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DOUBLE BICATEGORIES AND DOUBLE COSPANS

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Abstract

Interest in weak cubical n-categories arises in various contexts, in particular in topological field theories. In this paper, we describe a concept of *double bicategory* in terms of bicategories internal to **Bicat**. We show that in a special case one can reduce this to what we call a *Verity double bicategory*, after Domenic Verity. This is a weakened version of a double category, in the sense that composition in both horizontal and vertical directions satisfy associativity and unit laws only up to (coherent) isomorphisms. We describe examples in the form of double bicategories of "double cospans" (or "double spans") in any category with pushouts (pullbacks, respectively). We also give a construction from this which involves taking isomorphism classes of objects, and gives a Verity double bicategory of double cospans. Finally, we describe how to use a minor variation on this to describe cobordism of manifolds with boundary.

1. Introduction

The need to generalize the concept of a category was implicit from the beginning of the subject. Saunders Mac Lane stated that the concept of category was introduced to study not categories themselves, nor even functors from one category to another, but natural transformations between functors, which are naturally seen as 2-morphisms in a 2-category of all categories. This was an early seed of the notion of higher categories. Once explicitly recognized, however, the concept proved to be ambiguous.

There has been considerable work toward a general definition of a (weak) n-category. This has $(n+1)$ layers of structure, including objects, morphisms between

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objects, 2-morphisms between morphisms, and so on up to n-morphisms. Several possible alternative definitions exist, as discussed by Cheng and Lauda [**CL**], and by Leinster [**Le2**]. One of the features which varies among such definitions is the *shape* of higher-dimensional morphisms, with different choices suitable to different applications. Our aim in this paper is to develop one particular notion of higher category, in particular a *double bicategory*, which we shall define. We also show that there is a broad class of examples of this type in the form of *double cospans*.

The author's original motivation here was to describe rigorously a bicategory of *cobordisms with corners*. The most natural development of this idea turned out to be a special case of such double cospans. This in turn made it clear that the most natural structure for such things is not a bicategory, but the double bicategories discussed here. However, as we will prove in Theorem 3.4.1, given a double bicategory satisfying some simple conditions, one can get a bicategory, which is a better-understood and simpler structure. Our class of double span examples can be made to be of this type. The development of the topological material involved in cobordisms with corners will be given in a companion paper, but here we aim to be accessible to readers of that paper seeking background, and thus will give a relatively expository description of double bicategories and their double cospan examples.

The related concept of a "weak double category", or "pseudo double category" has also been defined (for further discussion, see e.g Marco Grandis and Robert Paré [**GP1**], Thomas Fiore [**Fi**], or Richard Garner [**Ga**]). In this setting, the weakening only occurs in only one direction, say the horizontal. That is, the associativity of composition, and unit laws, in the vertical direction apply only up to certain higher *associator* and *unitor* isomorphisms. In the horizontal direction, category axioms hold strictly. In fact, this must be so when weakening uses just the square 2-cells of the double category. For the composition in a double bicategory to be weak in both directions, it must be that the associator isomorphisms are (globular) 2-morphisms, rather than (square) 2-cells.

We note here that the feature that one direction is strict also appears in the *weak n-cubical categories* discussed by Grandis ([**Gr1**], [**Gr3**]), but that these are well defined for any dimension n. In particular, they are defined so as to have one direction in which composition is strict, while all others are weak. However, a Verity double bicategory can be taken to be a weak 3-cubical category with no nontrivial morphisms in the strict direction; on the other hand, a weak 2-cubical category can be seen as a Verity double bicategory in which the composition in one direction happens to be strict. For some purposes, this asymmetry is useful, but for our motivating application to cobordisms with corners, we want to define a "fully" weak cubical 2-category. We shall comment on the structures described by Grandis again when we consider double cospans, which give examples of both double bicategories and weak n-cubical categories.

In Section 2 we briefly describe some of the necessary category-theoretic background for readers who may be unfamiliar with it. This includes the concepts of *enrichment* and *internalization* in category theory, which give rise to bicategories and double categories respectively. The concept of a double bicategory is a mutual generalization of that of *bicategory* and of *double category*, and is best understood in this light.

Bicategories gave the first precise, explicit notion of weak higher categories. They were described by Bénabou [Be] in 1967, introducing the concept of 2-morphisms between morphisms:

$$\begin{tikzcd} x \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & y \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd} \tag{1}$$

Equations in the axioms for a category are replaced by 2-isomorphisms, which themselves satisfy coherence laws given in equations. This is known as *weakening*.

The original example used to illustrate this concept was the bicategory of *spans* in a suitable category $\mathbf{C}$, namely diagrams of the form:

$$X \to S \leftarrow Y \tag{2}$$

There is a natural concept of a map of spans, and an operation of composition for spans which is not strictly associative—rather, it is only associative up to isomorphism. Weakening thus appeared naturally in the setting of spans. The double cospans defined here lead to weakening in just the same way, but the concept being weakened is that of a double category.

Double categories, introduced by Ehresmann [Eh1, Eh2], have objects, horizontal and vertical morphisms which can be represented diagrammatically as edges, and squares:

$$\begin{tikzcd} x \arrow[r, "\phi"] \arrow[d, "f"'] \arrow[dr, phantom, "\Searrow F"] & x' \arrow[d, "f'"] \\ y \arrow[r, "\hat{\phi}"'] & y' \end{tikzcd} \tag{3}$$

These can be composed in geometrically obvious ways.

Double categories distinguish between horizontal and vertical 1-morphisms, which in general can only be composed with other morphisms of the same type. On the other hand, 2-morphisms are "squares", with both horizontal and vertical source and target, which can be composed in either direction with other squares having a common boundary.

Moskaliuk and Vlassov [MV] discuss the application of double categories to mathematical physics, and particularly to topological quantum field theories (TQFT's), and to dynamical systems with changing boundary conditions—that is, with inputs and outputs. Kerler and Lyubashenko [KL] describe "extended" TQFT's as "double pseudofunctors" between double categories. This formulation involves, among other things, a double category of cobordisms with corners. This sort of topological category has manifolds for objects, and manifolds with boundary or with corners as higher morphisms. This makes it possible to describe systems with changing boundary conditions, and the most natural way to do this is by allowing both initial and final states, and changing boundary conditions, as part of the boundary in a more general sense. This is one of the main motivations for the concepts we describe here, and we shall return to it in a subsequent paper. Double categories are too strict to be really natural for our purpose, however. Composition in a double category must be strictly associative, and in order to achieve this, one considers only *equivalence classes* of cobordisms as morphisms.

Thus, the principle here is to weaken the definition of a double category. This had previously been done in the definition of a *pseudocategory*, as described, for instance, by Fiore [**Fi**]. However, in a pseudocategory, just one direction of composition is weak: that is, the associative and unit laws satisfied by composition are replaced by associator and unitor isomorphisms. In a double bicategory, composition is weak in both directions.

In Section 3 we introduce double bicategories using a form of internalization, analogous to that which gives double categories as categories internal to **Cat**. In Section 3.2, we describe a somewhat different concept of double bicategory, due to Dominic Verity (which we denote a *Verity double bicategory* for clarity), a structure which also has both horizontal and vertical bicategories, and square 2-cells, with weak composition in both directions. In Section 3.3, we explain how a special case of our double bicategories can be reduced to Verity's definition. In turn, we show in Section 3.4 how a Verity double bicategory satisfying certain conditions, in turn yields a bicategory in the usual sense. Thus, this presents a series of increasingly manageable simplifications.

We finish in Section 4 by describing a rather broad general class of examples of double bicategories, which arise in rather the same way as the fact that Span($\mathbf{C}$) is a bicategory. A "double cospan" in a category with pushouts is a diagram of the following form:

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & S \arrow[d] & X_2 \arrow[l] \arrow[d] \\
T_1 \arrow[r] & M & T_2 \arrow[l] \\
X_1' \arrow[r] \arrow[u] & S' \arrow[u] & X_2' \arrow[l] \arrow[u]
\end{tikzcd} \qquad (4)$$

These diagrams can be composed horizontally and vertically, and in either case composition is by pushout, just as with ordinary spans or cospans. In Section 4.1 we describe double cospans and their composition in detail, and show that they naturally form a double bicategory in our original sense. In Section 4.2 we show how reducing to certain natural equivalence classes of double cospans yields a Verity double bicategory, following the procedure in Section 3.3.

2. Bicategories and Double Categories

Since this paper is intended as a companion to another of a more topological nature, we will recall for the reader with less category-theoretic background some relevant ideas about bicategories and double categories. Other readers may wish to skip to Section 3 when we introduce double bicategories.

We want to weaken the notion of a double category. Weakening a concept X in category theory generally involves creating a new concept in which defining equations in the original concept (such as associativity) are replaced by specified isomorphisms. Thus, one says that the defining equations hold with equality in a *strict* X and hold only "up to" isomorphism in a *weak* X.

Before describing our weakened concept of double category, we recall how this process works, and examine the strict form of the concept we want to weaken. So we begin by recalling some facts about bicategories, to illustrate weakening, and double categories, to provide a starting point.

2.1. Bicategories

A **bicategory** is a "weak globular 2-category". That is, if **B** is a bicategory, and $x, y \in \mathbf{B}$, then $\hom(x, y) \in \mathbf{Cat}$, allowing isomorphisms between morphisms where formerly we had equations. The morphisms in $\hom(x, y)$ are thought of as "2-morphisms" in **B**. Moreover, the strict version of a bicategory, usually called a "2-category", has the same unit and associativity axioms as a category. However, the weak form replaces these with 2-isomorphisms satisfying some coherence properties. So in particular, we have the following definition, due to Bénabou [**Be**], and discussed in more detail, for instance, in [**Le**].

Definition 2.1.1. *A* ***bicategory*** $\mathcal{B}$ *consists of the following data:*

- *A collection of* ***objects*** **Obj**
- *For each pair* $x, y \in \mathbf{Obj}$*, a category* $\hom(x, y)$ *whose objects are called* ***morphisms*** *of* $\mathcal{B}$ *and whose morphisms are called* ***2-morphisms*** *of* $\mathcal{B}$
- *For each object* $x \in \mathbf{Obj}$*, an identity* $1_x \in \hom(x, x)$
- *For each triple* x*,* y*,* z *of objects, a* ***composition*** *functor* $\circ : \hom(x, y) \times \hom(y, z) \to \hom(x, z)$
- *For each composable triple* f*,* g*,* h *of morphisms, a 2-isomorphism (i.e. invertible 2-morphism)* $\alpha_{f,g,h} : h \circ (g \circ f) \to (h \circ g) \circ f$ *called the* ***associator***
- *For each morphism* $f : x \to y$*, left and right* ***unitor*** *2-isomorphisms* $l_f : 1_y \circ f \to f$ *and* $r_f : f \circ 1_x \to f$

The associator is subject to the Pentagon identity, namely that the following *diagram commutes for any 4-tuple of composable morphisms (f, g, h, j):*

$$\begin{tikzcd}[column sep=small]
 & & (f\circ g)\circ(h\circ j) \arrow[drr, "a_{f,g,h\circ j}"] & & \\
((f\circ g)\circ h)\circ j \arrow[urr, "a_{f\circ g,h,j}"] \arrow[dr, "a_{f,g,h}\circ 1_j"'] & & & & f\circ(g\circ(h\circ j)) \\
 & (f\circ(g\circ h))\circ j \arrow[rr, "a_{f,g\circ h,j}"'] & & f\circ((g\circ h)\circ j) \arrow[ur, "1_f\circ a_{g,h,j}"'] &
\end{tikzcd} \tag{5}$$

Also, the unitors and associator make the following commute for all composable g, f:

$$\begin{tikzcd}
(g\circ 1_y)\circ f \arrow[r, "a_{g,1_y,f}"] \arrow[d, "r_g\circ 1_f"'] & g\circ(1\circ f) \arrow[dl, "1_g\circ l_f"] \\
g\circ f &
\end{tikzcd} \tag{6}$$

(where $y = t(f) = s(g)$).

