gamma\theta'_{\gamma'}}\right) \\
&= \sum_{v,v'\in V} A(\Gamma)^{vv'}\left(\alpha_v\cdot\beta_v\right)i\left(\gamma_{v'}\right) - \sum_{v,v'\in V'} A(\Gamma')^{vv'}\left(\alpha'_v\cdot\beta'_v\right)i'\left(\gamma'_{v'}\right) \\
&- \sum_{v,v'\in V} A(\Gamma)^{vv'}\left(\beta_v\cdot\gamma_v\right)i\left(\alpha_{v'}\right) + \sum_{v,v'\in V'} A(\Gamma')^{vv'}\left(\beta'_v\cdot\gamma'_v\right)i'\left(\alpha'_{v'}\right) \\
&\in \mathcal{R}_1\left(\Gamma,\Gamma',\phi\right).
\end{aligned}$$

In fact, we obtain the following criterion by Lemma 32.

Theorem 13. *Let $\Gamma=(V,E,\omega)$, $\Gamma'=(V',E',\omega')$ be two tree Seifert graphs satisfying the condition (Ndeg).*

Let $M(\Gamma)$ and $M(\Gamma')$ be the associated plumbed 3-manifolds, and $\phi=(L_*;i,i',\bullet)$ be a morphism between homology rings $(H_*(M(\Gamma)),\bullet)$, $(H_*(M(\Gamma')),\bullet)$ in the $\mathcal{L}_3$ category. If the triple product is not zero,

$$t\left(\Gamma,\Gamma',\phi\right)\left(\widetilde{\theta_\alpha\theta'_{\alpha'}}\otimes\widetilde{\theta_\beta\theta'_{\beta'}}\otimes\widetilde{\theta_\gamma\theta'_{\gamma'}}\right)\not\equiv 0\in\mathcal{R}_1\left(\Gamma,\Gamma',\phi\right)$$

for some triple $\widetilde{\theta_\alpha\theta'_{\alpha'}},\widetilde{\theta_\beta\theta'_{\beta'}},\widetilde{\theta_\gamma\theta'_{\gamma'}}\in R\left(\Gamma,\Gamma',\phi\right)$, then there exists no cobordism W between $M\left(\Gamma\right)$ and $M\left(\Gamma'\right)$ inducing ϕ such that $L_*\left(W;\mathbb{Z}\right)=H_*\left(W;\mathbb{Z}\right)$.

Example 14. *In the above Example 3, we obtain*

$$t\left(\Gamma,\Gamma',\phi\right)\left(\widetilde{\theta_{\alpha_{11}}\theta'_{\alpha'_{11}}}\otimes\widetilde{\theta_{\alpha_{12}}\theta'_{\alpha'_{12}}}\otimes\widetilde{\theta_{\alpha_{13}}\theta'_{\alpha'_{21}}}\right)=5\delta_3\neq 0.$$

In fact, by Definition 12, we have

$$\begin{aligned}
& t\left(\Gamma,\Gamma',\phi\right)\left(\widetilde{\theta_{\alpha_{11}}\theta'_{\alpha'_{11}}}\otimes\widetilde{\theta_{\alpha_{12}}\theta'_{\alpha'_{12}}}\otimes\widetilde{\theta_{\alpha_{13}}\theta'_{\alpha'_{21}}}\right) \\
&= A(\Gamma)^{11}\left(\alpha_{11}\cdot\alpha_{12}\right)i\left(\alpha_{13}\right) - A(\Gamma')^{12}\left(\alpha'_{11}\cdot\alpha'_{12}\right)i'\left(\alpha'_{21}\right) \\
&- A(\Gamma)^{11}\left(\alpha_{12}\cdot\alpha_{13}\right)i\left(\alpha_{11}\right) + A(\Gamma')^{11}\left(\alpha'_{12}\cdot 0\right)i'\left(\alpha'_{11}\right) + A(\Gamma')^{21}\left(0\cdot\alpha'_{21}\right)i'\left(\alpha'_{11}\right) \\
&= 2\cdot 1\cdot i\left(\alpha_{13}\right) - (-3)\cdot 1\cdot i'\left(\alpha'_{21}\right) = 5i\left(\alpha_{13}\right) \\
&= 5\delta_3\neq 0.
\end{aligned}$$

Hence by Theorem 13 we see that there exists no cobordism W between $M\left(\Gamma\right)\ \sharp\ s\cdot\Sigma$ and $M\left(\Gamma'\right)\ \sharp\ s\cdot\Sigma$ inducing $\phi=(L_*;i,i')$ such that $L\left(W;\mathbb{Z}\right)=H_*\left(W;\mathbb{Z}\right)$ for any rational homology 3-sphere Σ and non-negative integer s.

3. Cobordism category of 3-manifolds and an abstract category of graded commutative rings

In this section, we give a precise definition of the $\mathcal{L}_3$ category, a category of graded commutative ring with certain algebraic morphisms which was introduced in order to investigate the cobordism category $\mathcal{C}_3$ of 3-manifolds.

3.1. Cobordism category of 3-manifolds

Let $(\mathcal{C}_3, \sqcup, \emptyset)$ be the category whose set of objects $\mathrm{ob}(\mathcal{C}_3)$ is the set of all disjoint unions M of closed oriented 3-manifolds and whose set of morphisms $\mathcal{C}_3(M, M')$ between $M \in \mathrm{ob}(\mathcal{C}_3)$ and $M' \in \mathrm{ob}(\mathcal{C}_3)$ is the set of all cobordisms $(W; M, M')$ between M and M', that is, W is a compact oriented smooth 4-manifold with boundary $\partial W \cong M \sqcup (-M')$. If M, M', and M'' are three objects in $\mathcal{C}_3$, then the composite operation on morphisms $\mathcal{C}_3(M, M') \times \mathcal{C}_3(M', M'') \to \mathcal{C}_3(M, M'')$, $(W, W') \longmapsto W \cup_{M'} W'$ is defined by gluing the 4-manifolds W and W' along the boundary component M'. There exists a bifunctor $\sqcup : \mathcal{C}_3 \times \mathcal{C}_3 \to \mathcal{C}_3$ defined by the disjoint union $\sqcup : \mathrm{ob}(\mathcal{C}_3) \times \mathrm{ob}(\mathcal{C}_3) \to \mathrm{ob}(\mathcal{C}_3)$, $(M_1, M_2) \longmapsto M_1 \sqcup M_2$ and $\sqcup : \mathcal{C}_3(M_1, M_1') \times \mathcal{C}_3(M_2, M_2') \to \mathcal{C}_3(M_1 \sqcup M_2, M_1' \sqcup M_2')$, $(W_1, W_2) \longmapsto W_1 \sqcup W_2$, and the empty set $\emptyset \in \mathrm{ob}(\mathcal{C}_3)$ defines the unit element. Hence $(\mathcal{C}_3, \sqcup, \emptyset)$ defines a monoidal category.

Let $\mathcal{C}_3^{\mathrm{spin}}$ be the category whose set of objects $\mathrm{ob}(\mathcal{C}_3^{\mathrm{spin}})$ is the set of all disjoint unions (M, c) of closed oriented 3-manifolds with spin structures c and whose set of morphisms $\mathcal{C}_3^{\mathrm{spin}}((M, c), (M', c'))$ between $(M, c) \in \mathrm{ob}(\mathcal{C}_3^{\mathrm{spin}})$ and $(M', c') \in \mathrm{ob}(\mathcal{C}_3^{\mathrm{spin}})$ is the set $((W, \tilde{c}); (M, c), (M', c'))$ of all spin cobordisms between (M, c) and (M', c'), that is, $(W, \tilde{c})$ is a compact spin smooth 4-manifold with boundary $\partial(W, \tilde{c}) \cong (M, c) \sqcup (-(M', c'))$. Then the monoidal category of 3-manifolds with spin structures $(\mathcal{C}_3^{\mathrm{spin}}, \sqcup, \emptyset)$ is defined similarly.

3.2. A category of graded commutative rings

Let $(\mathcal{L}_3, \oplus, 0)$ be a category whose set of objects $\mathrm{ob}(\mathcal{L}_3)$ is the set of all pairs $(H_*, \bullet)$ composed of

1. a graded $\mathbb{Z}$-module $H_* = \bigoplus_{k=0}^{3} H_k$ of dimension 3 such that $H_0 \cong \mathbb{Z}^c$ for some $c \in \mathbb{Z}_{\geqslant 0}$,
2. a graded commutative product $\bullet : H_k \otimes H_\ell \to H_{k+\ell-3}$, i.e. $\alpha_k \cdot \beta_\ell = (-1)^{(3-k)(3-\ell)} \beta_\ell \cdot \alpha_k$ for $\alpha_k \in H_k$, $\beta_\ell \in H_\ell$, satisfying the following conditions:
   (a) $\bullet : H_k \otimes H_{3-k} \to H_0 \cong \mathbb{Z}^c \xrightarrow{\varepsilon} \mathbb{Z}$ induces an isomorphism $H_{3-k} \otimes \mathbb{Q} \cong (H_k \otimes \mathbb{Q})^*$, where $\varepsilon : \mathbb{Z}^c \to \mathbb{Z}$ is given by $\varepsilon(\oplus_{i=1}^{c} m_i) = \sum_{i=1}^{c} m_i$,
   (b) $H_3 \cong \mathbb{Z}^c$ and the element $\mu \in H_3 \cong \mathbb{Z}^c$ corresponding to $\oplus_{i=1}^{c} 1 \in \mathbb{Z}^c$ is a unit element with respect to the product $\bullet : H_k \otimes H_3 \to H_k$,

and whose set of morphisms $\mathcal{L}_3((H_*, \bullet), (H'_*, \bullet))$ between two objects $(H_*, \bullet)$ and $(H'_*, \bullet)$ is the set of all quadruples $(L_*; i, i', \bullet)$ composed of

1. a graded $\mathbb{Z}$-module $L_* = \bigoplus_{k=0}^{4} L_*$ of dimension 4 such that $L_0 \cong \mathbb{Z}^d$ for some $d \in \mathbb{Z}_{\geqslant 0}$,
2. two homomorphisms $H_* \xrightarrow{i} L_* \xleftarrow{i'} H'_*$ such that
   (a) $L_k = \mathrm{Im}\left(H_k \oplus H'_k \overset{i+i'}{\to} L_k\right)$,
   (b) if R_k are $\mathbb{Z}$-modules defined by

$$R_k = \mathrm{Ker}\left(H_{k-1} \oplus H'_{k-1} \overset{i+i'}{\to} L_{k-1}\right),$$

then the bilinear pairings $\bullet : R_k \otimes L_\ell \to L_{k+\ell-4}$, $\bullet : L_\ell \otimes R_k \to L_{k+\ell-4}$ induced by $\bullet : H_k \otimes H_\ell \to H_{k+\ell-3}$ and $\bullet : H'_k \otimes H'_\ell \to H'_{k+\ell-3}$ are graded-commutative with each other in the sense that $\alpha_k \cdot \beta_\ell = (-1)^{(4-k)(4-\ell)}\, \beta_\ell \cdot \alpha_k$ for $\alpha_k \in R_k$, $\beta_\ell \in L_\ell$, and the pairing $\bullet$ satisfies the following conditions:

i. the following commutative diagram holds,

$$\begin{array}{ccccc}
H_k & \otimes & H_\ell & \overset{\bullet}{\to} & H_{k+\ell-3} \\
{\scriptstyle \partial\otimes \mathrm{id}_{H_\ell}} & \uparrow & & & \\
R_{k+1} & \otimes & H_\ell & & \downarrow i \\
{\scriptstyle \mathrm{id}_{R_{k+1}}\otimes i} & \downarrow & & & \\
R_{k+1} & \otimes & L_\ell & \overset{\bullet}{\to} & L_{k+\ell-3} \\
{\scriptstyle \mathrm{id}_{R_{k+1}}\otimes i'} & \uparrow & & & \\
R_{k+1} & \otimes & H'_\ell & & \uparrow i' \\
{\scriptstyle \partial'\otimes \mathrm{id}_{H'_\ell}} & \downarrow & & & \\
H'_k & \otimes & H'_\ell & \overset{\bullet}{\to} & H'_{k+\ell-3}
\end{array},$$

where ∂, ∂' are the natural projections,

$$\partial \oplus \partial' : R_k = \mathrm{Ker}\left(H_{k-1} \oplus H'_{k-1} \overset{i+i'}{\to} L_{k-1} \right) \to H_{k-1} \oplus H'_{k-1}.$$

ii. $\bullet : R_k \otimes L_{4-k} \to L_0 \cong \mathbb{Z}^d \overset{\varepsilon}{\to} \mathbb{Z}$ induces an isomorphism $L_{4-k} \otimes \mathbb{Q} \cong (R_k \otimes \mathbb{Q})^*$, where $\varepsilon : \mathbb{Z}^d \to \mathbb{Z}$ is given by $\varepsilon\left(\oplus_{i=1}^d m_i\right) = \sum_{i=1}^d m_i$,

iii. $R_4 \cong \mathbb{Z}^d$ and the element $\nu \in R_4 \cong \mathbb{Z}^d$ corresponding to $\oplus_{i=1}^d 1 \in \mathbb{Z}^d$ is a unit element with respect to the product $\bullet : R_4 \otimes L_k \to L_k$.

Let $(H, \bullet)$, $(H', \bullet)$, and $(H'', \bullet)$ be three objects in $\mathcal{L}_3$. Then there exists a composite operation on morphisms

$$\begin{aligned}
\mathcal{L}_3\left((H,\bullet),(H',\bullet)\right) \times \mathcal{L}_3\left((H',\bullet),(H'',\bullet)\right) &\to \mathcal{L}_3\left((H,\bullet),(H'',\bullet)\right), \\
\left((L_*; i, i', \bullet), (L'_*; i'', i''', \bullet)\right) &\longmapsto \left((L \circ L')_*; i, i'', \bullet\right),
\end{aligned}$$

where $(L \circ L')_* = \bigoplus_{k=0}^3 (L \circ L')_k$ is the graded $\mathbb{Z}$-module defined by

$$(L \circ L')_k \cong \mathrm{Im}\left(H_k \oplus H''_k \overset{i'+i'''}{\to} \mathrm{Coker}\left(H'_k \overset{i' \oplus (-i'')}{\to} L_k \oplus L'_k \right)\right),$$

and the inclusions $H_k \overset{i}{\to} (L \circ L')_k \overset{i'''}{\leftarrow} H''_k$ are defined by the natural map

$$i \oplus i''' : H_k \oplus H''_k \overset{i \oplus i'''}{\to} L_k \oplus L'_k \to \mathrm{Coker}\left(H'_k \overset{i' \oplus (-i'')}{\to} L_k \oplus L'_k \right).$$

We define $R \circ R'$ in the same way as R is defined by L:

$$\begin{aligned}(R\circ R')_k &= \operatorname{Ker}\left(H_{k-1}\oplus H''_{k-1} \overset{i'+i'''}{\to} (L\circ L')_k\right)\\ &\cong \operatorname{Ker}\left(H_{k-1}\oplus H''_{k-1} \overset{i'+i'''}{\to} \frac{L_{k-1}\oplus L'_{k-1}}{(i'\oplus(-i''))\,H'_{k-1}}\right)\\ &\cong \operatorname{Im}\left(\operatorname{Ker}\left(R_k\oplus R'_k \overset{\partial'-\partial''}{\to} H'_{k-1}\right) \overset{\partial+\partial'''}{\to} H_{k-1}\oplus H''_{k-1}\right).\end{aligned}$$

Then the product $\bullet : (R\circ R')_k \otimes (L\circ L')_\ell \to (L\circ L')_{k+\ell-4}$ is induced by the natural product

$$\begin{aligned}\bullet\ :\ &\operatorname{Ker}\left(R_k\oplus R'_k \overset{\partial'-\partial''}{\to} H'_{k-1}\right)\otimes \operatorname{Coker}\left(H'_\ell \overset{i'\oplus(-i'')}{\to} L_\ell\oplus L'_\ell\right)\\ &\to \operatorname{Coker}\left(H'_{k+\ell-1} \overset{i'\oplus(-i'')}{\to} L_{k+\ell-5}\oplus L'_{k+\ell-5}\right)\end{aligned}$$

which is defined by using the products $\bullet : R_k\otimes L_\ell \to L_{k+\ell-4}$ and $\bullet : R'_k\otimes L'_\ell \to L'_{k+\ell-4}$. Then we have the following

Proposition 15. *The product* $\bullet : (R\circ R')_k\otimes(L\circ L')_{4-k} \to (L\circ L')_0 \cong \mathbb{Z}^{d''} \overset{\varepsilon}{\to} \mathbb{Z}$ *induces an isomorphism* $(R\circ R')_k\otimes\mathbb{Q} \cong ((L\circ L')_{4-k}\otimes\mathbb{Q})^*$.

Proof. Note that if $D : R_{k+1}\otimes\mathbb{Q} \to (L_{3-k}\otimes\mathbb{Q})^*$ and $D : H_k\otimes\mathbb{Q} \to (H_{3-k}\otimes\mathbb{Q})^*$ are the duality isomorphisms, then $D\circ\partial = i^*\circ D$, so we have

$$\begin{aligned}&\left(\operatorname{Ker}\left(R_k\oplus R'_k \overset{\partial'-\partial''}{\to} H'_{k-1}\right)\otimes\mathbb{Q}\right)^*\\ &\cong \operatorname{Coker}\left((R_k\otimes\mathbb{Q})^*\oplus(R'_k\otimes\mathbb{Q})^* \overset{(\partial'-\partial'')^*}{\leftarrow} (H'_{k-1}\otimes\mathbb{Q})^*\right)\\ &\cong \operatorname{Coker}\left(L_{4-k}\oplus L'_{4-k} \overset{i'\oplus(-i'')}{\leftarrow} H'_{4-k}\right)\otimes\mathbb{Q}.\end{aligned}$$

Note also that the identity map $\mathrm{id} : H_{3-k}\oplus H''_{3-k} \to H_{3-k}\oplus H''_{3-k}$ induces a natural isomorphism

$$\begin{aligned}&\operatorname{Ker}\left(H_{3-k}\oplus H''_{3-k} \overset{i+i'''}{\to} \operatorname{Coker}\left(H'_{3-k} \overset{i'\oplus(-i'')}{\to} L_{3-k}\oplus L'_{3-k}\right)\right)\\ &\cong \operatorname{Im}\left(H_{3-k}\oplus H''_{3-k} \overset{\partial+\partial'''}{\leftarrow} \operatorname{Ker}\left(R_{4-k}\oplus R'_{4-k} \overset{\partial'-\partial''}{\to} H'_{3-k}\right)\right).\end{aligned}$$

Then, using these isomorphisms and the fact that

$$\operatorname{Im}(f : V \to W)^* \cong \operatorname{Im}(f^* : V^* \leftarrow W^*)$$

for any homomorphism $f : V \to W$ of $\mathbb{Q}$-vector spaces, we have

$$\begin{aligned}
&((L \circ L')_k \otimes \mathbb{Q})^* \\
&= \left(\operatorname{Im}\left(H_k \oplus H''_k \overset{i+i'''}{\to} \operatorname{Coker}\left(H'_k \overset{i'\oplus(-i'')}{\to} L_k \oplus L'_k\right)\right) \otimes \mathbb{Q}\right)^* \\
&\cong \operatorname{Im}\left((H_{3-k} \otimes \mathbb{Q}) \oplus (H''_{3-k} \otimes \mathbb{Q}) \overset{\partial+\partial'''}{\leftarrow} \operatorname{Ker}\left(R_{4-k} \oplus R'_{4-k} \overset{\partial'-\partial''}{\to} H'_{3-k}\right) \otimes \mathbb{Q}\right) \\
&\cong \operatorname{Ker}\left((H_{3-k} \otimes \mathbb{Q}) \oplus (H''_{3-k} \otimes \mathbb{Q}) \overset{i+i'''}{\to} \operatorname{Coker}\left(H'_{3-k} \overset{i'\oplus(-i'')}{\to} L_{3-k} \oplus L'_{3-k}\right) \otimes \mathbb{Q}\right) \\
&= (R \circ R')_{4-k} \otimes \mathbb{Q}.
\end{aligned}$$

□

Remark 16. *The composition*

$$\begin{aligned}
\mathcal{L}_3\left((H_*, \bullet), (H'_*, \bullet)\right) \times \mathcal{L}_3\left((H'_*, \bullet), (H''_*, \bullet)\right) &\to \mathcal{L}_3\left((H_*, \bullet), (H''_*, \bullet)\right), \\
\left((L_*; i, i', \bullet), (L'_*; i'', i''', \bullet)\right) &\longmapsto \left((L \circ L')_*; i, i'', \bullet\right)
\end{aligned}$$

satisfies the associativity law, and the unit morphism is defined by $(H_* \overset{1}{\to} H_* \overset{1}{\leftarrow} H_*) \in \mathcal{L}_3\left((H_*, \bullet), (H_*, \bullet)\right)$. *Hence* $\mathcal{L}_3$ *defines a category.*

Remark 17. *The condition of the Poincaré duality should be replaced with the condition that the product* $\bullet : R_k \otimes L_{4-k} \to L_0 \cong \mathbb{Z}^d \overset{\varepsilon}{\to} \mathbb{Z}$ *induces an isomorphism* $\overline{L}_{4-k} \cong \operatorname{Hom}\left(\overline{R}_k, \mathbb{Z}\right)$ *on the free parts in the integral coefficients. But this condition may not be preserved under composition, and we need to impose certain torsion-free conditions on the boundary. For example, we may introduce a category* $\mathcal{L}_3^0$ *whose objects are* $(H_*, \bullet, F_*)$ *with additional submodules* $F_* \subset H_*$ *and whose morphisms between* $(H_*, \bullet, F_*)$ *and* $(H'_*, \bullet, F'_*)$ *are* $(L_*; i, i', \bullet)$, *with the following additional torsion-free conditions:*

1. *there exists a lifting* $\sigma : \overline{R}_k \to R_k$ *such that* $\partial \circ \sigma\left(\overline{R}_k\right) \subset H_{k-1}$ *and* $\partial' \circ \sigma\left(\overline{R}_k\right) \subset H'_{k-1}$ *are torsion-free,*
2. F_* *and* F'_* *satisfies the following condition:*
   (a) $L_k / \left(i\left(H_k\right) + i'\left(F'_k\right)\right)$ *and* $L_k / \left(i\left(H'_k\right) + i'\left(F_k\right)\right)$ *are torsion-free,*
   (b) $F_k = \partial R_{k+1}$ *and* $F'_k = \partial' R_{k+1}$.

But these conditions may cause extra complications, which are not essential for our discussion. So we will discuss this matter elsewhere.