Remark 2.1.2. This is a compact definition of a bicategory, but it is possible to describe the same data in different ways, which will be more directly relevant to subsequent discussion of double bicategories. In particular, this definition is related to the definition of a *strict* bicategory (a *2-category*) as a *category enriched in* **Cat**. That is, for any objects x and y, there is a category $\hom(x, y)$. However, there is a more elementary, although perhaps less elegant, way of describing the data of a bicategory.

One can form the collection $\mathbf{Mor} = \coprod \mathrm{ob}(\hom(x, y))$ of all morphisms of $\mathcal{B}$, and $\mathbf{2Mor} = \coprod \mathrm{mor}(\hom(x, y))$ of all 2-morphisms of $\mathcal{B}$. Then the existence of composition functors imply that there is, just as in categories, a partially defined composition function $\circ : \mathbf{Mor} \times \mathbf{Mor} \to \mathbf{Mor}$, which is defined for pairs (f, g) for which $t(f) = s(g)$ (and two such functions giving composition of 2-morphisms). These functions will have properties determined by the fact that they must give composition functors as defined above. The existence of identity morphisms means that there is an **identity** map $i : \mathbf{Obj} \to \mathbf{Mor}$, and this satisfies the usual relations with the source and target maps, and the composition.

As well as the source and target maps

$$s, t : \mathbf{Mor} \to \mathbf{Obj}$$

given in this definition, there are the source and target maps in each $\hom(x, y)$. These imply the existence of $s, t : \mathbf{2Mor} \to \mathbf{Mor}$, with the property that for any 2-morphism α, $s(s(\alpha)) = s(t(\alpha))$, and $t(s(\alpha)) = t(t(\alpha))$. A similar condition will apply to double categories, and indeed double bicategories, as we shall see. Together these describe the picture summarized in the diagram (1), depicting 2-morphisms as 2-dimensional cells between arrow-shaped 1-morphisms.

Jean Bénabou [**Be**] introduced bicategories in a 1967 paper, and one broad class of examples introduced there comes from the notion of a *span*. Since we will want to use a similar construction later, we remark on this here:

Definition 2.1.3. ***(Bénabou)*** *Given any category* $\mathbf{C}$*, a* ***span*** (S,π_1,π_2) *between objects* $X_1, X_2 \in \mathbf{C}$ *is a diagram in* $\mathbf{C}$ *of the form*

$$P_1 \xleftarrow{\pi_1} S \xrightarrow{\pi_2} P_2 \tag{7}$$

Given two spans (S,s,t) *and* (S',s',t') *between* X_1 *and* X_2 *a* ***morphism of spans*** *is a morphism* $g : S \to S'$ *making the following diagram commute:*

$$\begin{tikzcd} & S \arrow[dl, "\pi_1"'] \arrow[d, "g"] \arrow[dr, "\pi_2"] & \\ X_1 & S' \arrow[l, "\pi'_1"] \arrow[r, "\pi'_2"'] & X_2 \end{tikzcd} \tag{8}$$

Composition of spans S *from* X_1 *to* X_2 *and* S' *from* X_2 *to* X_3 *is given by pullback: that is, an object* R *with maps* f_1 *and* f_2 *making the following diagram commute:*

$$\begin{tikzcd} & & R \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & \\ & S \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & S' \arrow[dl, "\pi'_2"'] \arrow[dr, "\pi'_3"] & \\ X_1 & & X_2 & & X_3 \end{tikzcd} \tag{9}$$

which is terminal among all such objects. That is, given any other Q *with maps* g_1 *and* g_2 *which make the analogous diagram commute, these maps factor through a unique map* $Q \to R$*.* R *becomes a span from* X_1 *to* X_3 *with the maps* $\pi_1 \circ f_1$ *and* $\pi_2 \circ f_2$*.*

The span construction has a dual concept:

Definition 2.1.4. *A* ***cospan*** *in* $\mathbf{C}$ *is a span in* $\mathbf{C}^{\mathrm{op}}$*, morphisms of cospans are morphisms of spans in* $\mathbf{C}^{\mathrm{op}}$*, and composition of cospans is given by pullback in* $\mathbf{C}^{\mathrm{op}}$*—that is, by pushout in* $\mathbf{C}$*.*

Remark 2.1.5. ([**Be, ex. 2.6**]) Given any category $\mathbf{C}$ with all limits, there is a bicategory $\mathrm{Span}(\mathbf{C})$, whose objects are the objects of $\mathbf{C}$, whose *hom*-sets of morphisms $\mathrm{Span}(\mathbf{C})(X_1, X_2)$ consist of all spans between X_1 and X_2 with composition as defined above, and whose 2-morphisms are morphisms of spans. $\mathrm{Span}(\mathbf{C})$ as defined above forms a bicategory (dually, there is a bicategory $\mathrm{Cosp}(\mathbf{C})$ of cospans).

One should note that there is some choice in the precise definition of this bicategory since pushout (or pullback) is only defined up to isomorphism. However, one can make a particular choice of pushout (or pullback) as a given composite, and given this choice get corresponding associators and unitors. Different choices of composite will of course give different such maps. However, all such choices are equivalent. This is due in part to the fact that the pullback is a universal construction (universal properties of Span(**C**) are discussed by Dawson, Paré and Pronk [**DPP**]).

We briefly describe the proof of Bénabou that Span(**C**) is a bicategory:

The identity for X is $X \stackrel{id}{\leftarrow} X \stackrel{id}{\rightarrow} X$, which has an obvious unitor whose properties are easy to check.

The associator arises from the fact that the pullback is a *universal* construction. Given morphisms $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$ in Span(**C**), the composites $((f \circ g) \circ h)$ and $(f \circ (g \circ h))$ are pullbacks consisting of objects O_1 and O_2 with maps into X and W. The universal property of pullbacks gives an isomorphism between O_1 and O_2 as follows.

The universal property of pullback means that any object with maps into the objects f and $(g \circ h)$ will have a map into O_2 which they factor through. We have maps into the objects f, g, and h from O_1, and therefore a unique compatible map into $g \circ h$ by the universal property for that pullback. Therefore, there is again a unique compatible map into O_2. This we take to be the associator. We notice that in particular, the same argument works in reverse, and so the two maps we get are inverses, hence isomorphisms.

These associators satisfy the pentagon identity since they are unique (in particular, both sides of the pentagon give the same isomorphism).

It is easy to check that $\hom(X_1, X_2)$ is a category, since it inherits all the usual properties from **C**.

We will generalize the construction of bicategories of spans to give examples of double bicategories in Section 4. This development of bicategories illustrates the sort of weakening we want to apply to the concept of a double category. So we will first describe the strict notion in Section 2.2, before considering how to weaken it, in Section 3.2.

2.2. Double Categories

The concept of a double category extends that of a category in a different way than the concept of bicategory. Both, however, can be visualized as having both "arrow-like" morphisms, and also two-dimensional cells thought of as higher morphisms.

Just as with bicategories, we recall the definition here first by giving an abstract definition, then showing an equivalent, more concrete, version. The first definition of a bicategory highlighted its relation to the idea of an enriched category. Here we begin by illustrating how double categories illustrate *internalization*, which we will return to in Section 3 when describing double bicategories.

Definition 2.2.1. *A double category is a category internal to* **Cat**.

This is a generalization of the more broadly familiar terminology in which, for instance, a group internal to **Top** is called a *group object* in **Top**, or topological group. However, not all structures we might want to internalize are determined by a single object. In particular, a category (by default, internal to **Set**) consists of not one but two sets, namely the sets of *objects* and *morphisms*. A category internal to **C** (or "in **C**") has two objects of **C** playing the same roles.

So a category in **Cat** is a structure having a category **Ob** of objects and a category **Mor** of morphisms, with functors such as s and t satisfying the usual category axioms. Note that these axioms give conditions at both the object and morphism level, in addition to those which follow from the fact that they are functors.

We thus have sets of objects and morphisms in **Ob**, which satisfy the usual axioms for a category. The same is true for **Mor**. In addition, the category axioms for the double category are imposed on the composition and identity functors, and these must be compatible with the category axioms in the other direction. Thus we can think of both the objects in **Mor** and the morphisms in **Obj** as morphisms between the objects in **Obj**. A double category is often thought of as including the morphisms of two (potentially) different categories on the same collection of objects. These are the *horizontal* and *vertical* morphisms.

Here, the objects in the diagram can be thought of as objects in **Obj**, the vertical morphisms f and f' can be thought of as morphisms in **Obj** and the horizontal morphisms ϕ and $\hat{\phi}$ as objects in **Mor**. Vertical composition is given by composition in **Mor**, and horizontal composition by the morphism map of the composition functor $\circ$. (In fact, we can adopt either convention for distinguishing horizontal and vertical morphisms). However, we also have morphisms in **Mor**. We represent these as 2-cells, or *squares*, like the 2-cell S represented in (11). The fact that the composition map $\circ$ is a functor means that horizontal and vertical composition of square 2-cells commutes.

A double category can therefore be seen more directly. It consists of:

- a set O of objects
- *horizontal* and *vertical* categories, whose sets of objects are both O
- for any diagram of the form

$$\begin{tikzcd} x \arrow[r, "\phi"] \arrow[d, "f"'] & x' \arrow[d, "f'"] \\ y \arrow[r, "\phi'"'] & y' \end{tikzcd} \tag{10}$$

  a collection of *2-cells*, having horizontal source and target f and f', and vertical source and target ϕ and ϕ'

along with additional data such as the source and target maps, identities, and so forth, all satisfying category-like axioms in both horizontal and vertical directions. In particular, the 2-cells can be composed either horizontally or vertically in the obvious way. We denote a 2-cell filling the above diagram like this:

$$\begin{tikzcd} x \arrow[r, "\phi"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow S"] & x' \arrow[d, "f'"] \\ y \arrow[r, "\phi'"'] & y' \end{tikzcd} \tag{11}$$

and think of the composition of 2-cells as pasting them along an edge. The resulting 2-cell fills a square whose boundaries are the corresponding composites of the morphisms along its edges.

Next, in Section 3 we take our descriptions of double categories and bicategories, and see how to find some common generalizations of both.

3. Double Bicategories

We wish to describe a structure which is sufficient to reproduce the various types of composition found in a double category, but in such a way that all are weakened. This means we should have horizontal composition for horizontal morphisms and vertical composition for vertical morphisms. Square 2-cells should be composable in both directions. Composition of morphisms in each direction is to be "weak", in the sense of having associator and unitor isomorphisms rather than associativity and unit laws. This means there will also be 2-morphisms of some appropriate shape to act as unitors and associators (and, of course, there will in general be other 2-morphisms as well). In particular, in place of the mere categories found in a double category, we have horizontal and vertical *bicategories*, with their (globular) 2-morphisms, as well as (square) 2-cells.

The natural choice of name for such a structure is a *double bicategory.* This term seems to have been originally introduced by Dominic Verity [**Ve**]. There is some ambiguity here. By analogy with "double category", the term "double bicategory" might be expected to describe is an internal bicategory in **Bicat**, the category of all bicategories . Indeed, it is what we will mean by a double bicategory here. However, this is not quite the concept given by Verity. The two turn out to be closely related, and both will be important, so we will refer to double bicategories in the sense of Verity by the term *Verity double bicategories*, while reserving *double bicategory* for internal bicategories in **Bicat**. For more discussion of the relation between these, see Section 3.2.

3.1. Double Bicategories and Internalization

Here we present a more precise definition of the concept of a double bicategory as a bicategory internal to **Bicat**. This will be somewhat more complex than the analogous process for a double category, but runs along similar lines.

Thus, we will have bicategories Obj, Mor and 2Mor. Then one can consider a bicategory internal to **Bicat**. It is straightforward to treat $F(\mathrm{Obj})$ as a horizontal bicategory, and the objects of Obj, Mor and 2Mor as forming a vertical bicategory. Note, however, that a diagrammatic representation of, for instance, 2-morphisms in

2Mor would require a 4-dimensional diagram element. The comparison can be seen by contrasting tables 1 and 2 in Section 3.3.