Note that there exists a functor $H_* : \mathcal{C}_3 \to \mathcal{L}_3$ given by $M \longmapsto (H_*(M; \mathbb{Z}), \bullet)$ and $(W; M, M')$
$\longmapsto (L_*(W; \mathbb{Z}); i_*, i'_*, \bullet)$, where i_* and i'_* are the induced homomorphisms

$$H_*(M; \mathbb{Z}) \overset{i_*}{\to} L_*(W; \mathbb{Z}) \overset{i'_*}{\leftarrow} H_*(M'; \mathbb{Z}),$$

to the $\mathbb{Z}$-module $L_*(W; \mathbb{Z})$ defined by

$$L_*(W; \mathbb{Z}) = \operatorname{Im}\left(H_*(M; \mathbb{Z}) \oplus H_*(M'; \mathbb{Z}) \overset{i_* + i'_*}{\to} H_*(W; \mathbb{Z})\right).$$

Remark 18. *If we consider the category $\mathcal{L}_3^0$ instead of $\mathcal{L}_3$, we need to replace the category $\mathcal{C}_3$ with a category $\mathcal{C}_3^0$ to define a functor $H_* : \mathcal{C}_3^0 \to \mathcal{L}_3^0$. The objects of $\mathcal{C}_3^0$ are 3-manifolds M with an additional submodule- $F_* \subset H_*(M;\mathbb{Z})$, and the morphisms between (M, F_*) and (M', F'_*) are cobordisms $(W; M, M')$ with the following additional torsion-free conditions:*

1. $\overline{H}_k(W;\mathbb{Z}) / (i'_* + i''_*)\left(\overline{H}_k(M;\mathbb{Z}) \oplus \overline{H}_k(M';\mathbb{Z})\right)$ *is torsion-free,*
2. *there exists a lifting* $\sigma : \overline{R}_k(W;\mathbb{Z}) \to R_k(W;\mathbb{Z})$ *such that* $\partial_* \circ \sigma\left(\overline{R}_k(W;\mathbb{Z})\right) \subset H_{k-1}(M;\mathbb{Z})$ *and* $\partial'_* \circ \sigma\left(\overline{R}_k(W;\mathbb{Z})\right) \subset H_{k-1}(M';\mathbb{Z})$ *are torsion-free,*
3. F_* *and* F'_* *satisfies the following conditions:*
   (a) $L_k(W;\mathbb{Z}) / (i_*(H_k(M;\mathbb{Z})) + i'_*(F'_k))$ *and* $L_k(W;\mathbb{Z}) / (i_*(F_k) + i'(H_k(M';\mathbb{Z})))$ *are tor-sion-free,*
   (b) $F_k = \partial_* R_{k+1}(W;\mathbb{Z})$ *and* $F'_k = \partial'_* R_{k+1}(W;\mathbb{Z})$*.*

There exists a bifunctor $\oplus : \mathcal{L}_3 \times \mathcal{L}_3 \to \mathcal{L}_3$ defined by the direct sum

$$\begin{aligned}
&\oplus : \mathrm{ob}(\mathcal{L}_3) \times \mathrm{ob}(\mathcal{L}_3) \to \mathrm{ob}(\mathcal{L}_3), ((H_{1*}, \bullet), (H_{2*}, \bullet)) \longmapsto (H_{1*}, \bullet) \oplus (H_{2*}, \bullet)\\
&\oplus : \mathcal{L}_3((H_{1*}, \bullet), (H'_{1*}, \bullet)) \times \mathcal{L}_3((H_{2*}, \bullet), (H'_{2*}, \bullet)) \to\\
&\quad \mathcal{L}_3((H_{1*}, \bullet) \oplus (H_{2*}, \bullet), (H'_{1*}, \bullet) \oplus (H'_{2*}, \bullet)),\\
&\quad ((L_{1*}; i_1, i'_1, \bullet), (L_{2*}; i_2, i'_2, \bullet)) \longmapsto (L_{1*} \oplus L_{2*}; i_1 \oplus i_2, i'_1 \oplus i'_2, \bullet),
\end{aligned}$$

and the zero $\mathbb{Z}$-module $0 \in \mathrm{ob}(\mathcal{L}_3)$ defines the unit element. Hence $(\mathcal{L}_3, \oplus, 0)$ defines a monoidal category.

Similarly, a functor $H_*^{\mathrm{spin}} : \mathcal{C}_3^{\mathrm{spin}} \to \mathcal{L}_3$ is defined by $(M, c) \longmapsto (H_*(M;\mathbb{Z}), \bullet)$ and

$((W, \tilde{c}); (M, c), (M', c')) \longmapsto (L_*(W;\mathbb{Z}); i_*, i'_*, \bullet)$.

We call two objects $(H_{1*}, \bullet)$ and $(H_{2*}, \bullet)$ equivalent if and only if there exists a graded ring isomorphism $f = \{f_k\} : H_{1*} \to H_{2*}$, i.e. $f_k : H_{1,k} \to H_{2,k}$ are $\mathbb{Z}$-module isomorphisms such that $f_{k+\ell-3}(x \cdot y) = f_k(x) \cdot f_\ell(y)$ for any $x \in H_{1k}$, $y \in H_{2\ell}$. We also call two morphisms $\phi_1 = (L_{1*}; i_1, i'_1, \bullet)$ (resp. $\phi_2 = (L_{2*}; i_2, i'_2, \bullet)$) between H_{1*} and H'_{1*} (resp. H_{2*} and H'_{2*}) are equivalent if and only if there exist two ring isomorphisms $f : H_{1*} \to H_{2*}$, $f' : H'_{1*} \to H'_{2*}$ and a graded $\mathbb{Z}$-module isomorphism $g : L_{1*} \to L_{2*}$ such that the following diagrams commute.

$$\begin{array}{ccccc}
H_{1*} & \xrightarrow{i_1} & L_{1*} & \xleftarrow{i'_1} & H'_{1*}\\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f'}\downarrow\\
H_{2*} & \xrightarrow{i_2} & L_{2*} & \xleftarrow{i'_2} & H'_{2*}
\end{array},$$

$$\begin{array}{ccccc}
R_{1,k+1} & \otimes & L_{1,\ell} & \xrightarrow{\bullet} & L_{1,k+\ell-3}\\
{\scriptstyle (f_{k+1}\oplus f'_{k+1})\otimes g_\ell} & \downarrow & & & {\scriptstyle g_{k+\ell-3}}\downarrow\\
R_{2,k+1} & \otimes & L_{2,\ell} & \xrightarrow{\bullet} & L_{2,k+\ell-3}
\end{array}.$$

4. Homology cobordism category of 3-manifolds and a category of isomorphisms of graded commutative rings

In this section, we consider the homology cobordism category $\mathcal{C}_3^H$ of 3-manifolds and then reduce the category $\mathcal{L}_3$ to a category $\mathcal{L}_3^H$ of isomorphisms of graded commutative rings.

4.1. Homology cobordism category of 3-manifolds

Let $\left(\mathcal{C}_3^H, \sharp, S^3\right)$ be the category whose set of objects $\mathrm{ob}\left(\mathcal{C}_3^H\right)$ is the set of all closed oriented 3-manifolds M and whose set of morphisms $\mathcal{C}_3^H\left(M, M'\right)$ between $M \in \mathrm{ob}\left(\mathcal{C}_3^H\right)$ and $M' \in \mathrm{ob}\left(\mathcal{C}_3^H\right)$ is the set of all homology cobordisms $(W; M, M')$ between M and M'. If M, M', and M'' are three objects in $\mathcal{C}_3^H$, then the composite operation on morphisms $\mathcal{C}_3^H\left(M, M'\right) \times \mathcal{C}_3^H\left(M', M''\right) \to \mathcal{C}_3^H\left(M, M''\right)$, $(W, W') \longmapsto W \cup_{M'} W'$ is defined by gluing 4-manifolds W and W' along the boundary component M'. There exists a bifunctor $\sharp : \mathcal{C}_3^H \times \mathcal{C}_3^H \to \mathcal{C}_3^H$ defined by the connected sum $\sharp : \mathrm{ob}\left(\mathcal{C}_3^H\right) \times \mathrm{ob}\left(\mathcal{C}_3^H\right) \to \mathrm{ob}\left(\mathcal{C}_3^H\right)$, $(M_1, M_2) \longmapsto M_1 \sharp M_2$ and the boundary connected sum $\natural : \mathcal{C}_3^H\left(M_1, M_1'\right) \times \mathcal{C}_3^H\left(M_2, M_2'\right) \to \mathcal{C}_3^H\left(M_1 \sharp\, M_2, M_1' \sharp\, M_2'\right)$, $(W_1, W_2) \longmapsto W_1 \natural W_2$, and the 3-sphere $S^3 \in \mathrm{ob}\left(\mathcal{C}_3^H\right)$ defines the unit element. Hence $\left(\mathcal{C}_3^H, \sharp, S^3\right)$ defines a monoidal category. The monoidal category $(\mathcal{C}_3^{H,\mathrm{spin}}, \sharp, S^3)$, whose objects (M, c) are 3-manifolds M with spin structures c and morphisms between (M, c) and (M', c') are homology spin cobordisms $\left((W, \tilde{c}); (M, c), (M', c')\right)$, can be defined similarly. Note that there exists a natural functor $\mathcal{C}_3^H \to \mathcal{C}_3$, but the monoidal operations $\sharp$ and $\sqcup$ are not compatible. Let $\left(\mathcal{S}_3^H, \sharp, S^3\right)$ be the subcategory of $\left(\mathcal{C}_3^H, \sharp, S^3\right)$ generated by the set $\mathrm{ob}\left(\mathcal{S}_3^H\right)$ of all objects $\Sigma \in \mathrm{ob}\left(\mathcal{C}_3^H\right)$ such that $H_*\left(\Sigma; \mathbb{Z}\right) \cong H_*\left(S^3; \mathbb{Z}\right)$. In particular, there exists a monoidal operation $\sharp : \mathcal{S}_3^H \times \mathcal{C}_3^H \to \mathcal{C}_3^H$. Since spin structures on homology 3-spheres are unique, the corresponding subcategory $(\mathcal{S}_3^{H,\mathrm{spin}}, \sharp, S^3)$ of $(\mathcal{C}_3^{H,\mathrm{spin}}, \sharp, S^3)$ is equivalent to $(\mathcal{S}_3^H, \sharp, S^3)$.

Fix $H_* \in \mathrm{ob}\left(\mathcal{L}_3\right)$ and let $\mathcal{C}_3^H\left(H_*\right)$ be the subcategory of $\mathcal{C}_3^H$ generated by the set $\mathrm{ob}\left(\mathcal{C}_3^H\left(H_*\right)\right)$ of all objects $M \in \mathrm{ob}\left(\mathcal{C}_3^H\right)$ such that $H_*\left(M; \mathbb{Z}\right) \cong H_*$. Then $\mathcal{C}_3^H = \bigcup_{H_* \in \mathrm{ob}(\mathcal{L}_3)} \mathcal{C}_3^H\left(H_*\right)$ and the monoidal operation $\sharp : \mathcal{S}_3^H \times \mathcal{C}_3^H \to \mathcal{C}_3^H$ induces $\sharp : \mathcal{S}_3^H \times \mathcal{C}_3^H\left(H_*\right) \to \mathcal{C}_3^H\left(H_*\right)$. Similarly, let $\mathcal{C}_3^{H,\mathrm{spin}}\left(H_*\right)$ be the subcategory of $\mathcal{C}_3^{H,\mathrm{spin}}$ generated by the set $\mathrm{ob}(\mathcal{C}_3^{H,\mathrm{spin}}\left(H_*\right))$ of all objects $(M, c) \in \mathrm{ob}(\mathcal{C}_3^{H,\mathrm{spin}})$ such that $H_*\left(M\right) \cong H_*$. Then we have $\mathcal{C}_3^{H,\mathrm{spin}} = \bigcup_{H_* \in \mathrm{ob}(\mathcal{L}_3)} \mathcal{C}_3^{H,\mathrm{spin}}\left(H_*\right)$ and $\sharp : \mathcal{S}_3^H \times \mathcal{C}_3^{H,\mathrm{spin}}\left(H_*\right) \to \mathcal{C}_3^{H,\mathrm{spin}}\left(H_*\right)$.