Definition 3.1.1. *A double bicategory consists of:*

- *bicategories* **Obj** *of* ***objects***, **Mor** *of* ***morphisms***, **2Mor** *of* ***2-morphisms***
- ***source*** *and* ***target*** *2-functors*
  - $s, t : \mathbf{Mor} \to \mathbf{Obj}$
  - $s, t : \mathbf{2Mor} \to \mathbf{Obj}$
  - $s, t : \mathbf{2Mor} \to \mathbf{Mor}$
- *Composition 2-functors:*
  - $\circ : \mathrm{MPairs} \longrightarrow \mathbf{Mor}$
  - $\circ : \mathrm{HPairs} \longrightarrow \mathbf{2Mor}$
  - $\cdot : \mathrm{VPairs} \longrightarrow \mathbf{2Mor}$

  satisfying the interchange law, where
  - $\mathrm{MPairs} = \mathbf{Mor} \times_{\mathbf{Obj}} \mathbf{Mor}$
  - $\mathrm{HPairs} = \mathbf{2Mor} \times_{\mathbf{Mor}} \mathbf{2Mor}$
  - $\mathrm{VPairs} = \mathbf{2Mor} \times_{\mathbf{Obj}} \mathbf{2Mor}$

  are (strict) pullbacks
- *an* ***associator*** *2-functor*
  - $a : \mathrm{Triples} \longrightarrow \mathrm{2Mor}$

  where
  - $\mathrm{Triples} = \mathbf{Mor} \times_{\mathbf{Obj}} \mathbf{Mor} \times_{\mathbf{Obj}} \mathbf{Mor}$
- ***unitors***
  - $l, r : \mathbf{Obj} \longrightarrow \mathbf{Mor}$

such that a makes the following diagram commute:

$$\begin{tikzcd}
\mathrm{Pairs} \arrow[d, "\circ"] & \mathrm{Triples} \arrow[l, "\circ\times 1"'] \arrow[r, "1\times\circ"] \arrow[d, "a"] & \mathrm{Pairs} \arrow[d, "\circ"] \\
\mathrm{Mor} & \mathrm{2Mor} \arrow[l, "s"] \arrow[r, "t"'] & \mathrm{Mor}
\end{tikzcd}$$

and additional diagrams with the interpretation that a gives invertible 2-morphisms. The unitors must satisfy $s(l(x)) = t(l(x)) = x$ and $s(r(x)) = t(r(x)) = x$, and the associator should satisfy the pentagon identity (5), and the unitors should satisfy the unitor laws (6).

We interpret these morphisms involving pullbacks (the fibred products) as giving *partially defined* composition 2-functors $\circ : \mathbf{Mor}^2 \to \mathbf{Mor}$, $\circ : \mathbf{2Mor}^2 \longrightarrow \mathbf{2Mor}$ and $\cdot : \mathbf{2Mor}^2 \longrightarrow \mathbf{2Mor}$, and associator 2-functor $a : \mathbf{Mor}^3 \longrightarrow \mathbf{2Mor}$.

Remark 3.1.2. The Pentagon identity is shown in (5) for a bicategory (i.e. a bicategory in **Sets**). In **Bicat**, this holds for objects, morphisms, and 2-morphisms. We can express this condition formally, in any category (with pullbacks), building from composable quadruples, so that the pentagon identity is expressed in a commuting diagram which includes the one built by pasting the two following diagrams together along the outside edges:

$$
\begin{tikzcd}
\mathrm{Mor} \arrow[ddddddrrr, bend right, "s"'] & & & & & & \mathrm{Mor} \arrow[ddddddlll, bend left, "t"] \\
& & & \mathrm{4tuples} \arrow[ulll, "\pi_1 \circ i"] \arrow[urrr, "\pi_4 \circ i"'] \arrow[dll, bend right] \arrow[d, "(id \times \circ \times id) i"] \arrow[drr, bend left] & & & \\
& \mathrm{Triples} \times \mathrm{Mor} \arrow[d, "a \times id"] & & \mathrm{Triples} \arrow[d, "a"] & & \mathrm{Mor} \times \mathrm{Triples} \arrow[d, "id \times a"] & \\
& 2\mathrm{Mor}^2 \arrow[dr, "\circ"] & & 2\mathrm{Mor} & & 2\mathrm{Mor}^2 \arrow[dl, "\circ"] & \\
& & 2\mathrm{Mor} & & 2\mathrm{Mor} & & \\
& & & \mathrm{VTriples} \arrow[ul, "p_1"] \arrow[uu, "p_2"] \arrow[ur, "p_3"'] \arrow[d, "\circ"] & & & \\
& & & 2\mathrm{Mor} & & &
\end{tikzcd}
\tag{12}
$$

and

$$
\begin{tikzcd}
\mathrm{Mor} \arrow[ddddddrrr, bend right, "s"'] & & & & & & \mathrm{Mor} \arrow[ddddddlll, bend left, "t"] \\
& & & \mathrm{4tuples} \arrow[ulll, "\pi_1 \circ i"] \arrow[urrr, "\pi_4 \circ i"'] \arrow[dl] \arrow[dr] & & & \\
& & \mathrm{Triples} \arrow[dl, "a"] & & \mathrm{Triples} \arrow[dr, "a"] & & \\
& 2\mathrm{Mor}^2 \arrow[dr, "\circ"] & & & & 2\mathrm{Mor}^2 \arrow[dl, "\circ"] & \\
& & 2\mathrm{Mor} & & 2\mathrm{Mor} & & \\
& & & \mathrm{VPairs} \arrow[ul, "\pi_1 \circ i"] \arrow[ur, "\pi_2 \circ i"'] \arrow[d, "\circ"] & & & \\
& & & 2\mathrm{Mor} & & &
\end{tikzcd}
\tag{13}
$$

Note that diagram (12) denotes the three 2-morphism sequence in the pentagon, and (13) the sequence of two 2-morphisms.

Similar remarks apply to give “element-free” versions of the interchange laws for composition of 2-morphisms and the unitor laws shown in (6).

To fully expand this definition without assuming the concept of a bicategory would be much longer than the form given here. One would have to specify nine types of data - objects, morphisms, and 2-morphisms in each of **Obj**, **Mor**, and **2Mor**, and describe all the axioms in detail, such as the conditions implied by the fact that $\circ$ and $\cdot$ are functors. This is a rather complicated structure, as we see in more detail when we return to it in Section 3.3 (and in particular we show the types of data implied by this definition in Table 2).

In particular, the most natural geometric representation of a 2-morphism in **2Mor** is as a 4-dimensional object. We had hoped for a common generalization of double categories and bicategories, each of which is represented graphically with morphisms being cells having dimension at most 2. One could hope that such a structure would also have at most 2-dimensional morphisms. The definition of a Verity double bicategory satisfies this, as we describe in Section 3.2, and in Section 3.3 we show how it is related to the definition we have given here.

First, we briefly remark that one can cast this description of internal bicategories in terms of models of the *finite limit theory* of bicategories, **Th(Bicat)**. This is a category with finite limits, which can be described in terms of its generators and relations. It is generated by objects O, M and B, together with morphisms, and subject relations, as given in the definition (where in that case these are in **Bicat**. A model of such a theory in a category **C** with finite limits is a functor $F : \mathbf{Th(Bicat)} \to \mathbf{C}$.

Here we are considering *strict* models of the theory of categories in **Cat**, and bicategories in **Bicat**, rather than a weak model, which one might also consider. In particular, **Bicat** is a tricategory (defined by Gordon, Power and Street [**GPS**]): it has objects which are bicategories, morphisms which are (weak) 2-functors between bicategories, 2-morphisms which are natural transformations between bifunctors, and 3-morphisms which are modifications of such transformations. However, for both double categories and double bicategories, we ignore the higher morphisms in this setting and think of **Bicat** as a mere category, taking equivalence classes of morphisms between bicategories (that is, 2-functors) where needed. Thus, equations in the theory are mapped to equations (not isomorphisms) in **Bicat**.

Even such strict models, however, are fairly complex structures, so we now consider one way to simplify them.

3.2. Verity Double Bicategories

The following definition of a “double bicategory” is due to Dominic Verity [**Ve**], and will henceforth be referred to as a Verity double bicategory. It is readily seen as a natural weakening of the definition of a double category. Just as the concept of *bicategory* is weaker than that of *2-category* by weakening the associative and unit laws, Verity double bicategories will be weaker than double categories.

Definition 3.2.1. *(Verity) A* ***Verity double bicategory*** **C** *is a structure* $\mathcal{V}$ *consisting of the following data:*

- *a class of* ***objects*** Obj*,*

- ***horizontal*** *and* ***vertical bicategories*** **Hor** *and* **Ver** *having* Obj *as their objects*
- *for every square of horizontal and vertical morphisms of the form*

$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, "v"'] & b \arrow[d, "v'"] \\ c \arrow[r, "h'"] & d \end{tikzcd} \tag{14}$$

a class of ***squares*** **Squ***, with maps* s_h, t_h : **Squ** $\to$ Mor(**Hor**) *and* s_v, t_v : **Squ** $\to$ Mor(**Ver**)*, satisfying an equation for each corner, namely:*

$$\begin{aligned} s(s_h) &= s(s_v) \\ t(s_h) &= s(t_v) \\ s(t_h) &= t(s_v) \\ t(t_h) &= t(t_v) \end{aligned} \tag{15}$$

The squares should have horizontal and vertical composition operations, defining the vertical composite $F \otimes_V G$

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow F"] & x' \arrow[d] \\ y \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow G"] & y' \arrow[d] \\ z \arrow[r] & z' \end{tikzcd} \quad = \quad \begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow F \otimes_V G"] & x' \arrow[d] \\ z \arrow[r] & z' \end{tikzcd} \tag{16}$$

and horizontal composite $F \otimes_H G$*:*

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow F"] & y \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow G"] & z \arrow[d] \\ x' \arrow[r] & y' \arrow[r] & z' \end{tikzcd} \quad = \quad \begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow F \otimes_H G"] & z \arrow[d] \\ x' \arrow[r] & z' \end{tikzcd} \tag{17}$$

The composites have the usual relation to source and target maps, satisfy the interchange law

$$(F \otimes_V F') \otimes_H (G \otimes_V G') = (F \otimes_H G) \otimes_V (F' \otimes_H G') \tag{18}$$

and there is a unit for composition of squares:

$$\begin{tikzcd} x \arrow[r, "1_x"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow 1_f"] & x \arrow[d, "f"] \\ y \arrow[r, "1_y"] & y \end{tikzcd} \tag{19}$$

(and similarly for vertical composition).

There is a left and right action by the horizontal and vertical 2-morphisms on

Squ, *giving* $F \star_V \alpha$,

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow F"] & y \arrow[d] \arrow[d, bend left=60, ""{name=C}] \\ x' \arrow[r] & y' \end{tikzcd} \quad = \quad \begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow F \star_V \alpha"] & y \arrow[d] \\ x' \arrow[r] & y' \end{tikzcd} \tag{20}$$

(and similarly on the left) and $F \star_H \alpha$,

$$\begin{tikzcd} x \arrow[r] \arrow[r, bend left=60, "\Downarrow\alpha"] \arrow[d] \arrow[dr, phantom, "\Swarrow F"] & y \arrow[d] \\ x' \arrow[r] & y' \end{tikzcd} \quad = \quad \begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow \alpha \star_H F"] & y \arrow[d] \\ x' \arrow[r] & y' \end{tikzcd} \tag{21}$$

The actions also satisfy interchange laws:

$$(F \otimes_H F') \star_H (\alpha \otimes_V \alpha') = (F \star_H \alpha) \otimes_h (F' \star_H \alpha') \tag{22}$$

(and similarly for the vertical case) and are compatible with composition:

$$(F \otimes_H G) \star_V \alpha = F \otimes_H (G \star_V \alpha) \tag{23}$$

(and analogously for vertical composition). They also satisfy additional compatibility conditions: the left and right actions of both vertical and horizontal 2-morphisms satisfy the "associativity" properties,

$$\alpha \star (S \star \beta) = (\alpha \star S) \star \beta \tag{24}$$

for both $\star_H$ *and* $\star_V$*. Moreover, horizontal and vertical actions are independent:*

$$\alpha \star_H (\beta \star_V S) = \beta \star_V (\alpha \star_H S) \tag{25}$$

and similarly for the right action.

Finally, the composition of squares agrees with the associators for composition *by the action in the sense that given three composable squares F, G, and H:*

$$
\begin{tikzcd}
x \arrow[rr, "h\circ(g\circ f)"] \arrow[rr, bend left=50, "(h\circ g)\circ f"] \arrow[rr, phantom, bend left=25, "\Uparrow a_{f,g,h}"] \arrow[d] \arrow[drr, phantom, "(F\otimes_H G)\otimes_H H \,\Downarrow"] & & y \arrow[d] \\
x' \arrow[rr, "h'\circ(g'\circ f')"'] & & y'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x \arrow[rr, "(h\circ g)\circ f"] \arrow[d] \arrow[drr, phantom, "F\otimes_H(G\otimes_H H) \,\Downarrow"] & & y \arrow[d] \\
x' \arrow[rr, "(h'\circ g')\circ f"] \arrow[rr, bend right=50, "h'\circ(g'\circ f)"'] \arrow[rr, phantom, bend right=25, "\Downarrow a_{f',g',h'}"] & & y'
\end{tikzcd}
\tag{26}
$$

and similarly for vertical composition. Likewise, unitors in the horizontal and vertical bicategories agree with the identity for composition of squares:

$$
\begin{tikzcd}
x \arrow[r, "f"'] \arrow[rr, bend left=50, "f"] \arrow[rr, phantom, bend left=25, "\Uparrow l_f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow F"] & y \arrow[r, "1_y"'] \arrow[d, "g'"] \arrow[dr, phantom, "\Downarrow 1_{g'}"] & y \arrow[d, "g'"] \\
x' \arrow[r, "f'"'] & y' \arrow[r, "1_{y'}"'] & y'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x \arrow[r, "1_x"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow 1_g"] & x \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "\Downarrow F"] & y \arrow[d, "g'"] \\
x' \arrow[r, "1_{x'}"] \arrow[rr, bend right=50, "f'"'] \arrow[rr, phantom, bend right=25, "\Downarrow r_{f'}"] & x' \arrow[r, "f'"] & y'
\end{tikzcd}
\tag{27}
$$

and similarly for vertical unitors.

Remark 3.2.2. This is rather more unwieldy than the definition of either a bicategory or a double category, but is simpler than a similarly elementary description of a double bicategory (in the sense of Section 3.1) would be. In particular, where there are compatibility conditions involving equations in this definition, a double bicategory would have only higher isomorphisms, themselves satisfying additional coherence laws. In particular, in Verity double bicategories, the action of 2-morphisms on squares is described by strict equations, rather than by a specified isomorphism satisfying coherence laws.

To help make sense of this definition, we note that it is possible (following [**Ve**, sec. 1.4]) to define categories $\mathbf{Cyl_H}$ (respectively, $\mathbf{Cyl_V}$) of *cylinders*. The objects of these categories are squares, and maps are pairs of vertical (respectively, horizontal) 2-morphisms joining the vertical (respectively, horizontal) source and targets of pairs of squares which share the other two sides (this is shown in Table 2, in Section 3.3: the cylinders are "thin" versions of higher morphisms appearing there). These are categories in the usual sense, with strict associativity and unit laws. These conditions would be weakened in a double bicategory (in which maps would include not just pairs of 2-morphisms, but also a 3-dimensional interior of the cylinder, which is a morphism in $2\,\mathrm{Mor}$, or 2-morphism in Mor, satisfying properties only up to a 4-dimensional 2-morphism in $2\,\mathrm{Mor}$).

3.3. Decategorification

The main idea we are pursuing in this section is that of "decategorification". This rather vague term refers to a process in which category-theoretic information is discarded from a structure. Typically, it refers to replacing isomorphisms with equations—for example, a decategorification of the category of finite sets is the set of their cardinalities, $\mathbb{N}$. Similarly, one can turn a bicategory into a category by taking new morphisms to be 2-isomorphism classes of old morphisms, and discarding all 2-morphisms.

To establish a relationship between the apparently conflicting terms for the two types of "double bicategory", we will now show how a Verity double bicategory can arise as a decategorification of a double bicategory satisfying some conditions. We will show later that this can be done with the double cospan examples of Section 4. The conditions which are needed allow us to speak of the "action of 2-cells upon squares". To see what these are, we first consider a "lower dimensional" example of a similar process. What we want to do to obtain Verity double bicategories has an analog in the case of double categories.

	Obj	**Mor**
Objects	$\bullet^x$	$\bullet \xrightarrow{f} \bullet$
Morphisms	$\begin{array}{c}\bullet\\ \downarrow g\\ \bullet\end{array}$	$\begin{array}{ccc}\bullet & \longrightarrow & \bullet\\ \downarrow & \Downarrow F & \downarrow\\ \bullet & \longrightarrow & \bullet\end{array}$

Table 1: Data of a Double Category

In a double category, thought of as an internal category in **Cat**, we have data of four sorts, as shown in Table 1. That is, a double category **DC** has categories **Obj** of objects and **Mor** of morphisms. The first column of the table shows the data of **Obj**: its objects are the objects of **DC**; its morphisms are the *vertical* morphisms. The second column shows the data of **Mor**: its objects are the *horizontal* morphisms of **DC**; its morphisms are the squares of **DC**.

There is a condition we can impose which effectively turns the double category into a category, where the horizontal and vertical morphisms are composable, and the squares can be ignored. The sort of condition involved is similar to the *horn-filling conditions* introduced by Ross Street [**St**] in his first introduction of the idea of weak ω-categories, or *quasicategories*, in which all morphisms are n-simplexes for some n. A horn filling condition says that, given some hollow simplex with just one face (morphism) missing from the boundary, there will be a morphism to fill that face, and a compatible "filler" for the inside of the simplex. In a double category, there is an analogous "niche-filler" condition.

Definition 3.3.1. *A double category $\mathcal{DC}$ satisfies the* ***composability condition*** *if the following holds. For any pair (f, g) of a horizontal and vertical morphism where* *the target object of f is the source object of g, there is a unique pair $(h, \star)$ consisting of a unique vertical morphism h and unique invertible square $\star$ making the following diagram commute:*

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, dashed, "h"'] \arrow[dr, phantom, "\Downarrow\star"] & y \arrow[d, "g"] \\ z \arrow[r, "1_z"'] & z \end{tikzcd} \tag{28}$$

and similarly when the source of f is the target of g.

Notice that taking f to be the identity in this condition implies $\star$ is the identity square. This defines a composition:

Theorem 3.3.2. *If $\mathcal{DC}$ satisfies the composability condition and there are no other squares in $\mathcal{DC}$, there is a category $\mathcal{DC}_0$ with the same objects as $\mathcal{DC}$, and all horizontal and vertical morphisms as its morphisms.*

Proof. Begin by defining composition from $\star$, so that if f is horizontal and g is vertical, then $g \circ f = h$. If f and g are both horizontal (or both vertical), then define $g \circ f$ to be the usual composite. Then this composition is associative and has identities. We only need to check this for the composition using $\star$. For example, given morphisms as in the diagram:

$$\begin{tikzcd} w \arrow[r, "f"] & x \arrow[r, "f'"] & y \arrow[d, "g"] \\ z \arrow[r, "1_z"'] & z \arrow[r, "1_z"'] & z \end{tikzcd} \tag{29}$$

there are two ways to use the unique-filler principle to fill this rectangle. One way is to first compose the pairs of horizontal morphisms on the top and bottom, then fill the resulting square. The square we get is unique, and the morphism is denoted $g \circ (f' \circ f)$. The second way is to first fill the right-hand square, and then using the unique morphism we call $g \circ f'$, we get another square on the left hand side, which our principle allows us to fill as well. The square is unique, and the resulting morphism is called $(g \circ f') \circ f$. Composing the two squares obtained this way must give the square obtained the other way, since both make the diagram commute, and both are unique. So we have:

$$\begin{tikzcd} w \arrow[r, "f"] \arrow[d, dashed, "(g\circ f')\circ f"'] \arrow[dr, phantom, "\Downarrow\star"] & x \arrow[r, "f'"] \arrow[d, dashed] \arrow[dr, phantom, "\Downarrow\star"] & y \arrow[d, "g"] \\ z \arrow[r, "1_z"'] & z \arrow[r, "1_z"'] & z \end{tikzcd} \quad = \quad \begin{tikzcd} w \arrow[r, "f'\circ f"] \arrow[d, dashed, "g\circ(f'\circ f)"'] \arrow[dr, phantom, "\Downarrow\star"] & y \arrow[d, "g"] \\ z \arrow[r, "1_z"'] & z \end{tikzcd} \tag{30}$$

$\square$

Remark 3.3.3. Note that the composability condition does not require a square for every possible combination of source and target morphisms. In particular, there must be an identity morphism on the boundary of the square—on the bottom in (28). If instead of the identity 1_z, one could have any morphism h, then by choosing f and g to be identities, this would imply that every morphism must be invertible (at least weakly), since there must then be an h^{-1} with $h^{-1} \circ h$ isomorphic to the identity, but of course we do not insist that all morphisms should have inverses. When a filler square does exist, it indicates there is a commuting square in $\mathcal{DC}_0$: the square $\star$ becomes an equation between the composites along the upper right and lower left.

The decategorification of a double bicategory to give a Verity double bicategory is similar, except that with a double category we were removing only the squares (the lower-right quadrant of Table 1). There will be a similar condition to satisfy, but we need to do more with a double bicategory, since there are more sorts of data (and, therefore, a more complex condition). These fall into a similar arrangement, as shown in Table 2.

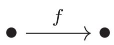
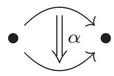
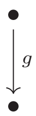
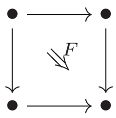
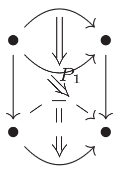
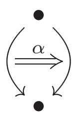
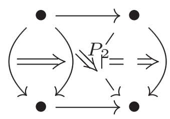
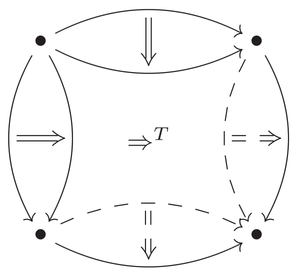
	Obj	**Mor**	**2Mor**
Objects			
Morphisms			
2-Cells			

Table 2: The data of a double bicategory

This table shows the data of the bicategories **Obj**, **Mor**, and **2Mor**, each of which has objects, morphisms, and 2-cells. Note that the morphisms in the three entries in the lower right hand corner—2-cells in **Mor**, and morphisms and 2-cells in **2Mor**—are not 2-dimensional. The 2-cells in **Mor** and morphisms in **2Mor** are the three-dimensional "filling" inside the illustrated cylinders, which each have two square faces and two bigonal faces. The 2-cells in **2Mor** should be drawn as 4-dimensional. The picture illustrated can be thought of as taking both square faces of one cylinder P_1 to those of another, P_2, by means of two other cylinders (S_1 and S_2, say), in such a way that P_1 and P_2 share their bigonal faces. This description works whether we consider the P_i to be horizontal and the S_j vertical, or vice versa. These describe the "frame" of this sort of morphism: the filling is the 4-dimensional "track" taking P_1 to P_2, or equivalently, S_1 to S_2, just as a square in a double category can be read horizontally or vertically. (Not all relevant parts of the diagrams have been labeled here, for clarity.)