4.2. A category of isomorphisms of graded commutative rings

Let $\left(\mathcal{L}_3^H, \sharp, S\right)$ be a category whose objects $\mathrm{ob}\left(\mathcal{L}_3^H\right)$ are the same as $\mathrm{ob}\left(\mathcal{L}_3\right)$, except that the object $\left(H_*, \bullet\right)$ satisfies $H_0 = \mathbb{Z}$ and whose morphisms $\mathcal{L}_3^H\left(\left(H_*, \bullet\right), \left(H_*', \bullet\right)\right)$ between two objects $\left(H_*, \bullet\right)$ and $\left(H_*', \bullet\right)$ are the same as $\mathcal{L}_3\left(\left(H_*, \bullet\right), \left(H_*', \bullet\right)\right)$, except that the two homomorphisms $H_* \xrightarrow{i} L_* \xleftarrow{i'} H_*'$ are isomorphisms. This implies that $L_k \cong H_k \cong H'_k \cong R_{k+1}$ and the following diagram

$$\begin{array}{ccccc} H_k & \otimes & H_\ell & \stackrel{\bullet}{\to} & H_{k+\ell-3} \\ {\scriptstyle \partial\otimes \mathrm{id}_{H_\ell}} & \uparrow\cong & & & \\ R_{k+1} & \otimes & H_\ell & & \downarrow\cong i \\ {\scriptstyle \mathrm{id}_{R_{k+1}}\otimes i} & \downarrow\cong & & & \\ R_{k+1} & \otimes & L_\ell & \stackrel{\bullet}{\to} & L_{k+\ell-3} \\ {\scriptstyle \mathrm{id}_{R_{k+1}}\otimes i'} & \uparrow\cong & & & \\ R_{k+1} & \otimes & H'_\ell & & \uparrow\cong i' \\ {\scriptstyle \partial'\otimes \mathrm{id}_{H'_\ell}} & \downarrow\cong & & & \\ H'_k & \otimes & H'_\ell & \stackrel{\bullet}{\to} & H'_{k+\ell-3} \end{array}$$

commutes, and hence $H_* \stackrel{i}{\to} L_* \stackrel{i'}{\leftarrow} H'_*$ are in fact ring isomorphisms. Therefore we may define $\mathcal{L}_3^H\left(\left(H_*,\bullet\right),\left(H'_*,\bullet\right)\right)$ to be the set of all graded ring isomorphisms $\phi : H_* = \bigoplus_{k=0}^3 H_k \to H'_* = \bigoplus_{k=0}^3 H'_k$, $\phi\left(x\cdot y\right) = \phi\left(x\right)\cdot\phi\left(y\right)$ for any $x, y \in H_*$. If $(H,\bullet)$, $(H',\bullet)$, and $(H'',\bullet)$ are three objects in $\mathcal{L}_3^H$, then the composite operation on morphisms is defined as follows:

$$\begin{array}{ccc} \mathcal{L}_3^H\left(\left(H_*,\bullet\right),\left(H'_*,\bullet\right)\right)\times\mathcal{L}_3^H\left(\left(H'_*,\bullet\right),\left(H''_*,\bullet\right)\right) & \to & \mathcal{L}_3^H\left(\left(H_*,\bullet\right),\left(H''_*,\bullet\right)\right) \\ (\phi_1,\phi_2) & \longmapsto & \phi_2\circ\phi_1 \end{array}$$

Note that there exists a functor $H_* : \mathcal{C}_3^H \to \mathcal{L}_3^H$ given by $M \longmapsto \left(H_*\left(M;\mathbb{Z}\right),\bullet\right)$ and $(W;M,M') \longmapsto \phi_W = i_*^{-1}\circ i'_*$.

There exists a bifunctor $\sharp : \mathcal{L}_3^H\times\mathcal{L}_3^H \to \mathcal{L}_3^H$ defined by the "connected sum",

$$\sharp : \mathrm{ob}\left(\mathcal{L}_3^H\right)\times\mathrm{ob}\left(\mathcal{L}_3^H\right)\to\mathrm{ob}\left(\mathcal{L}_3^H\right), \left(\left(H_{1*},\bullet\right),\left(H_{2*},\bullet\right)\right)\longmapsto\left(H_{1*}\sharp\, H_{2*},\bullet\right)$$

$$\begin{array}{ccc} \natural : \mathcal{L}_3^H\left((H_{1*},\bullet),(H'_{1*},\bullet)\right)\times\mathcal{L}_3^H\left((H_{2*},\bullet),(H'_{2*},\bullet)\right) & \to & \mathcal{L}_3^H((H_{1*}\sharp\, H_{2*},\bullet),(H'_{1*}\sharp\, H'_{2*},\bullet)) \\ (\phi_1,\phi_2) & \longmapsto & \phi_1\natural\,\phi_2 \end{array}$$

where

$$\left(H_{1*}\sharp\, H_{2*}\right)_k = \begin{cases} \mathbb{Z} & (k=0,3) \\ H_{1k}\oplus H_{2k} & (k=1,2) \end{cases},$$

with the product structure $\bullet$ defined naturally, and

$$\left(\phi_1\natural\,\phi_2\right)_k = \begin{cases} \mathrm{id}_{\mathbb{Z}} & (k=0,3) \\ \phi_{1k}\oplus\phi_{2k} & (k=1,2) \end{cases} : H_{1k}\sharp\, H_{2k}\to H'_{1k}\sharp\, H'_{2k}.$$

Note that $1\in\mathbb{Z} = \left(H_{1*}\sharp\, H_{2*}\right)_3$ satisfies $1\cdot x = x$ for any $x\in H_{1*}\sharp\, H_{2*}$. The "3-sphere" $S\in\mathrm{ob}\left(\mathcal{L}_3^H\right)$,

$$S_k = \begin{cases} \mathbb{Z} & (k=0,3) \\ 0 & (k=1,2) \end{cases}$$

defines the unit element. Hence $\left(\mathcal{L}_3^H,\sharp,S\right)$ defines a monoidal category.

4.3. Homology cobordism monoid

Let $\left(\mathcal{C}_3^H, \sharp, S^3\right)$ be the homology cobordism monoidal category. We define an equivalence relation $M \sim^H M'$, $M, M' \in \mathrm{ob}\left(\mathcal{C}_3^H\right)$ if and only if $\mathcal{C}_3^H(M, M') \neq \emptyset$. Let C_3^H be the abelian monoid defined by the quotient of $\left(\mathcal{C}_3^H, \sharp, S^3\right)$ by the equivalence relation $\sim^H$, and we call C_3^H the homology cobordism monoid. Note that the monoid corresponding to the subcategory $\left(\mathcal{S}_3^H, \sharp, S^3\right) \subset \left(\mathcal{C}_3^H, \sharp, S^3\right)$ is exactly the homology cobordism group Θ_3^H of homology 3-spheres. Then the monoidal operation $\sharp : \mathcal{S}_3^H \times \mathcal{C}_3^H \to \mathcal{C}_3^H$ induces the action $\sharp : \Theta_3^H \times C_3^H \to C_3^H$, and hence the homology cobordism monoid C_3^H is a Θ_3^H-space. Then for any $M \in C_3^H$, the inertia group $\left(\Theta_3^H\right)_M$,

$$\left(\Theta_3^H\right)_M = \{\Sigma \in \Theta_3^H \mid \Sigma \sharp M \sim^H M\}$$

is well-defined. Similarly, let $C_3^{H,\mathrm{spin}}$ be the abelian monoid obtained as the quotient of $(\mathcal{C}_3^{H,\mathrm{spin}}, \sharp, S^3)$ by the equivalence relation $\sim^{H,\mathrm{spin}}$ of homology spin cobordism. Then for any $(M, c) \in \mathcal{C}_3^{H,\mathrm{spin}}$, the inertia group $\left(\Theta_3^H\right)_{(M,c)}^{\mathrm{spin}}$,

$$\left(\Theta_3^H\right)_{(M,c)}^{\mathrm{spin}} = \{\Sigma \in \Theta_3^H \mid \Sigma \sharp (M, c) \sim^{H,\mathrm{spin}} (M, c)\}$$

can be defined.

5. w-invariants

Let (M, c) be a closed oriented 3-manifold M with spin structure c, and let $(X, \hat{c})$ be a compact oriented 4-V-manifold with V-spin structure $\hat{c}$, satisfying $\partial(X, \hat{c}) = (M, c)$. Since the 3-dimensional spin cobordism group Ω_3^{spin} is zero, we can take a compact oriented 4-manifold W with a spin structure $\tilde{c}$, satisfying $\partial(W, \tilde{c}) = (-M, -c)$. Then we glue them along the boundary and obtain a closed oriented 4-V-manifold $Z = X \cup_M W$ with spin structure $\hat{c} = \hat{c} \cup_c \tilde{c}$. We fix a Riemannian V-metric on Z, and let $\mathcal{D}(Z)$ be the Dirac operator on Z associated with the V-spin structure $\hat{c}$. Then we define an invariant for the pair $((M, c), (X, \hat{c}))$ as follows. This invariant is an extension of the definition of the w-invariant [4], [5] for homology 3-spheres to the case of closed oriented 3-manifolds with $b_1 > 0$.

Definition 19.

$$w((M, c), (X, \hat{c})) = 8\,\mathrm{ind}_V \mathcal{D}(Z) + \mathrm{Sign}\,(W) \in \mathbb{Z}.$$

Remark 20. *By the excision property of the indices of the Dirac operators, and the Novikov additivity of the signature, this invariant does not depend on the choice of (W, c_W).*

Since the V-index of the Dirac operator is always divisible by 4, we see that the following proposition holds.

Proposition 21. *Let $\mu(M,c) \in \mathbb{Z}/16$ be the Rochlin invariant; then we have*

$$w((M,c),(X,\hat{c})) \equiv -\mu(M,c) \bmod 16.$$

By the excision properties of the indices of the Dirac operators, and the vanishing of the kernel of the Dirac operator on a round sphere, this invariant is additive under connected sums.

Proposition 22.

$$\begin{aligned} &w((M_1,c_1)\sharp(M_2,c_2),(X_1,\hat{c}_1)\natural(X_2,\hat{c}_2)) \\ &= w((M_1,c_1),(X_1,\hat{c}_1)) + w((M_2,c_2),(X_2,\hat{c}_2)) \end{aligned}$$

To state some properties of the invariant, we first recall some notation [2].

Definition 23. *Let k^+, k^- and r be non-negative integers. We define the set $\mathcal{X}(k^+,k^-;r)$ of all pairs $((M,c),(X,\hat{c}))$ composed of*

1. (M,c) *: a closed 3-manifold with a spin structure, and*
2. $(X,\hat{c})$ *: a compact oriented 4-V-manifold with spin structure satisfying*
   - (a) $\partial(X,\hat{c}) = (M,c)$,
   - (b) $b_2^+(X) \leqslant k^+$, $b_2^-(X) \leqslant k^-$, *and*
   - (c) *rank* $\mathrm{Ker}\,(i_* : H_1(M;\mathbb{Q}) \to H_1(X;\mathbb{Q})) \leqslant r$.