Next we want to describe a condition similar to the composability condition for a double category. In that case, we got a condition which effectively allowed us to treat any square as an identity, so that we only had objects and morphisms. Here, we want a condition which lets us throw away the three entries of dimension greater than two in Table 2 in the bottom right. This condition, when satisfied, should allow us to treat a double bicategory as a Verity double bicategory. It comes in three parts, one for each type of data we want to discard:

Definition 3.3.4. *We say that a double bicategory satisfies the* ***vertical action condition*** *if, for any morphism $F_1 \in$ **Mor** and 2-morphism $\alpha \in$ **Obj** such that $s(F_1) = t(\alpha)$, there is a unique morphism $F_2 \in$ **Mor** and unique invertible 2-morphism $P \in$ **Mor** such that P fills the "pillow diagram":*

$$\begin{tikzcd} x \arrow[r] \arrow[r, bend left=80, "\Downarrow\alpha"] \arrow[d] \arrow[dr, phantom, "\Swarrow F_1"] & y \arrow[d] \\ x' \arrow[r] & y' \end{tikzcd} \quad \Rightarrow_P \quad \begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow F_2"] & y \arrow[d] \\ x' \arrow[r] \arrow[r, bend right=80, "\Downarrow id"'] & y' \end{tikzcd} \tag{31}$$

*where F_2 is the back face of this diagram, and the 2-morphism in **Obj** at the bottom is the identity.*

A double bicategory satisfies the ***horizontal action condition*** *if for any morphism $F_1 \in$ **Mor** and object α in **2Mor** with $s(F_1) = t(\alpha)$ there is a unique morphism $F_2 \in$ **Mor** and unique invertible morphism $P \in$ **2Mor** such that P fill the pillow diagram:*

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[d, bend right=80, "\Rightarrow\alpha"'] \arrow[dr, phantom, "\Swarrow F_1"] & y \arrow[d] \\ x' \arrow[r] & y' \end{tikzcd} \quad \Rightarrow_P \quad \begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow F_2"] & y \arrow[d] \arrow[d, bend left=80, "\Rightarrow id"] \\ x' \arrow[r] & y' \end{tikzcd} \tag{32}$$

In (31), F_2 is the square which will eventually be named $F_1 \star_H \alpha$ when we define an action of 2-cells on squares, and in (32), F_2 is the square will eventually be named $F_1 \star_V \alpha$.

Remark 3.3.5. One can see that this condition is analogous to the filler condition (28) in a double category by imagining the diagram (31) viewed obliquely. The diagram says that given a square with two bigons—the top one arbitrary and the bottom one the identity—there is another square F_2 (the back face of a pillow diagram) and a filler 2-morphism $P \in \mathbf{2Mor}$ which fills the diagram. If one imagines turning this diagram on its side and viewing it obliquely, one sees precisely (28), as a dimension has been suppressed. The role played by cylinders (2-morphism in **2Mor**) in (31) and (32 is played by a square in (28); the roles of both squares and bigons in (31) and (32) are played by arrows in (28); the role of arrows in (31) and (32) is filled by point-like objects in (28).

This gives horizontal and vertical actions, but to get the compatibility between them, we need a further condition. In particular, since these conditions involve both horizontal and vertical cylinders, the compatibility condition must correspond to the 4-dimensional 2-cells in **2Mor**, shown in the lower right corner of Table 2.

To draw the necessary condition is difficult, since the necessary diagram is four-dimensional, but we can describe it as follows:

Definition 3.3.6. *We say a double bicategory satisfies the* ***action compatibility condition*** *if the following holds. Suppose we are given*

- *a morphism $F \in$* **Mor**
- *an object $\alpha \in$* **2Mor** *whose target in* **Mor** *is a source of F*
- *a 2-cell $\beta \in$* **Obj** *whose target morphism is a source of F*
- *an invertible morphism $P_1 \in$* **2Mor** *with F as source, and the objects α and* id *in* **2Mor** *as source and target*
- *an invertible 2-cell $P_2 \in$* **Mor** *with F as source, and the 2-cells β and* id *in* **Mor** *as source and target*

where P_1 and P_2 have, as targets, morphisms in **Mor** *we call $\alpha \star F$ and $\beta \star F$ respectively. Then there is a unique morphism $\hat{F}$ in* **Mor** *and unique invertible 2-cell T in* **2Mor** *having all of the above as sources and targets.*

Geometrically, the unique 2-cell in **2Mor** looks like the structure in the bottom right corner of Table 2. This can be seen as taking one horizontal cylinder to another in a way that fixes the (vertical) bigons on its sides. It does this by means of a translation which acts on the front and back faces with a pair of vertical cylinders (have the same top and bottom bigonal faces). Alternatively, it can be seen as taking one vertical cylinder to another, acting on the faces with a pair of horizontal cylinders. In either case, the cylinders involved in the translation act on the faces, but the four-dimensional interior, T, acts on the original cylinder to give another. The simplest interpretation of this condition is that it is precisely the condition needed to give the compatibility condition (25).

Remark 3.3.7. Notice that the two conditions given imply the existence of unique data of three different sorts in our double bicategory. If these are the only data of these kinds, we can effectively omit them (since it suffices to know information about their sources and targets). This omission is part of a decategorification of the same kind we saw for a double category **DC**.

In particular, we show how a double bicategory **D** satisfying the above conditions gives a Verity double bicategory. We know that **D** consists of bicategories (**Obj**, **Mor**, **2Mor**) together with all required maps (three kinds of source and target maps, two kinds of identity, three partially-defined compositions, left and right unitors, and the associator), satisfying the usual properties. To begin with, we describe how the elements of a Verity double bicategory **V** (Definition 3.2.1) arise from this:

Definition 3.3.8. *If* **D** *is a double bicategory satisfying the horizontal and vertical action conditions and the action compatibility condition, then* $\mathbf{V(D)}$ *is the Verity double bicategory with:*

- *The objects* Obj *are the objects of* **Obj**.
- *The horizontal bicategory* **Hor** *of* $\mathbf{V(D)}$ *is* **Obj**
- *The vertical bicategory* **Ver** *of* $\mathbf{V(D)}$ *has:*
  - *Objects: Objects of* **Obj**
  - *Morphisms: Objects of* **Mor**
  - *2-morphisms: Objects of* **2Mor**

  The source, target and composition maps for **Ver** *are the object maps from the source, target, and composition 2-functors for* **D**.
- *The squares* **Squ** *of* $\mathbf{V(D)}$ *are isomorphism classes of morphisms of* **Mor**. *These are equipped with:*
  - *Vertical source and target maps: the morphism maps from the functors* $s, t : \mathbf{Mor} \to \mathbf{Obj}$.
  - *Horizontal source and target maps: the internal ones in* **Mor**.
  - *Horizontal composition (17): the composition of morphisms in* **Mor**.
  - *Vertical composition (16): the morphism maps for the partially defined functor* $\circ$ *for* **Mor**,
  - *Horizontal Identity: The identity square for a morphism* g *in* **Ver** *(i.e.* g *an object in* **Mor** *is* $1_f \in \mathbf{Mor}$.
  - *Vertical Identity: The identity square for a morphism* f *in* **Hor** *(i.e. a morphism* f *in* **Obj***) is given by* $\mathrm{id}(f)$ *for the unit functor* $\mathrm{id} : \mathbf{Obj} \to \mathrm{Mor}$.
- *The horizontal action defines* $F \star_H \alpha$ *to be (the isomorphism class of) the unique morphism in* **Mor** *whose existence is required by the horizontal action condition.*
- *The vertical action defines* $F \star_V \alpha$ *to be (the isomorphism class of) the unique morphism in* **Mor** *whose existence is required by the vertical action condition.*

Of course, we must check this is really a Verity double bicategory:

Theorem 3.3.9. *Suppose* **D** *is a double bicategory satisfying the horizontal and vertical action conditions and the action compatibility condition. Then* **V(D)** *is a Verity double bicategory.*

Proof. We check all the properties in the definition of a Verity double bicategory:

- By assumption, **Hor** is a bicategory.
- **Ver** is a bicategory since the source and target functors in **D** for **Ver** satisfy all the usual axioms for a bicategory, hence their object maps do also. Similarly, the composition maps have natural isomorphisms giving associators and unitors: they are just object maps of functors which satisfy the same conditions: in **D**, the associator a satisfies the pentagon identity. The object maps for a give the associator in **Ver**. Since the associator 2-natural transformation satisfies the pentagon identity, so do these object maps. The other properties are shown similarly, so that **Ver** is a bicategory.
- The source and target maps for **Squ** satisfies equations (15) because the source and target maps of **D** are functors.
- The composition lawsfor squares have the usual relation to source and target maps because, by assumption, **Mor** is a bicategory, but taking **Squ** to be 2-isomorphism classes of morphisms in **Mor**, and disregarding all other 2-morphisms, we get that horizontal composition in **Squ** is exactly associative and has exact identities, so the squares are the morphisms of a category with respect to horizontal composition.
  Vertical composition for squares in **D** satisfies the axioms for a bicategory by the same argument as given above for **Ver**, since it is the morphism map for the functor $\circ$. In particular, it has an associator and a unitor: but these must be morphisms in **2Mor** since we take the morphism maps from the associator and unitor functors for $\circ$. These are 2-isomorphisms, but since we defined squares to be 2-isomorphism classes (any isomorphism in **2Mor** becomes an equation), this composition is exactly associative and has a unit. Also, we disregard any morphisms in **2Mor**, so the squares are the morphisms of a category under vertical composition.
- The interchange rule (18) follows from functoriality of the composition functors.
- The actions $\star_H$ and $\star_V$ defined by the horizontal and vertical action conditions is well defined. In particular, by composition of in **Mor** or **2Mor**, we guarantee the existence of the categories of horizontal and vertical cylinders $\mathbf{Cyl_H}$ and $\mathbf{Cyl_V}$, respectively. These come from the 2-morphisms in **Mor** or morphisms in **2Mor** respectively which those conditions demand must exist. Taking these to be identities, the cylinders consist of commuting cylindrical diagrams with two bigons and two squares.
  In the case where one bigon is the identity, and the other is any bigon α, the conditions guarantee the existence of an invertible cylinder, which is now the identity because we have taken squares to be isomorphism classes. This defines the effect of the action of α on the square whose source is the target of α. If this square is F, we denote the other square $\alpha \star_H F$ or $\alpha \star_V F$ as appropriate.

- The horizontal action condition gives a well-defined action satisfies (22) and (23) by an argument exactly analogous to that in the proof of Proposition 3.3.2. That is, the horizontal action condition means that certain fillers are unique. When they can be obtained in two ways, these are equal.
- The vertical action satisfies the vertical equivalent of (22) and (23) for the same reason.
- The condition (25) guaranteeing independence of the horizontal and vertical actions follows from the action compatibility condition. For suppose we have a square F whose horizontal and vertical source arrows are the targets of 2-cells α and β, and attach to its opposite faces two identity 2-cells. Then the horizontal and vertical action conditions mean that there will be a square $\alpha \star_H F$ and a square $\beta \star_V F$). Then the action compatibility condition applies (the P_i are the identities we get from the action condition), and there is a morphism in **Mor**, namely a square in **V** and a 2-cell $T \in$ **2Mor**. Consider the remaining face, which the action condition suggests we call $\alpha \star_H (\beta \star_V F)$ or $\beta \star_V (\alpha \star_H F)$, depending on the order in which we apply them. The compatibility condition says that there is a unique square which fills this spot so the two must be equal.
- We next check that composition for squares agrees with composition as in (26). Suppose we have three composable squares—that is, morphisms F, G, and H in **Mor**, which are composable along shared source and target objects in **Mor**. The associator functor has an object map, giving objects in **2Mor** at the "top" and "bottom" of the squares. It also has a morphism map, giving morphisms in **2Mor**. But by assumption there is only a unique such map between , these associators must be the unique morphism in **2Mor** with source $(H \circ G) \circ F$ and target $H \circ (G \circ F)$. Then by the vertical action condition, we have a filler 2-morphism in **Mor** for the action on the composite square by the top associator, and then, taking the result and composing with the bottom associator, we get another filler. This must be the unique map between the two composites, which is the identity since they have the same sources and targets. So we get a commuting cylinder. Composing squares along source and target morphisms in **Obj** works the same way by a symmetric argument.
- The condition (27) is similar. The unitor functor will give the unique morphism in **2Mor**, and the action compatibility condition gives the commuting cylinder for unitors on the composite of squares.

So indeed the construction of **V(D)** defines a Verity double bicategory. □

Next, in Section 3.4, we continue the process of reducing the complexity of these structures. In particular, we see how Verity double bicategories can give rise to ordinary bicategories, which are frequently easier to use.

3.4. Bicategories from Double Bicategories

It is well known that double categories can yield 2-categories in three different ways. Two obvious cases are when there are only identity horizontal morphisms, or only identity vertical morphisms, so that squares simply collapse into bigons with the two nontrivial sides. Notice that it is also true that a Verity double bicategory in which **Hor** is trivial (equivalently, if **Ver** is trivial) is again a bicategory. The squares become 2-morphisms in the obvious way, the action of 2-morphisms on squares is then just composition, and the composition rules for squares in the double category become the rules for composing 2-morphisms, and the result is clearly a bicategory.

The other, less obvious, case, is when the horizontal and vertical categories on the objects are the same: this is the case of *path-symmetric* double categories, and the recovery of a bicategory was shown by Brown and Spencer [**BS**]. Fiore [**Fi**] shows how their demonstration of this fact is equivalent to one involving *folding structures*.

In this case we can interpret squares as bigons by composing the top and right edges, and the left and bottom edges. Introducing identity bigons completes the structure. These new bigons have a natural composition inherited from that for squares. It turns out that this yields a bicategory. Here, our goal will be to show half of an analogous result, that a Verity double bicategory similarly gives rise to a bicategory when the horizontal and vertical bicategories are equal. We will also show that a double bicategory for which the horizontal (or vertical) bicategory is trivial can be seen as a bicategory. The condition that **Hor** = **Ver** will hold in our general example of double cospans.

Theorem 3.4.1. *Any Verity double bicategory*

$$\mathbf{V} = (\mathrm{Obj}, \mathbf{Hor}, \mathbf{Ver}, \mathbf{Squ}, \otimes_H, \otimes_V, \star_H, \star_V)$$

for which **Hor** = **Ver** *produces a bicategory* **B** *by taking the 2-morphisms to be 2-morphisms in* **Hor** *and squares in* **Squ**.

Proof. We begin by defining the data of **B**. Its objects and morphisms are the same as those of **Hor** (equivalently, **Ver**). We describe the 2-morphisms by observing that **B** must contain all those in **Hor** (equivalently, **Ver**), but also some others, which correspond to the squares in **Squ**.

In particular, given a square

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Searrow S"] & b \arrow[d, "g'"] \\ c \arrow[r, "f'"'] & d \end{tikzcd} \tag{33}$$

there should be a 2-morphism

$$\begin{tikzcd} a \arrow[r, bend left, "g'\circ f"] \arrow[r, bend right, "f'\circ g"'] \arrow[r, phantom, "\Downarrow S"] & d \end{tikzcd} \tag{34}$$

The composition of squares corresponds to either horizontal or vertical composition of 2-morphisms in **B**, and the relation between these two is given in terms of the interchange law in a bicategory:

Given a composite of squares,

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\phi_x"'] \arrow[dr, phantom, "\Downarrow F"] & y \arrow[r, "g"] \arrow[d, "\phi_y"] \arrow[dr, phantom, "\Downarrow G"] & z \arrow[d, "\phi_z"] \\ x' \arrow[r, "f'"'] & y' \arrow[r, "g'"'] & z' \end{tikzcd} \tag{35}$$

there will be a corresponding diagram in **B**:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right=60, "\phi_x \circ f'"'] & y \arrow[r, "\phi_y"] \arrow[rr, bend left=60, "\phi_z \circ g"] \arrow[d, phantom, "\Downarrow F"] & y' \arrow[r, "g'"] \arrow[u, phantom, "\Downarrow G"] & z' \end{tikzcd} \tag{36}$$

Using horizontal composition with identity 2-morphisms ("whiskering"), we can write this as a vertical composition:

$$\begin{tikzcd} x \arrow[r, bend left=50, "\phi_z \circ g \circ f"] \arrow[r, "g' \circ \phi_y \circ f"' description] \arrow[r, bend right=50, "g' \circ f' \circ \phi_x"'] & z' \end{tikzcd} \qquad \Downarrow G \circ 1_f, \quad \Downarrow 1_{g'} \circ F \tag{37}$$

So the square $F \otimes_H G$ corresponds to $(1 \circ G) \cdot (F \circ 1)$ for appropriate identities 1. Similarly, the vertical composite of $F' \otimes_V G'$ must be the same as $(1 \circ F) \cdot (G \circ 1)$. Thus, every composite of squares which can be built from horizontal and vertical composition, gives a corresponding composite of 2-morphisms in **B**, which are generated by those corresponding to squares in **Squ**, subject to the relations imposed by the composition rules in a bicategory.

Now we want to show that Verity double bicategory **V** gives the entire bicategory **B**. That is, that **B** has no other 2-morphisms than those which arise by the above process. It suffices to show that all such 2-morphisms not already in **Hor** arise as squares (that is, the structure is closed under composition). So suppose we have any composable pair of 2-morphisms which arise from squares F and G. If F and G have an edge in common, then we have the situation depicted above (or possibly the corresponding form in the vertical direction). In this case, the composite 2-morphism corresponds exactly to the composite of squares, and the axioms for composition of squares ensure that all 2-morphisms generated this way are already in our bicategory. In particular, the unit squares become unit 2-morphisms when composed with left and right unitors.

Now, if there is no edge in common to two squares, the 2-morphisms in **B** must be made composable by whiskering with identities. In this case, all the identities can be derived from 2-morphisms in **Hor**, or from identity squares in **Squ** (inside commuting diagrams). Clearly, any identity 2-morphism can be factored this way. Then, again, the composite 2-morphisms in $\mathbf{B}$ will correspond exactly to the composite of all such squares in **Squ** and 2-morphisms **Hor**.

Finally, the associativity condition (26) for the action of 2-morphisms on squares ensures that composition of squares agrees with that for 2-morphisms, so there are no extra squares from composites of more than two squares. $\square$

Remark 3.4.2. When producing the bicategory $\mathbf{B}$ from $\mathbf{V}$, we made a particular choice of orientation for the 2-morphisms obtained from squares. The square $S \in$ **Squ** shown in (33) has vertical source f and target f', and horizontal source g and target g'. However, the corresponding 2-morphism $S \in \mathbf{B}$ has source $g' \circ f$, which combines vertical source and horizontal target; on the other hand, the target of $S \in \mathbf{B}$ is $f' \circ g$, combining vertical target and horizontal source. We could equally well have chosen the opposite convention. This would give $\mathbf{B}^{\mathrm{co}}$, which is $\mathbf{B}$ with the orientation of its 2-morphisms reversed. (See, e.g. [**Le**]).

It is also worth considering here the situation of a double bicategory in which all horizontal morphisms and 2-morphisms are identities. In this case, one can define a 2-morphism from a square with and bottom edges being identities, whose source is the object whose identity is the corresponding edge, and similarly for the target. The composition rules for squares in the vertical direction, then, are just the same as those for a bicategory. Likewise, the axioms for action of a 2-morphism on a square reduce to the composition laws for a bicategory if one replaces the square by a 2-cell.

Next we describe a broad class of examples of double bicategories, in the spirit of the use of spans to give examples of bicategories.

4. Double Cospans

In Remark 2.1.5 we described Bénabou's demonstration that $\mathrm{Span}(\mathbf{C})$ is a bicategory for any category $\mathbf{C}$ with pullbacks. Similarly, there is a bicategory of cospans in a category $\mathbf{C}$ with pushouts. There will be an analogous fact giving a double bicategory of double spans. In fact, we describe this in terms of double *cospans*, since our aim in a subsequent paper will be to use these to describe cobordisms, which have a natural description as cospans. Since cospans in $\mathbf{C}$ are the same as spans in the opposite category, $\mathbf{C}^{\mathrm{op}}$, this distinction is a matter of taste.

We remark here that similar constructions are described by Grandis [**Gr3**], and related "profunctor-based examples" of pseudo-double categories are described by Grandis and Paré [**GP2**].

4.1. The Double Cospan Example

We begin by defining a double bicategory of double cospans:

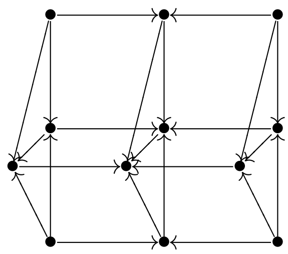
Definition 4.1.1. $2\mathrm{Cosp}(\mathbf{C})$ *is a double bicategory of* ***double cospans*** *in* $\mathbf{C}$*, consisting of the following:*

- *the bicategory of objects is* $\mathbf{Obj} = \mathrm{Cosp}(\mathbf{C})$
- *the bicategory of morphisms* **Mor** *has:*

– *as objects, cospans in* **C***;*
– *as morphisms, commuting diagrams of the form*

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & S \arrow[d] & X_2 \arrow[l] \arrow[d] \\
T_1 \arrow[r] & M & T_2 \arrow[l] \\
X_1' \arrow[r] \arrow[u] & S' \arrow[u] & X_2' \arrow[l] \arrow[u]
\end{tikzcd} \tag{38}$$

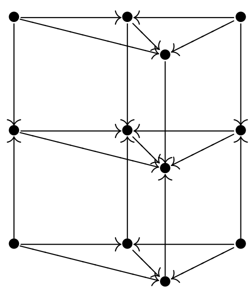
(in subsequent diagrams we suppress the labels for clarity);
– *as 2-morphisms, cospans of cospan maps, namely commuting diagrams of the following shape:*



(39)

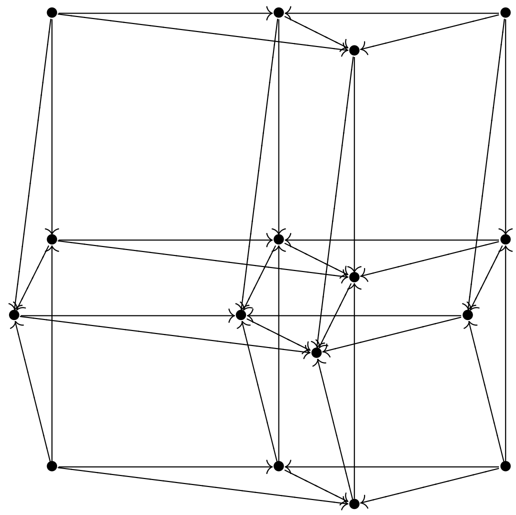
- *the bicategory of 2-morphisms has:*

– *as objects, cospan maps in* **C** *as in (8)*
– *as morphisms, cospan maps of cospans:*



(40)

– *as 2-morphisms, cospan maps of cospan maps:*



(41)

All composition operations are by pushout; source and target operations are the same as those for cospans. The associators and unitors in the horizontal and vertical bicategories are the maps which come from the universal property of pushouts.

Remark 4.1.2. Just as 2-morphisms in **Mor** and morphisms in **2Mor** can be seen as diagrams which are "products" of a cospan with a map of cospans, 2-morphisms in **2Mor** are given by diagrams which are products (as diagrams) of horizontal and vertical cospan maps. These have, in either direction, four maps of cospans, with objects joined by maps of cospans. Composition again is by pushout in composable pairs of diagrams.