Then we define a set of 3-manifolds as follows.

$$\mathcal{Y}(k^+,k^-;r) = \{(M,c) | ((M,c),(X,\hat{c})) \in \mathcal{X}(k^+,k^-;r) \text{ for some } (X,\hat{c})\}.$$

Remark 24. *$\mathcal{Y}(k^+,k^-;r)$ is not closed under connected sums. In fact, the connected sum defines a map*

$$\sharp : \mathcal{Y}(k_1^+,k_1^-;r_1) \times \mathcal{Y}(k_2^+,k_2^-;r_2) \to \mathcal{Y}(k_1^+ + k_2^+, k_1^- + k_2^-; \min(r_1,r_2)).$$

Then we have the following theorem [2].

Theorem 25. *Let k^+, k^- and r be non-negative integers satisfying $k^+ + k^- + r \leqslant 2$. Then the map*

$$w(k^+,k^-;r) : \mathcal{Y}(k^+,k^-;r) \ni (M,c) \longmapsto w((M,c),(X,\hat{c})) \in \mathbb{Z}$$

gives a homology spin cobordism invariant.

Theorem 26. *Suppose $(M,c) \in \mathrm{ob}(\mathcal{C}_3^{H,\mathrm{spin}})$ belongs to the class $\mathcal{Y}(k^+,k^-;r)$. If $\Sigma \in \left(\Theta_3^H\right)_M^{\mathrm{spin}}$ is in the class $\mathcal{Y}(l^+,l^-;0)$ with $k^+ + l^+ + k^- + l^- + r \leqslant 2$ then $w(l^+,l^-,0)(\Sigma) = 0$.*

Proof. Let (M,c) and Σ be as above. Then (M,c) and $\Sigma \sharp (M,c)$ belong to the class

$$\mathcal{Y}\left(k^+ + l^+, k^- + l^-; r\right).$$

Since $k^+ + l^+ + k^- + l^- + r \leqslant 2$, we apply Theorem 25 to $w(k^+ + l^+, k^- + l^-; r)$ and obtain

$$w\left(k^+ + l^+, k^- + l^-; r\right)(\Sigma \sharp (M,c)) = w\left(k^+ + l^+, k^- + l^-; r\right)(M,c).$$

By the additivity formula 22, we have

$$w\left(l^{+},l^{-};0\right)(\Sigma)+w\left(k^{+},k^{-};r\right)(M,c)=w\left(k^{+},k^{-};r\right)(M,c)$$

and therefore $w\left(l^{+},l^{-};0\right)(\Sigma)=0$. □

Example 27. *If one of a,b,c is even, then the Brieskorn homology 3-sphere $\Sigma(a,b,c)$ bounds a spin D^2-V-bundle X of Euler number $e=-1/(abc)$ associated to the S^1-fibration of $\Sigma(a,b,c)$ over a 2-sphere S^2. On the other hand, if all of a,b,c are odd, then $\Sigma(a,b,c)$ bounds a spin 4-V-manifold X with $b_2^{\pm}(X)=1$, constructed by using a "4-dimensional Seifert fibration," as in our joint work with M. Furuta and M. Ue [5]. Hence the pair $((\Sigma(a,b,c),c),(X,\hat{c}))$ belongs to $\mathcal{Y}(1,1;0)$. Therefore, if $(\Sigma(a,b,c),c)\in\left(\Theta_3^H\right)_M^{\mathrm{spin}}$, then $w(1,1;0)(\Sigma(a,b,c))=0$. Note that it is known that the w-invariant is equal to (-8)-times the Neumann-Siebenmann $\bar{\mu}$-invariant, $w((\Sigma(a,b,c),c),(X,\hat{c}))=-8\bar{\mu}(\Sigma(a,b,c))$, [12], [5]. Several sequence of the Brieskorn homology 3-spheres are known to bound contractible 4-manifolds due to the work of A. Casson and J. Harer [1].*

6. Plumbed 3-manifolds

Let $\Gamma=(V,E,\omega)$ be a Seifert graph. For simplicity, we assume that Γ is a tree graph. Let $P(\Gamma)$ be the plumbed 4-V-manifold with boundary obtained by plumbing according to Γ. For any vertices $v\in V$, we take the disk V-bundle $E_v\to\Sigma_v$ of Seifert invariant

$$\omega(v)=\{g_v;(a_{v1},b_{v1}),\ldots,(a_{vn_v},b_{vn_v})\},$$

where Σ_v is a closed V-surface of genus g_v, and if two vertices v and v' are connected by an edge $(v,v')\in E$ then we take a sufficiently small neighborhood $D^2\cong U_{v,v'}\subset S_v$ away from the singularity and glue the two disk V-bundles $E_v\to\Sigma_v$ by the map,

$$\phi_e: E_v|_{U_{v,v'}}\cong U_{v,v'}\times D^2\ni(z,w)\longmapsto(w,z)\in U_{v',v}\times D^2\cong E_{v'}|_{U_{v',v}}.$$

Remark 28. *We can also consider plumbing at singular points. In fact, this extension of the notion of plumbing is generalized to the notion of plumbed V-manifolds associated to decorated graphs by N. Saveliev [12].*

Then the surfaces Σ_v form a basis for the second homology $H_2(P(\Gamma),\mathbb{Q})$, and the intersection matrix $A(\Gamma)$ is given as follows:

$$A(\Gamma)_{vv'}=\begin{cases} e_v & v=v',\\ 1 & (v,v')\in E,\\ 0 & \text{otherwise},\end{cases}$$

where $e_v=\sum_{i=1}^{n_v}\frac{b_{vi}}{a_{vi}}$ is the Euler number of the disk V-bundle $E_v\to\Sigma_v$. The boundary $M(\Gamma)=\partial P(\Gamma)$ is a smooth 3-manifold, and $M(\Gamma)$ is a homology 3-sphere if and only if the following conditions hold:

$$\begin{cases} \Gamma: \text{a tree graph,} \\ g_v = 0 \text{ for any } v \in V, \\ a_{v1}, \dots, a_{vn_v}: \text{pairwise coprime, and} \\ \det A(\Gamma) = \pm \dfrac{1}{\prod_{v \in V} \alpha_v}, \quad \alpha_v = \prod_{i=1}^{n_v} a_{vi}. \end{cases} \cdots (HS)$$

If Γ satisfies the condition

$$\begin{cases} \exists a_{vi}: \text{even or} \\ \forall a_{vi}: \text{odd and } \sum_{i=1}^{n_v} b_{vi}: \text{even} \end{cases} \text{for any } v \in V \cdots (SP),$$

then $P(\Gamma)$ is a spin 4-V-manifold.

Let Γ, Γ' be two tree graphs with Seifert invariants and let $M(\Gamma)$, $M(\Gamma')$ be the corresponding plumbed 3-manifolds. Let $\phi = (L_*; i, i', \bullet)$ be an algebraic morphism between the homology rings $(H_*(M(\Gamma);\mathbb{Z}), \bullet)$, $(H_*(M(\Gamma');\mathbb{Z}), \bullet)$ of the corresponding 3-manifolds $M(\Gamma)$, $M(\Gamma')$. We assume that Γ, Γ' satisfy the condition (Ndeg). Suppose that there exists a cobordism $(W; M(\Gamma), M(\Gamma'))$ inducing $\phi = (L_*; i, i', \bullet)$. Then $L_* \cong \operatorname{Im}\left(H_2(M(\Gamma);\mathbb{Z}) \oplus H_2(M(\Gamma');\mathbb{Z}) \stackrel{i_*+i'_*}{\to} H_*(W;\mathbb{Z})\right)$. Let Z be a 4-V-manifold obtained by gluing the 4-V-manifolds $P(\Gamma)$, $P(\Gamma')$ along their boundaries $M(\Gamma)$, $M(\Gamma')$, respectively. Then we have the following lemma [2].

Lemma 29. *The second homology group $H_2(Z;\mathbb{Q})$ is isomorphic to*

$$H_2(P(\Gamma);\mathbb{Q}) \oplus H_2(P(\Gamma');\mathbb{Q}) \oplus \operatorname{Coker}\left(H_2(M(\Gamma);\mathbb{Z}) \oplus H_2(M(\Gamma');\mathbb{Z}) \stackrel{i_*+i'_*}{\to} H_2(W;\mathbb{Z})\right) \otimes \mathbb{Q}.$$

Let Z^0 be a smooth manifold obtained by removing the interiors of neighborhoods of the singularities. Then the boundary $\partial P(\Gamma)^0$ of $P(\Gamma)^0$ is composed of the disjoint union of the plumbed 3-manifold $M(\Gamma)$ and a disjoint union L of lens spaces. Then we have the following Lemma.

Lemma 30. *1. Let Γ be a Seifert graph and set $H(\Gamma) = \bigoplus_{v \in V} H_1(\bar{\Sigma}_v;\mathbb{Z})$.*

(a) There exists an injective homomorphism

$$H(\Gamma) \stackrel{\lambda}{\to} H_1(M(\Gamma);\mathbb{Z}),$$

(b) There exists the following natural commutative diagram,

$$\begin{array}{ccc} H(\Gamma) & \stackrel{\bar{\theta}}{\to} & H_3\left(P(\Gamma)^0, M(\Gamma) \sqcup L;\mathbb{Z}\right) \\ & {}_{\theta}\searrow & \downarrow \\ & & H_2(M(\Gamma);\mathbb{Z}) \end{array}.$$

2. Let Z^0 be the smooth 4-manifold obtained by removing the interiors of neighborhoods of singularities of Z.

(a) Set

$$L(\Gamma, \Gamma'; \phi) = \{i_*\lambda(\alpha) + i'_*\lambda'(\alpha') \in H_1(W;\mathbb{Z}) \,|\, \alpha \oplus \alpha' \in H(\Gamma) \oplus H(\Gamma')\}.$$

Then there exists an injective homomorphism

$$\begin{array}{ccc} L(\Gamma,\Gamma';\phi) & \to & H_1(Z^0,L;\mathbb{Z}) \\ \beta & \longmapsto & k_*(\beta) \end{array}$$

for the inclusion $k : W \hookrightarrow Z^0$.

(b) *Set*

$$R(\Gamma,\Gamma';\phi) = \{\alpha\oplus\alpha' \in H(\Gamma)\oplus H(\Gamma') \,|\, i_*\theta_\alpha + i'_*\theta'_{\alpha'} = 0 \in H_2(W;\mathbb{Z})\}.$$

Then there exists an injective homomorphism

$$\begin{array}{cccc} \widetilde{\theta\theta'}: & R(\Gamma,\Gamma';\phi) & \to & H_3(Z^0,L;\mathbb{Z}) \\ & (-\alpha)\oplus\alpha' & \longmapsto & \widetilde{\theta_\alpha\theta'_{\alpha'}} \end{array}$$

such that for any pair $(-\alpha)\oplus\alpha' \in R(\Gamma,\Gamma';\phi)$, *the corresponding* 3*-cycle* $\widetilde{\theta_\alpha\theta'_{\alpha'}} \in H_3(Z^0,\partial Z^0;\mathbb{Z})$ *defines* $-\theta_\alpha+\theta'_{\alpha'}$ *on* $H_2(M(\Gamma)\sqcup M(\Gamma'))$.

Proof. 1. Let V be the set of vertices in Γ. For a V-manifold X, we denote X^0 be the manifold with boundary obtained by removing the interior of a sufficiently small neighborhood of the singularity of X.