Note that all these diagrams are products of smaller diagrams, each of which is either a cospan, or a cospan map. This suggests that the horizontal and vertical directions should in some way behave like a bicategory of cospans. The next theorem shows this is indeed the case:

Theorem 4.1.3. *For any category* **C** *with pushouts,* 2Cosp(**C**) *forms a double bicategory.*

Proof. **Mor** and **2Mor** are bicategories since the composition functors act just like composition in Cosp(**C**), the bicategory of cospans in **C**, in each column, and therefore satisfies the same axioms.

Now, the horizontal and vertical directions have composition operations defined in the same way. Thus we can construct functors between **Obj**, **Mor**, and **2Mor** with the properties of a bicategory simply by using the same constructions that turn each into a bicategory in its own right. In particular, the source and target maps $s, t : \mathbf{Mor} \to \mathbf{Obj}$ and $s, t : \mathbf{2Mor} \to \mathbf{Mor}$ are the obvious maps giving the domains of the maps in (38). The partially defined (horizontal) composition maps $\circ : \mathbf{Mor}^2 \to \mathbf{Mor}$ and $\otimes_H : \mathbf{2Mor}^2 \to \mathbf{2Mor}$ are defined by taking pushouts of diagrams in **C**, which exist for any composable pairs of diagrams because **C** has pushouts. They are functorial since they are independent of composition in the horizontal direction. The associator for composition of morphisms is given in the pushout construction.

To see that this construction gives a double bicategory, we note that **Obj**, **Mor**, and **2Mor** as defined above are indeed bicategories. Certainly, **Obj** is a bicategory because Cosp(**C**) is a bicategory. **Mor** and **2Mor** are bicategories because the morphism and 2-morphism maps from the composition, associator, and other functors required for a double bicategory give them the structure of bicategories as well.

Moreover, the composition functors satisfy the properties of a bicategory for just the same reason that composition of cospans (and spans) does, since each of the three maps involved are given by this construction. Thus, we have a double bicategory. □

Our motivation for Theorem 4.1.3 is to show that double cospans in suitable categories **C** give examples of Verity double bicategories. We have described how to get a double bicategory of such structures, and we saw in Section 3.3 that given certain conditions, this gives a Verity double bicategory. In Section 4.2 we describe explicitly the modifications we must make to Cosp(**C**) to get these conditions.

4.2. A Verity Double Bicategory of Double Cospans

As we saw in Section 3.3, double bicategories have higher morphisms of dimension up to 4, but given certain conditions, these can be omitted to give a Verity double bicategory.

Definition 4.2.1. *For a category* **C** *with pushouts, the Verity double bicategory* $2\mathrm{Cosp}(\mathbf{C})_0$*, has:*

- *the objects are objects of* **C**
- *the horizontal and vertical bicategories* **Hor** = **Ver** *are both equal to a sub-bicategory of* Cosp(**C**)*, which includes only invertible cospan maps*
- *the squares are isomorphism classes of commuting diagrams of the form (38)*

where two diagrams of the form (38) are isomorphic if they differ only in the middle objects, say M and M', and the maps into these objects, and if there is an isomorphism $f : M \to M'$ making the combined diagram commute.

The action of 2-morphisms α in **Hor** *and* **Ver** *on squares is by composition in diagrams of the form:*

$$\begin{tikzcd}
 & \hat{S} \arrow[d, "\alpha"] & \\
X_1 \arrow[ur] \arrow[r] \arrow[d] & S \arrow[d, "s"] & X_2 \arrow[ul] \arrow[l] \arrow[d] \\
T_1 \arrow[r, "t_1"] & M & T_2 \arrow[l, "t_2"'] \\
X_1' \arrow[u] \arrow[r] & S' \arrow[u, "s'"] & X_2' \arrow[u] \arrow[l]
\end{tikzcd} \qquad (42)$$

(where the resulting square is as in 38, with $\hat{S}$ in place of S and $s \circ \alpha$ in place of s).

Composition (horizontal or vertical) of squares of cospans is, as in $2\mathrm{Cosp}(\mathbf{C})$*, given by composition (by pushout) of the three cospans of which the square is composed. The composition for diagrams of cospan maps are as usual in* $\mathrm{Cosp}(\mathbf{C})$*.*

Remark 4.2.2. Notice that **Hor** and **Ver** as defined are indeed bicategories: eliminating all but the invertible 2-morphisms leaves a collection which is closed under composition by pushouts.

We will show more fully that this is a Verity double bicategory in Theorem 4.2.4. First one must show that horizontal and vertical composition of squares is well defined is defined on equivalence classes. We will get this result indirectly as a result of Theorems 4.1.3 and 3.3.9, but it is instructive to see directly how this works in $\mathrm{Cosp}(\mathbf{C})$.

Theorem 4.2.3. *In any category with pushouts, composition of squares in Definition 4.2.1 is well-defined.*

Proof. Suppose we have two representatives of a square, bounded by horizontal cospans $X_1 \to S \leftarrow X_2$ and $X_1' \to S' \leftarrow X_2'$, and vertical cospans $X_1 \to T_1 \leftarrow X_1'$ and $X_2 \to T_2 \leftarrow X_2'$. Suppose the middle objects are M and $\hat{M}$ as in the diagram (38). Given a composable diagram which coincides along an edge (morphism in **Hor** or **Ver**) with the first, we need to know that the two pushouts of the different representatives are also isomorphic (that is, represent the same composite square).

In the horizontal and vertical composition of these squares, the maps to the middle object M of the new square from the middle objects of the new sides (given by composition of cospans) arise from the universal property of the pushouts on the sides being composed (and the induced maps from M to the corners, via the maps in the cospans on the other sides). Since the middle objects are defined only up to isomorphism class, so is the pushout: so the composition is well defined, since the result is again a square of the form (38). $\square$

We use this, together with Theorems 3.3.9 and 4.1.3, (proved in Section 3), to show the following:

Theorem 4.2.4. *If* $\mathbf{C}$ *is a category with pushouts, then* $2\mathrm{Cosp}(\mathbf{C})_0$ *is a Verity double bicategory.*

Proof. In the construction of $2\mathrm{Cosp}(\mathbf{C})_0$, we take isomorphism classes of double cospans as the squares. We also restrict to invertible cospan maps in the horizontal and vertical bicategories.

That is, take 2-isomorphism classes of morphisms in **Mor** in the double bicategory, where the 2-isomorphisms are invertible cospan maps, in both horizontal and vertical directions. We are then effectively discarding all non-invertible morphisms and 2-morphisms in **2Mor**, and all non-invertible 2-morphisms in **Mor**. In particular, there may be "squares" of the form (38) in $2\mathrm{Cosp}(\mathbf{C})$ with non-invertible maps joining their middle objects M, but we have ignored these, and also ignore non-invertible cospan maps in the horizontal and vertical bicategories. Thus, we consider no diagrams of the form (39) except those in which the span maps are invertible, in which case the middle objects are representatives of the same isomorphism class. Similar reasoning applies to the 2-morphisms in **2Mor**.

The structure we get from discarding these will again be a double bicategory. In particular, the new **Mor** and **2Mor** will be bicategories, since they are, respectively, just a category and a set made into a discrete bicategory by adding identity morphisms or 2-morphisms as needed. On the other hand, for the composition, source and target maps to be bifunctors, the structures built from the objects, morphisms, and 2-cells respectively must be bicategories. This is since the composition, source, and target maps are the object, morphism, and 2-morphism maps of these bifunctors, which satisfy the usual category axioms. But the same argument applies to those built from the morphisms and 2-cells as within **Mor** and **2Mor**. So we have a double bicategory.

Next we show that the horizontal and vertical action conditions (Definition 3.3.4 of Section 3.3) hold in $2\mathrm{Cosp}(\mathbf{C})$. A square in $2\mathrm{Cosp}(\mathbf{C})$ is a diagram of the form (38), and a 2-cell is a map of cospans. Given a square M and 2-cell α with compatible source and targets as in the action conditions, we have a diagram of the form shown in (42). Here, M is the square diagram at the bottom, whose top row is the cospan containing S. The 2-cell α is the cospan map including the arrow $\alpha : \hat{S} \to S$. There is a unique square built using the same objects as M, but using the cospan containing $\hat{S}$ as the top row. The map from $\hat{S}$ to M is then $s \circ \alpha$.

To satisfy the action condition, we want this square $\hat{M}$, which is the candidate for $M_1 \star_V \alpha$, to be unique. But suppose there were another $\hat{M}_2$ with a map from $\hat{S}$. Since we are in $2\mathrm{Cosp}(\mathbf{C})_0$, α must be invertible, which would give a map to $\hat{M}_2$ from S. We then find that $\hat{M}_2$ and $\hat{M}$ are representatives of the same isomorphism class, so in fact this is the same square. That is, there is a unique morphism in **2Mor** taking M to $\hat{M}$ (a diagram of the form 40) with invertible cospan maps in the middle and bottom rows. This is the unique filler for the pillow diagram required by definition 3.3.4.

The argument that $2\mathrm{Cosp}(\mathbf{C})_0$ satisfies the action compatibility condition is similar.

So $2\mathrm{Cosp}(\mathbf{C})_0$ is a double bicategory in which, there there is at most one unique morphism in **Mor**, and at most unique morphisms and 2-morphisms in **2Mor**, for any specified source and target, and the horizontal and vertical action conditions hold. So $2\mathrm{Cosp}(\mathbf{C})_0$ can be interpreted as a Verity double bicategory (Theorem 3.3.9). □

Remark 4.2.5. We observe here that the compatibility condition (26) relating the associator in the horizontal and vertical bicategories to composition for squares is due to the fact that the associators are maps which come from the universal property of pushouts. This is by the parallel argument to that we gave for spans in Remark 2.1.5. The same argument applies to the middle objects of the squares, and gives associator isomorphisms for that composition. When we reduce to isomorphism classes, these isomorphisms become identity maps, so we get a commuting pillow as in (26). A similar argument shows the compatibility condition for the unitor, (27).

It is interesting to note how the arguments in the proof of Theorem 3.3.9 apply to the case of $2\mathrm{Cosp}(\mathbf{C})$.

In particular, the interchange rules hold because the middle objects in the four squares being composed form the vertices of a new square. The pushouts in the vertical and horizontal direction form the middle objects of vertical and horizontal cospans over these. The interchange law means that the pushout (in the horizontal direction) of the objects from the vertical cospans is in the same isomorphism class as the pushout (in the vertical direction) of the objects from the horizontal cospans. This follows from the universal property of the pushout.

4.3. Example: Cobordisms with Corners

One important example of a category of cospans involves cobordism of manifolds, although to realize this example requires some additional structure. In particular, the category $\mathbf{nCob}_2$ of cobordisms with corners is not $2\mathrm{Cosp}(\mathbf{C})$ for a category $\mathbf{C}$ with pushouts, since the objects of this category are manifolds, and **Man** does not have pushouts.

Recall that two manifolds S_1, S_2 are *cobordant* if there is a compact manifold with boundary, M, such that ∂M is isomorphic to the disjoint union of S_1 and S_2. This M is called a *cobordism* between S_1 and S_2. So in particular, a cobordism is a cospan $S_1 \rightarrow M \leftarrow S_2$, where both maps are inclusions of the boundary components. A cobordism with corners is then a manifold with corners, where the boundary components and corners are included in a double cospan.

In particular, for topological cobordisms (i.e. cobordisms which are topological manifolds with boundary), all the pushouts required to compose such double cospans will still be topological manifolds. For smooth manifolds, to ensure that the result of gluing is smooth we need to specify an additional condition, using "collars" on the boundaries and corners.

4.3.1. Cobordisms and Collars

To begin with, recall that a smooth manifold with corners is a topological manifold with boundary, together with a maximal compatible set of coordinate charts $\phi : \Omega \rightarrow [0,\infty)^n$ - into the positive sector of $latex\mathbb{R}^n$. (where ϕ_1, ϕ_2 are compatible if $\phi_2 \circ \phi_1^{-1}$ is a diffeomorphism).