(a) There exists an isomorphism

$$\iota : H(\Gamma) = \bigoplus_{v\in V} H_1(\bar\Sigma_v;\mathbb{Z}) \to H_1(\overline{P(\Gamma)};\mathbb{Z}) \cong \bigoplus_{v\in V} H_1\left(\bar P_v;\mathbb{Z}\right),$$

and by the exact sequence of relative homology of the pair $(\overline{P(\Gamma)}, M(\Gamma))$, the induced homomorphism $H_1(M(\Gamma);\mathbb{Z}) \xrightarrow{j} H_1(\overline{P(\Gamma)};\mathbb{Z})$ is onto and we can take a splitting homomorphism $H_1(M(\Gamma);\mathbb{Z}) \xleftarrow{\sigma} H_1(\overline{P(\Gamma)};\mathbb{Z})$, which establishes an injective homomorphism $\lambda : H(\Gamma) \xrightarrow{\iota} H_1(\overline{P(\Gamma)};\mathbb{Z}) \xrightarrow{\sigma} H_1(M(\Gamma);\mathbb{Z})$.

(b) Now for each 1-cycle $\alpha \in \bigoplus_{v\in V} H_1(\bar\Sigma_v;\mathbb{Z})$, we can associate a relative 3-cycle $\bar\theta_\alpha \in H_3(P(\Gamma)^0, M(\Gamma)\sqcup L;\mathbb{Z})$ as follows, where L is a disjoint union of lens spaces such that $\partial P(\Gamma)^0 \cong M(\Gamma)\sqcup L$. Let $M_v \to \Sigma_v$ be the Seifert fibration of Seifert invariant $\omega(v)$, and let $P_v \to \Sigma_v$ be the associated disk V-bundle. Since $\bar P_v$ is deformation retract to $\bar\Sigma_v$, $H^1\left(\bar P_v;\mathbb{Z}\right) \cong H^1\left(\bar\Sigma_v;\mathbb{Z}\right)$. By the Meyer-Vietoris sequence for $\bar P_v = P_v^0 \cup \text{cone } L_v$ and Poincaré duality, we have $H^1\left(\bar P_v;\mathbb{Z}\right) \cong H^1\left(P_v^0;\mathbb{Z}\right) \cong H_3\left(P_v^0, M_v\sqcup L_v;\mathbb{Z}\right)$, where L_v is a disjoint union of lens spaces such that $\partial P_v^0 \cong M_v \sqcup L_v$ and cone L_v is the disjoint union of cones over the lens spaces L_v. On the other hand, we have $H^1\left(\bar\Sigma_v;\mathbb{Z}\right) \cong \text{Ker}\left(H^1\left(\Sigma_v^0;\mathbb{Z}\right) \to H^1\left(\partial\Sigma_v^0;\mathbb{Z}\right)\right)$, and by Poincaré duality and the exact sequence of relative homology, this is isomorphic to $\text{Im}\left(H_1\left(\Sigma_v^0;\mathbb{Z}\right) \to H_1\left(\Sigma_v^0,\partial\Sigma_v^0;\mathbb{Z}\right)\right) \cong H_1\left(\bar\Sigma_v;\mathbb{Z}\right)$. Therefore, we have $H_1\left(\bar\Sigma_v;\mathbb{Z}\right) \cong H_3\left(P_v^0, M_v\sqcup L_v;\mathbb{Z}\right)$. Let Γ' be the graph obtained by removing a terminal vertex in Γ and the edge adjacent to it. Then the plumbed V-manifold $P(\Gamma)$ is obtained by gluing $P(\Gamma')$ and P_v along a local trivialization $D^2\times D^2 \subset P(\Gamma')$. Then $\partial P(\Gamma')^0$ is a disjoint union of the plumbed 3-manifold $M(\Gamma')$ and a disjoint union

L' of lens spaces. Set $M(\Gamma')^0 = M(\Gamma') - D^2 \times \partial D^2$. Then by the exact sequence for triples $(P(\Gamma')^0, M(\Gamma') \sqcup L', M(\Gamma')^0 \sqcup L')$, we see that $H_3(P(\Gamma')^0, M(\Gamma') \sqcup L'; \mathbb{Z}) \cong H_3(P(\Gamma')^0, M(\Gamma')^0 \sqcup L'; \mathbb{Z})$, and similarly $H_3(P_v^0, M_v \sqcup L_v; \mathbb{Z}) \cong H_3(P_v^0, M_v^0 \sqcup L_v; \mathbb{Z})$. Now $P(\Gamma)^0 = P(\Gamma')^0 \cup P_v^0$, $M(\Gamma) \sqcup L = M(\Gamma')^0 \sqcup L' \cup M_v^0 \sqcup L_v$, $P(\Gamma')^0 \cap P_v^0 = D \cong D^2 \times D^2$, and $M(\Gamma')^0 \sqcup L' \cap M_v^0 \sqcup L_v = T \cong S^1 \times S^1$. Note that $H_3(D, T; \mathbb{Z}) \to H_3(P(\Gamma')^0, M(\Gamma')^0 \sqcup L'; \mathbb{Z}) \oplus H_3(P_v^0, M_v^0 \sqcup L_v; \mathbb{Z})$ is the zero map and $H_2(D, T; \mathbb{Z}) \to H_2(P(\Gamma')^0, M(\Gamma')^0 \sqcup L'; \mathbb{Z}) \oplus H_2(P_v^0, M_v^0 \sqcup L_v; \mathbb{Z})$ is injective. Then by the Meyer-Vietoris sequence we have

$$H_3(P(\Gamma)^0, M(\Gamma) \sqcup L; \mathbb{Z}) \cong H_3(P(\Gamma')^0, M(\Gamma') \sqcup L'; \mathbb{Z}) \oplus H_3(P_v^0, M_v^0 \sqcup L_v; \mathbb{Z}).$$

Therefore, by induction on the number of vertices, we obtain

$$H_3(P(\Gamma)^0, M(\Gamma) \sqcup L; \mathbb{Z}) \cong \bigoplus_{v \in V} H_3(P_v^0, M_v \sqcup L_v; \mathbb{Z}).$$

Hence we have the natural isomorphism,

$$\bar{\theta} : \bigoplus_{v \in V} H_1\left(\bar{\Sigma}_v; \mathbb{Z}\right) \cong \bigoplus_{v \in V} H_3\left(P_v^0, M_v \sqcup L_v; \mathbb{Z}\right) \cong H_3(P(\Gamma)^0, M(\Gamma) \sqcup L; \mathbb{Z}).$$

Combining with the boundary connecting homomorphisms $\partial_* : H_3\left(P_v^0, M_v \sqcup L_v; \mathbb{Z}\right) \to H_2(M_v; \mathbb{Z})$ and $\partial_* : H_3(P(\Gamma)^0, M(\Gamma) \sqcup L; \mathbb{Z}) \to H_2(M(\Gamma); \mathbb{Z})$, which are isomorphisms, we have a natural isomorphism

$$\theta : \bigoplus_{v \in V} H_1\left(\bar{\Sigma}_v; \mathbb{Z}\right) \to \bigoplus_{v \in V} H_2(M_v; \mathbb{Z}) \cong H_2(M(\Gamma); \mathbb{Z}).$$

For $\alpha = \sum_{v \in V} \alpha_v \in \bigoplus_{v \in V} H_1\left(\bar{\Sigma}_v; \mathbb{Z}\right)$, we denote the decomposition of relative 3-cycles $\bar{\theta}_\alpha \in H_3(P(\Gamma)^0, M(\Gamma) \sqcup L; \mathbb{Z}) \cong \bigoplus_{v \in V} H_3\left(P_v^0, M_v \sqcup L_v; \mathbb{Z}\right)$, and denote it by $\bar{\theta}_\alpha = \sum_{v \in V} \bar{\theta}_{\alpha_v}$ with $\bar{\theta}_{\alpha_v} \in H_3\left(P_v^0, M_v \sqcup L_v; \mathbb{Z}\right)$ and also denote that of the corresponding 2-cycles by $\theta_\alpha = \sum_{v \in V} \theta_{\alpha_v} \in H_2(M(\Gamma); \mathbb{Z})$ with $\theta_{\alpha_v} \in H_2(M_v; \mathbb{Z})$.

2. We have the following homomorphisms on homology ϕ.

$$\phi : \quad H_*(M(\Gamma); \mathbb{Z}) \quad \overset{i_*}{\to} \quad H_1(W; \mathbb{Z}) \quad \overset{i'_*}{\leftarrow} \quad H_*(M(\Gamma'); \mathbb{Z})$$

   (a) By the Meyer-Vietoris sequence

$$H_1(M(\Gamma) \sqcup M(\Gamma'); \mathbb{Z}) \overset{j_* \sqcup j'_* \oplus (i_* + i'_*)}{\to} H_1(\overline{P(\Gamma)} \sqcup \overline{P(\Gamma')}; \mathbb{Z}) \oplus H_1(W; \mathbb{Z}) \to H_1\left(\bar{Z}; \mathbb{Z}\right) \to 0,$$

   and the surjectivity of $H_1(M(\Gamma); \mathbb{Z}) \overset{j_*}{\to} H_1(\overline{P(\Gamma)}; \mathbb{Z})$, $H_1\left(\bar{Z}; \mathbb{Z}\right)$ is isomorphic to

$$\left(H_1(\overline{P(\Gamma)} \sqcup \overline{P(\Gamma')}; \mathbb{Z}) \oplus H_1(W; \mathbb{Z})\right) \Big/ \operatorname{Im}(j_* \sqcup j'_* \oplus (i_* + i'_*))$$
$$\cong \operatorname{Coker}\left(\operatorname{Ker} j_* \oplus \operatorname{Ker} j'_* \overset{i_* + i'_*}{\to} H_1(W; \mathbb{Z})\right).$$

Since there exists an injective homomorphism $H(\Gamma) \stackrel{\lambda}{\to} H_1(M(\Gamma);\mathbb{Z})$ which factors injective homomorphism $H(\Gamma) \stackrel{\iota}{\to} H_1(\overline{P(\Gamma)};\mathbb{Z})$, there exists an injective homomorphism from

$$L(\Gamma,\Gamma';\phi) =$$

$$\mathrm{Im}\left(H(\Gamma)\oplus H(\Gamma') \stackrel{\lambda\oplus\lambda'}{\to} H_1(M(\Gamma);\mathbb{Z})\oplus H_1(M(\Gamma);\mathbb{Z}) \stackrel{i_*+i'_*}{\to} H_1(W;\mathbb{Z})\right)$$

to Coker $\left(\mathrm{Ker}\ j_* \oplus \mathrm{Ker}\ j'_* \stackrel{i_*+i'_*}{\to} H_1(W;\mathbb{Z})\right) \cong H_1\left(\bar{Z};\mathbb{Z}\right)$. Note that by the Meyer-Vietoris sequence,

$$H_1(L;\mathbb{Z}) \stackrel{i_*\oplus(-j_*)}{\to} H_1\left(Z^0;\mathbb{Z}\right)\oplus H_1(V;\mathbb{Z}) \to H_1\left(\bar{Z};\mathbb{Z}\right)\to 0$$

and since $H_1(V)=0$, $H_1\left(\bar{Z}\right)\cong H_1\left(Z^0\right)/i_*H_1(L)$. On the other hand, by the exact sequence of the pair $\left(Z^0,L\right)$

$$H_1(L;\mathbb{Z}) \stackrel{i_*}{\to} H_1\left(Z^0;\mathbb{Z}\right) \stackrel{j_*}{\to} H_1\left(Z^0,L;\mathbb{Z}\right)\to$$
$$H_0(L;\mathbb{Z}) \stackrel{i_*}{\to} H_0\left(Z^0;\mathbb{Z}\right) \stackrel{j_*}{\to} H_0\left(Z^0,L;\mathbb{Z}\right)\to 0$$

and hence there exists a natural injective homomorphism $H_1\left(\bar{Z};\mathbb{Z}\right)\to H_1\left(Z^0,L;\mathbb{Z}\right)$.