Jänich [Ja] introduces the notion of $\langle n \rangle$-manifold, reviewed by Laures [Lau]. This is build on a manifold with corners, using the notion of a *face*:

Definition 4.3.1. *(**Jänich**)A **face** of a manifold with corners is the closure of some connected component of the set of points with exactly one zero component in any coordinate chart. An $\langle n \rangle$-manifold is a manifold with faces together with an n-tuple $(\partial_0 M, \ldots, \partial_{n-1} M)$ of faces of M, such that*

- $\partial_0 M \cup \ldots \partial_{n-1} M = \partial M$
- $\partial_i M \cap \partial_j M$ *is a face of* $\partial_i M$ *and* $\partial_j M$

The case we will be interested in here is the case of $\langle 2 \rangle$-manifolds. In this notation, a $\langle 0 \rangle$-manifold is just a manifold without boundary, a $\langle 1 \rangle$-manifold is a manifold with boundary, and a $\langle 2 \rangle$-manifold is a manifold with corners whose boundary decomposes into two components (of codimension 1), whose intersections form the corners (of codimension 2). We can think of $\partial_0 M$ and $\partial_1 M$ as the "horizontal" and "vertical" part of the boundary of M.

Now, for a point $x \in S$, there will be a neighborhood U of x which restricts to $U_1 \subset M_1$ and $U_2 \subset M_2$ with smooth maps $\phi_i : U_i \to [0,\infty)^n$ with $\phi_i(x)$ on the boundary of $[0,\infty)^n$ with exactly one coordinate equal to 0. One can easily combine these to give a homeomorphism $\phi : U \to \mathbb{R}^n$, but this will not necessarily be a diffeomorphism along the boundary S. While *topological* cobordisms can be composed along their boundaries, *smooth* cobordisms M_1 and M_2 should be composed differently, to ensure that every point—including points on the boundary of M_i—will have a neighborhood with a smooth coordinate chart. To solve this problem, we use *collars*, which is also done in the category **nCob**.

The *collaring theorem* says that for any smooth manifold with boundary M, ∂M has a *collar*: an embedding $f : \partial M \times [0,\infty) \to M$, with $(x,0) \mapsto x$ for $x \in \partial M$. This is a well-known result (for a proof, see e.g. [**Hi**], sec. 4.6). It is an easy corollary that we can choose to use the interval $[0,1]$ in place of $[0,\infty)$ here.

Laures ([**Lau**], Lemma 2.1.6) describes a generalization of this theorem to $\langle n\rangle$-manifolds, so that for any $\langle n\rangle$-manifold M, there is an n-dimensional cubical diagram ($\langle n\rangle$-*diagram*) of embeddings of cornered neighborhoods of the faces. It is then standard that one can compose two smooth cobordisms with corners, equipped with such smooth collars, by gluing along S. The composite is then the topological pushout of the two inclusions. Along the collars of S in M_1 and M_2, charts $\phi_i : U_i \to [0,\infty)^n$ are equivalent to charts mapping into $\mathbb{R}^{n-1} \times [0,\infty)$, and since the the composite has a smooth structure defined up to a diffeomorphism which is the identity along S. The precise smooth structure on this cobordism depends on the collar chosen, but there is always such a choice, and the resulting composites are all equivalent up to diffeomorphism.

Now, for each n, one can define:

Definition 4.3.2. *The bicategory* $\mathbf{nCob}_2$ *is given by the following data:*

- *The objects of* $\mathbf{nCob}_2$ *are of the form* $P = \hat{P} \times I^2$ *where* $\hat{P}$ *may be any* $(n-2)$ *manifolds without boundary and* $I = [0,1]$.
- *The morphisms of* $\mathbf{nCob}_2$ *are cobordisms* $P_1 \xrightarrow{i_1} S \xleftarrow{i_2} P_2$ *where* $S = \hat{S} \times I$ *and* $\hat{S}$ *is an* $(n-1)$*-dimensional collared cobordism with corners such that: the* $\hat{P}_i \times I$ *are objects, the maps are injections into* S*, a manifold with boundary, such that* $i_1(P_1) \cup i_2(P_2) = \partial S \times I$, $i_1(P_1) \cap i_2(P_2) = \emptyset$,
- *The 2-morphisms of* $\mathbf{nCob}_2$ *are generated by:*
  - *diffeomorphisms of the form* $f \times \mathrm{id} : T \times [0,1] \to T' \times [0,1]$ *where* T *and* T' *have a common boundary, and* f *is a diffeomorphism*$T \to T'$ *compatible with the source and target maps, i.e. fixing the collar.*
  - *2-cells: diffeomorphism classes of* n*-dimensional manifolds* M *with corners satisfying the properties of* M *in the diagram of equation (38), where isomorphisms are diffeomorphisms preserving the boundary*

  where the composite of the diffeomorphisms with the 2-cells (classes of manifolds M*) is given by composition of diffeomorphisms of the boundary cobordisms with the injection maps of the boundary* M

The source and target objects of any cobordism S are specified by saying that the source of S is the collection of components of $\partial S \times I$ for which the image of $(x, 0)$ lies on the boundary for $x \in \partial S$, and the target has the image of $(x, 1)$. The source and target objects are the collars, embedded in the cobordism in such a way that the source object $P = \hat{P} \times I^2$ is embedded in the cobordism $S = \hat{S} \times I$ by a map which is the identity on I taking the first interval in the object to the interval for a horizontal morphism, and the second to the interval for a vertical morphism. The same condition distinguishing source and target applies as above.

Composition of 2-cells works by gluing along common boundaries.

Lemma 4.3.3. *Composition of morphisms and 2-morphisms in* $\mathbf{nCob}_2$ *is well-defined and* $\mathbf{nCob}_2$ *is closed under composition.*

Proof. The horizontal and vertical morphisms are products of the interval I with $\langle 1 \rangle$-manifolds, whose boundary is $\partial_0 S$), equipped with collars. Suppose we are given two such cobordisms S_1 and S_2, and an identification of the source of S_2 with the target of S_1 (say this is $P = \hat{P} \times I$). Then the composite $S_2 \circ S_1$ is topologically the pushout of S_1 and S_2 over P. Now, P is smoothly embedded in S_1 and S_2, and any point in the pushout will be in the interior of either S_1 or S_2 since for any point on $\hat{P}$ each end of the interval I occurs as the boundary of only one of the two cobordisms. So the result is smooth. Thus, **2Cob** is closed under such composition of morphisms.

The same argument holds for 2-cells, since it holds for any representative of the equivalence class of some manifold with corners, M, and the differentiable structure will be the same, since we consider equivalence up to diffeomorphisms which preserve the collar exactly. ☐

Examples of such cobordisms with corners in 2 and 3 dimensions, as illustrations of (38), are shown in Figures 1 and 2, respectively.

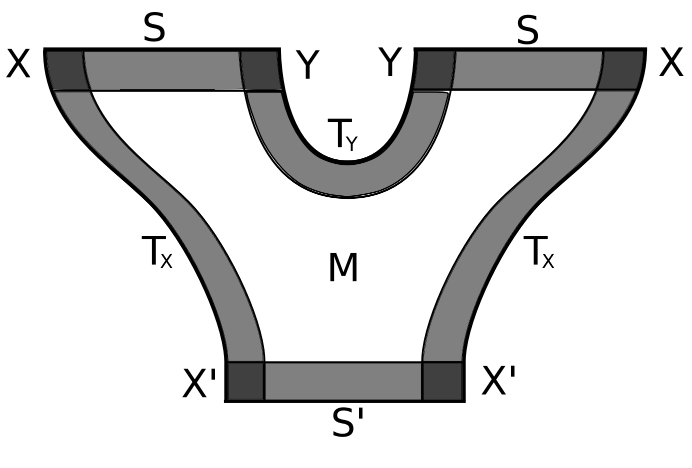


Figure 1: A 2-Dimensional Cobordism with Corners

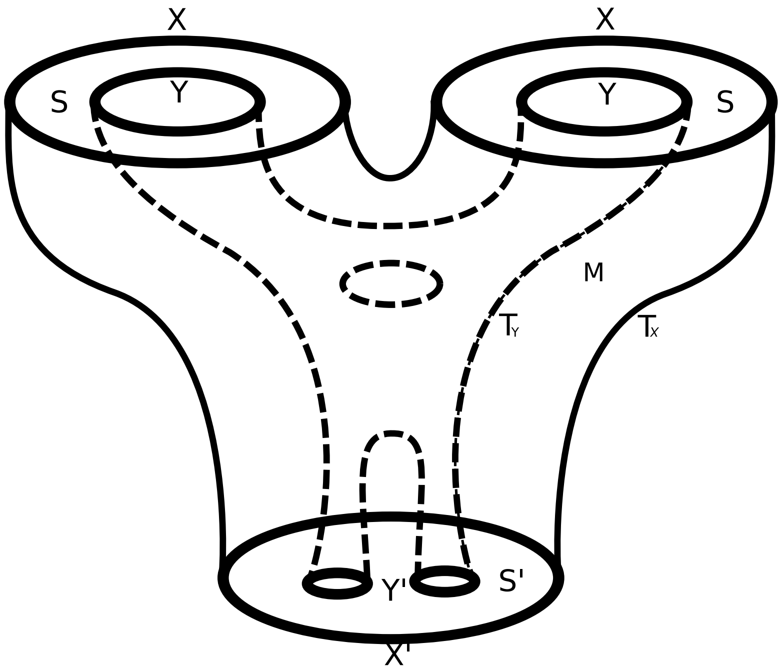


Figure 2: A 3-Dimensional Cobordism with Corners

4.4. Prospects for n-tuple Bicategories

We have discussed both double bicategories and Verity double bicategories, both of which can be seen as forms of *weak cubical* n-category. The broader question of various definitions of weak n-category is discussed in more detail by Tom Leinster [**Le**], and by Eugenia Cheng and Aaron Lauda [**CL**]. In light of this context, Theorem 4.1.3 suggests one direction of generalization for double bicategories, to "n-tuple bicategories" for any n. Moreover, our example of double (co)spans can be generalized to arbitrarily high dimension.

We have seen how to construct 2Cosp($\mathbf{C}$) for a category $\mathbf{C}$ with pushouts, and how we take a restricted form of this construction to yield a Verity double bicategory. We have chosen to stop the process of taking cospans in a category of cospans after two steps, but we could continue this construction. Taking cospans in this new category gives cubes of objects with maps from corners to the middles of edges, from middles of edges to middles of faces, and from middles of faces to the middle of the cube. Similarly, for any finite n, we can iterate the process of taking cospans to yield an n-dimensional cube.

In particular, we note that "Verity double bicategories" arise from special examples of bicategories internal to **Bicat**. There is a category of all such structures, namely the functor category of all maps $F : Th(\mathbf{Bicat}) \to \mathbf{Bicat}$, denoted $[Th(\mathbf{Bicat}), \mathbf{Bicat}]$. There will be an analogous concept of "triple bicategories", namely bicategories internal to $[Th(\mathbf{Bicat}), \mathbf{Bicat}]$. In general, a "$k$-tuple bicategory" will be a bicategory internal to the category of weak $(k-1)$-tuple categories.

We expect that for all k, a k-tuply iterated process of taking cospans of cospans (or similarly for spans) will yield examples of these structures. These k-dimensional (co)spans will naturally form a weak k-tuple category. Marco Grandis [**Gr3**] describes this in terms of a somewhat different description of weak n-cubical categories.

A further direction of generalization would be to substitute tricategories, tetracategories, and so forth in place of bicategories in the preceding construction, perhaps making different choices each stage. The question then arises what sort of structures it would be possible to define by selectively decategorifying, and what sorts of "filler" conditions this would need. Another potentially interesting question is whether the examples based on cospans also generalize—perhaps by taking cospans, not in a category, but in an n-category.

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