(b) Let us define $R(\Gamma,\Gamma';\phi)=\{\alpha\oplus\alpha'\in H(\Gamma)\oplus H(\Gamma')\,|i_*\theta_\alpha+i'_*\theta'_{\alpha'}=0\in H_2(W;\mathbb{Z})\}$. By using this ϕ, we can construct a 3-cycle $\widetilde{\theta_\alpha\theta'_{\alpha'}}$ as follows. Here we denote $X=P(\Gamma)$, $M=M(\Gamma)$, and $L=\partial Z^0$.

$$\begin{array}{ccccc}
H_3\left(X^0\sqcup X'^0,L;\mathbb{Z}\right) & \to & H_3(Z^0,L;\mathbb{Z}) & \to & H_3(Z^0,X^0\sqcup X'^0;\mathbb{Z})\\
 & & \widetilde{\theta_\alpha\theta'_{\alpha'}} & & \longmapsto\\
 & \stackrel{\cong}{\to} & H_3(W,M\sqcup M';\mathbb{Z}) & \to & H_2(M\sqcup M';\mathbb{Z})\\
 & & \overline{\theta_\alpha\theta'_{\alpha'}} & \longmapsto & -\theta_\alpha+\theta'_{\alpha'}
\end{array}$$

Note that $i_*\theta_\alpha+i'_*\theta'_{\alpha'}=0\in H_2(W;\mathbb{Z})$. Then by the exact sequence of the pair $(W,M\sqcup M')$ there exists a relative homology class $\overline{\theta_\alpha\theta'_{\alpha'}}\in H_3(W,M\sqcup M';\mathbb{Z})$ that maps to $-\theta_\alpha+\theta'_{\alpha'}$. Note that $\overline{\theta_\alpha\theta'_{\alpha'}}$ is only determined up to the image of $H_3(W;\mathbb{Z})\to H_3(W,M\sqcup M';\mathbb{Z})$. By the excision property $H_3(Z^0,X^0\sqcup X^0;\mathbb{Z})\cong H_3(W,M\sqcup M';\mathbb{Z})$, the surjectivity of the map $H_3(Z^0,L;\mathbb{Z})\to H_3(Z^0,X^0\sqcup X'^0;\mathbb{Z})$ (since $H_2\left(X^0\sqcup X'^0_{\,},L;\mathbb{Z}\right)$ $\to H_2(Z^0,L;\mathbb{Z})$ is injective), and $H_3(X^0\sqcup X^0,L;\mathbb{Z})\cong 0$, there exists a unique three-cycle $\widetilde{\theta_\alpha\theta'_{\alpha'}}$ in $H_3(Z^0,L,\mathbb{Z})$ which maps to $\overline{\theta_\alpha\theta'_{\alpha'}}$. This establishes a map $\widetilde{\theta\theta'}:R(\Gamma,\Gamma';\phi)\to H_3(Z^0,L;\mathbb{Z})$.

□

Remark 31. *Note that we can define the intersection pairing*

$$H_3(Z^0, L;\mathbb{Z}) \otimes H_3(Z^0, L;\mathbb{Z}) \to H_2(Z^0;\mathbb{Z})$$

by using

$$H_3(Z^0;\mathbb{Z}) \otimes H_3(Z^0, L;\mathbb{Z}) \to H_2(Z^0;\mathbb{Z})$$

since $H_3\left(Z^0;\mathbb{Z}\right) \to H_3\left(Z^0, L;\mathbb{Z}\right) \to H_2\left(L;\mathbb{Z}\right) = 0$, *and we can take a lift to* $H_3\left(Z^0;\mathbb{Z}\right)$ *up to the image of* $H_3\left(L;\mathbb{Z}\right)$. *Note that the pairings of elements in* $H_3(Z^0;\mathbb{Z})$ *and that of* $H_3\left(L;\mathbb{Z}\right)$ *are trivial.*

As a generalization of Lemma in [2], we have the following

Lemma 32. *Let* $\Gamma = (V, E, \omega)$, $\Gamma' = (V', E', \omega')$ *be two tree Seifert graphs satisfying the condition (Ndeg) and* $\phi = (L_*; i, i', \bullet)$ *be a morphism between* $(H_*\left(M\left(\Gamma\right);\mathbb{Z}\right), \bullet)$ *and* $(H_*\left(M\left(\Gamma'\right);\mathbb{Z}\right), \bullet)$. *Suppose that there exists a cobordism* $(W; M\left(\Gamma\right), M\left(\Gamma'\right))$ *between* $M\left(\Gamma\right)$ *and* $M\left(\Gamma'\right)$ *such that* $\left(L_*\left(W;\mathbb{Z}\right); i_*, i'_*, \bullet\right) \cong \phi$ *and* $L_*\left(W;\mathbb{Z}\right) = H_*\left(W;\mathbb{Z}\right)$. *Let* Z *be the closed 4-*V*-manifold obtained by gluing* $P(\Gamma)$, $P(\Gamma')$ *and* W *along the boundaries* $M(\Gamma)$, $M(\Gamma')$, *and let* Z_0 *be the 4-manifold obtained by removing the interiors of sufficiently small regular neighborhoods of the singularity in* Z. *For a pair of 1-cycles* $(\alpha, \alpha') \in L\left(\Gamma, \Gamma', \phi\right)$, *we can define a 3-cycle* $\widetilde{\theta_\alpha \theta'_{\alpha'}} = \sum_{v\in V, v'\in V'} \widetilde{\theta_{\alpha_v} \theta'_{\alpha'_{v'}}} \in H_3(Z_0;\mathbb{Z})$, *and the intersection products among these 3-cycles can be calculated by using the intersection pairings among the closed* V*-surfaces* Σ_v *in* $P\left(\Gamma\right)$ *and curves on* Σ_v*'s as follows.*

$$\begin{aligned}
&\widetilde{\theta_\alpha \theta'_{\alpha'}} \cdot \widetilde{\theta_\beta \theta'_{\beta'}} \\
&= -\sum_{v,v'\in V} A(\Gamma)^{vv'} \left(\alpha_v \cdot \beta_v\right) \Sigma_{v'} + \sum_{v,v'\in V'} A(\Gamma')^{vv'} \left(\alpha'_v \cdot \beta'_v\right) \Sigma'_{v'} \in H_2(Z;\mathbb{Q}),
\end{aligned}$$

$$\begin{aligned}
&\left(\widetilde{\theta_\alpha \theta'_{\alpha'}} \cdot \widetilde{\theta_\beta \theta'_{\beta'}}\right) \cdot \widetilde{\theta_\gamma \theta'_{\gamma'}} \\
&= \sum_{v,v'\in V} A(\Gamma)^{vv'} \left(\alpha_v \cdot \beta_v\right) i\left(\gamma_{v'}\right) - \sum_{v,v'\in V'} A(\Gamma')^{vv'} \left(\alpha'_v \cdot \beta'_v\right) i'\left(\gamma'_{v'}\right) \in H_1(Z;\mathbb{Q}),
\end{aligned}$$

$$\begin{aligned}
&\widetilde{\theta_\alpha \theta'_{\alpha'}} \cdot \left(\widetilde{\theta_\beta \theta'_{\beta'}} \cdot \widetilde{\theta_\gamma \theta'_{\gamma'}}\right) \\
&= \sum_{v,v'\in V} A(\Gamma)^{vv'} \left(\beta_v \cdot \gamma_v\right) i\left(\alpha_{v'}\right) - \sum_{v,v'\in V'} A(\Gamma')^{vv'} \left(\beta'_v \cdot \gamma'_v\right) i'\left(\alpha'_{v'}\right) \in H_1(Z;\mathbb{Q}),
\end{aligned}$$

$$\begin{aligned}
&\left(\left(\widetilde{\theta_\alpha \theta'_{\alpha'}} \cdot \widetilde{\theta_\beta \theta'_{\beta'}}\right) \cdot \widetilde{\theta_\gamma \theta'_{\gamma'}}\right) \cdot \widetilde{\theta_\delta \theta'_{\delta'}} \\
&= -\sum_{v,v'\in V} A(\Gamma)^{vv'} \left(\alpha_v \cdot \beta_v\right) \left(\gamma_{v'} \cdot \delta_{v'}\right) + \sum_{v,v'\in V'} A(\Gamma')^{vv'} \left(\alpha'_v \cdot \beta'_v\right) \left({\gamma'}_{v'} \cdot \delta'_{v'}\right) \\
&\in H_0(Z^0;\mathbb{Z}),
\end{aligned}$$

where $A(\Gamma_\ell)^{vv'}$ *are the inverses of the intersection matrices* $A(\Gamma)_{vv'} = \Sigma_v \cdot \Sigma_{v'}$, $v, v' \in V$ *in* $H_2(P\left(\Gamma\right);\mathbb{Q})$.

Proof. First we prove that the intersection pairing $\widetilde{\theta_\alpha\theta'_{\alpha'}} \cdot \widetilde{\theta_\beta\theta'_{\beta'}}$ is given by

$$\widetilde{\theta_\alpha\theta'_{\alpha'}} \cdot \widetilde{\theta_\beta\theta'_{\beta'}} = -\sum_{v,v'\in V} A(\Gamma)^{vv'} \left(\alpha_{v'} \cdot \beta_{v'}\right) \Sigma_v + \sum_{v,v'\in V'} A(\Gamma')^{vv'} \left(\alpha'_{v'} \cdot \beta'_{v'}\right) \Sigma'_v,$$

where $\alpha\alpha', \beta\beta' \in L\left(\Gamma, \Gamma'; \phi\right)$.

By Lemma 29 we can write

$$\widetilde{\theta_\alpha\theta'_{\alpha'}} \cdot \widetilde{\theta_\beta\theta'_{\beta'}} = \sum_{v\in V} c^v_{\alpha\beta}\Sigma_v + \sum_{v\in V'} c'^v_{\alpha\beta}\Sigma'_v$$

for some $c^v_{\alpha\beta}, c'^v_{\alpha\beta} \in \mathbb{Q}$. Now we multiply $\Sigma_{v'}$ from the left, we have

$$\Sigma_{v'} \cdot \left(\widetilde{\theta_\alpha\theta'_{\alpha'}} \cdot \widetilde{\theta_\beta\theta'_{\beta'}}\right) = \sum_{v\in V} c^v_{\alpha\beta}\Sigma_{v'} \cdot \Sigma_v = \sum_{v\in V} c^v_{\alpha\beta} \left(-A(\Gamma)_{v'v}\right)$$

On the other hand, by the associativity of the intersection pairing, we have

$$\begin{aligned}
\Sigma_{v'} \cdot \left(\widetilde{\theta_\alpha\theta'_{\alpha'}} \cdot \widetilde{\theta_\beta\theta'_{\beta'}}\right) &= \left(\Sigma_{v'} \cdot \widetilde{\theta_\alpha\theta'_{\alpha'}}\right) \cdot \widetilde{\theta_\beta\theta'_{\beta'}} \\
&= \left(\Sigma_{v'} \cdot \bar{\theta}_\alpha\right) \cdot \widetilde{\theta_\beta\theta'_{\beta'}} \\
&= \left(-i\left(\alpha_{v'}\right)\right) \cdot \widetilde{\theta_\beta\theta'_{\beta'}} \\
&= \left(-\alpha_{v'}\right) \cdot \bar{\theta}_\beta = \alpha_{v'} \cdot \beta_{v'}.
\end{aligned}$$

Therefore we have

$$c^{v''}_{\alpha\beta} = -\sum_{v'\in V} \left(\alpha_{v'} \cdot \beta_{v'}\right) A\left(\Gamma\right)^{v'v''}.$$

Similarly, if we multiply $\Sigma'_{v'}$ from the left, we have

$$\Sigma_{v'} \cdot \left(\widetilde{\theta_\alpha\theta'_{\alpha'}} \cdot \widetilde{\theta_\beta\theta'_{\beta'}}\right) = \sum_{v\in V'} c'^v_{\alpha\beta}\Sigma'_{v'} \cdot \Sigma'_v = \sum_{v\in V} c'^v_{\alpha\beta} A\left(\Gamma'\right)_{v'v}$$

On the other hand, by the associativity of intersection pairings, we have

$$\begin{aligned}
\Sigma'_{v'} \cdot \left(\widetilde{\theta_\alpha\theta'_{\alpha'}} \cdot \widetilde{\theta_\beta\theta'_{\beta'}}\right) &= \left(\Sigma'_{v'} \cdot \widetilde{\theta_\alpha\theta'_{\alpha'}}\right) \cdot \widetilde{\theta_\beta\theta'_{\beta'}} = \left(\Sigma'_{v'} \cdot \left(-\bar{\theta}'_{\alpha'}\right)\right) \cdot \widetilde{\theta_\beta\theta'_{\beta'}} \\
&= -i'\left(\alpha'_{v'}\right) \cdot \widetilde{\theta_\beta\theta'_{\beta'}} = -\alpha'{}_{v'} \cdot \left(-\bar{\theta}_{\beta'}\right) = \alpha'_{v'} \cdot \beta'_{v'}.
\end{aligned}$$

Therefore we have

$$c'^{v''}_{\alpha\beta} = \sum_{v'\in V} \left(\alpha'_{v'} \cdot \beta'_{v'}\right) A\left(\Gamma'\right)^{v'v''}.$$

Hence the assertion on the double products follows.

Next we calculate the triple products. Note that the intersections of Σ_v, Σ'_v and $\widetilde{\theta_\gamma \theta'_{\gamma'}}$ are calculated to be

$$\Sigma_v \cdot \widetilde{\theta_\gamma \theta'_{\gamma'}} = \Sigma_v \cdot \bar{\theta}_\gamma = -i\left(\gamma_v\right), \quad \Sigma'_v \cdot \widetilde{\theta_\gamma \theta'_{\gamma'}} = \Sigma'_v \cdot \left(-\bar{\theta}'_{\gamma'}\right) = -i'\left(\gamma'_v\right).$$

Then we can calculate $\left(\widetilde{\theta_\alpha \theta'_{\alpha'}} \cdot \widetilde{\theta_\beta \theta'_{\beta'}}\right) \cdot \widetilde{\theta_\gamma \theta'_{\gamma'}}$ and $\widetilde{\theta_\alpha \theta'_{\alpha'}} \cdot \left(\widetilde{\theta_\beta \theta'_{\beta'}} \cdot \widetilde{\theta_\gamma \theta'_{\gamma'}}\right)$ as follows.

$$\begin{aligned}
&\left(\widetilde{\theta_\alpha \theta'_{\alpha'}} \cdot \widetilde{\theta_\beta \theta'_{\beta'}}\right) \cdot \widetilde{\theta_\gamma \theta'_{\gamma'}} \\
&= \sum_{v \in V} c^v_{\alpha\beta} \Sigma_v \cdot \widetilde{\theta_\gamma \theta'_{\gamma'}} + \sum_{v \in V'} c'^v_{\alpha\beta} \Sigma'_v \cdot \widetilde{\theta_\gamma \theta'_{\gamma'}} \\
&= \sum_{v \in V} c^v_{\alpha\beta} \left(-i\left(\gamma_v\right)\right) + \sum_{v \in V'} c'^v_{\alpha\beta} \left(-i'\left(\gamma'_v\right)\right) \\
&= -\sum_{v \in V} \left(-\sum_{v' \in V} \left(\alpha_{v'} \cdot \beta_{v'}\right) A\left(\Gamma\right)^{v'v}\right) i\left(\gamma_v\right) \\
&\quad + \sum_{v \in V} \left(\sum_{v' \in V} \left({\alpha'}_{v'} \cdot \beta'_{v'}\right) A\left(\Gamma'\right)^{v'v}\right) \left(-i'\left(\gamma'_v\right)\right) \\
&= \sum_{v, v' \in V} A\left(\Gamma\right)^{v'v} \left(\alpha_{v'} \cdot \beta_{v'}\right) i\left(\gamma_v\right) - \sum_{v, v' \in V'} A\left(\Gamma'\right)^{v'v} \left(\alpha'_{v'} \cdot \beta'_{v'}\right) i'\left(\gamma'_v\right).
\end{aligned}$$

$$\begin{aligned}
&\widetilde{\theta_\alpha \theta'_{\alpha'}} \cdot \left(\widetilde{\theta_\beta \theta'_{\beta'}} \cdot \widetilde{\theta_\gamma \theta'_{\gamma'}}\right) \\
&= \sum_{v \in V} c^v_{\beta\gamma} \widetilde{\theta_\alpha \theta'_{\alpha'}} \cdot \Sigma_v + \sum_{v \in V'} c'^v_{\beta\gamma} \widetilde{\theta_\alpha \theta'_{\alpha'}} \cdot \Sigma'_v \\
&= \sum_{v \in V} c^v_{\beta\gamma} \left(-i\left(\alpha_v\right)\right) + \sum_{v \in V'} c'^v_{\beta\gamma} \left(-i'\left(\alpha'_v\right)\right) \\
&= -\sum_{v \in V} \left(-\sum_{v' \in V} \left(\beta_{v'} \cdot \gamma_{v'}\right) A\left(\Gamma\right)^{v'v}\right) i\left(\alpha_v\right) \\
&\quad + \sum_{v \in V'} \left(\sum_{v' \in V'} \left(\beta'_{v'} \cdot \gamma'_{v'}\right) A\left(\Gamma'\right)^{v'v}\right) \left(-i'\left(\alpha'_v\right)\right) \\
&= \sum_{v, v' \in V} A\left(\Gamma\right)^{v'v} \left(\beta_{v'} \cdot \gamma_{v'}\right) i\left(\alpha_v\right) - \sum_{v, v' \in V'} A\left(\Gamma'\right)^{v'v} \left(\beta'_{v'} \cdot \gamma'_{v'}\right) i'\left(\alpha'_v\right).
\end{aligned}$$

The quadruple products can be calculated as follows.

$$\begin{aligned}
&\widetilde{\theta_\alpha\theta'_{\alpha'}}\cdot\widetilde{\theta_\beta\theta'_{\beta'}}\cdot\widetilde{\theta_\gamma\theta'_{\gamma'}}\cdot\widetilde{\theta_\delta\theta'_{\delta'}}\\
&=\left(\left(\widetilde{\theta_\alpha\theta'_{\alpha'}}\cdot\widetilde{\theta_\beta\theta'_{\beta'}}\right)\cdot\widetilde{\theta_\gamma\theta'_{\gamma'}}\right)\cdot\widetilde{\theta_\delta\theta'_{\delta'}}\\
&=\sum_{v\in V}\sum_{v'\in V}A\left(\Gamma\right)^{v'v}\left(\alpha_{v'}\cdot\beta_{v'}\right)i\left(\gamma_v\right)\cdot\widetilde{\theta_\delta\theta'_{\delta'}}\\
&\quad-\sum_{v\in V'}\sum_{v'\in V'}A\left(\Gamma'\right)^{v'v}\left(\alpha'_{v'}\cdot{\beta'}_{v'}\right)i'\left(\gamma'_v\right)\cdot\widetilde{\theta_\delta\theta'_{\delta'}}\\
&=\sum_{v\in V}\sum_{v'\in V}A\left(\Gamma\right)^{v'v}\left(\alpha_{v'}\cdot\beta_{v'}\right)\left(-\gamma_v\cdot\delta_v\right)\\
&-\sum_{v\in V'}\sum_{v'\in V'}A\left(\Gamma'\right)^{v'v}\left(\alpha'_{v'}\cdot\beta'_{v'}\right)\left(\gamma'_v\cdot\left(-\delta'_v\right)\right)\\
&=-\sum_{v,v'\in V}A\left(\Gamma\right)^{v'v}\left(\alpha_{v'}\cdot\beta_{v'}\right)\left(\gamma_v\cdot\delta_v\right)\ +\sum_{v,v'\in V'}A\left(\Gamma'\right)^{v'v}\left(\alpha'_{v'}\cdot\beta'_{v'}\right)\left(\gamma'_v\cdot\delta'_v\right).
\end{aligned}$$

□

Remark 33. *The quadruple product can be calculated in different ways. If we denote the above formula for the quadruple product* $\left(\left(\widetilde{\theta_\alpha\theta'_{\alpha'}}^\phi\cdot\widetilde{\theta_\beta\theta'_{\beta'}}^\phi\right)\cdot\widetilde{\theta_\gamma\theta'_{\gamma'}}^\phi\right)\cdot\widetilde{\theta_\delta\theta'_{\delta'}}^\phi$ *by* $\tilde\theta^\phi_{\alpha\alpha'\beta\beta'\gamma\gamma'\delta\delta'}$*, then we obtain the following formula.*

$$\begin{aligned}
\left(\widetilde{\theta_\alpha\theta'_{\alpha'}}^\phi\cdot\left(\widetilde{\theta_\beta\theta'_{\beta'}}^\phi\cdot\widetilde{\theta_\gamma\theta'_{\gamma'}}^\phi\right)\right)\cdot\widetilde{\theta_\delta\theta'_{\delta'}}^\phi&=\tilde\theta^\phi_{\beta\beta'\gamma\gamma'\alpha\alpha'\delta\delta'},\\
\widetilde{\theta_\alpha\theta'_{\alpha'}}^\phi\cdot\left(\left(\widetilde{\theta_\beta\theta'_{\beta'}}^\phi\cdot\widetilde{\theta_\gamma\theta'_{\gamma'}}^\phi\right)\cdot\widetilde{\theta_\delta\theta'_{\delta'}}^\phi\right)&=\tilde\theta^\phi_{\beta\beta'\gamma\gamma'\alpha\alpha'\delta\delta'},\\
\widetilde{\theta_\alpha\theta'_{\alpha'}}^\phi\cdot\left(\widetilde{\theta_\beta\theta'_{\beta'}}^\phi\cdot\left(\widetilde{\theta_\gamma\theta'_{\gamma'}}^\phi\cdot\widetilde{\theta_\delta\theta'_{\delta'}}^\phi\right)\right)&=\tilde\theta^\phi_{\gamma\gamma'\delta\delta'\alpha\alpha'\beta\beta'},\\
\left(\widetilde{\theta_\alpha\theta'_{\alpha'}}^\phi\cdot\widetilde{\theta_\beta\theta'_{\beta'}}^\phi\right)\cdot\left(\widetilde{\theta_\gamma\theta'_{\gamma'}}^\phi\cdot\widetilde{\theta_\delta\theta'_{\delta'}}^\phi\right)&=\tilde\theta^\phi_{\alpha\alpha'\beta\beta'\gamma\gamma'\delta\delta'}.
\end{aligned}$$

These formulas can be used to check the associativity in Theorem 13.

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