

EMBEDDING THE FLAG REPRESENTATION IN DIVIDED POWERS

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Abstract

A generalization of a theorem of Crabb and Hubbuck concerning the embedding of flag representations in divided powers is given, working over an arbitrary finite field \mathbb{F} , using the category of functors from finite-dimensional \mathbb{F} -vector spaces to \mathbb{F} -vector spaces.

1. Introduction

Let V be a finite-dimensional \mathbb{F} -vector space over a finite field \mathbb{F} ; the flag variety of complete flags of length r in V induces a permutation representation $\mathbb{F}[\mathfrak{F}(\mathbf{a}g_r)](V)$ of the general linear group $GL(V)$, which is of interest in representation theory. The notation is derived from the fact that the flag representation arises as the evaluation on the space V of a functor $\mathbb{F}[\mathfrak{F}(\mathbf{a}g_r)]$ in the category \mathcal{F} of functors from finite-dimensional \mathbb{F} -vector spaces to \mathbb{F} -vector spaces; similarly, the divided power functors Γ^n induce $GL(V)$ -representations $\Gamma^n(V)$. Motivated by questions from algebraic topology (and working over the prime field \mathbb{F}_2), Crabb and Hubbuck [1, 3] associated to a sequence \underline{s} of integers, $s_1 \geq \dots \geq s_r > s_{r+1} = 0$, a morphism

$$\mathbb{F}_2[\mathfrak{F}(\mathbf{a}g_r)](V) \rightarrow \Gamma^{[\underline{s}]^2}(V) \quad (1)$$

or $GL(V)$ -modules, where $[\underline{s}]_q$ is the integer $\sum_{i=1}^r (q^{s_i} - 1)$. The motivating observation of this paper is that this is defined globally as a natural transformation

$$\phi_{\underline{s}} : \mathbb{F}[\mathfrak{F}(\mathbf{a}g_r)] \rightarrow \Gamma^{[\underline{s}]_q} \quad (2)$$

in the functor category \mathcal{F} and for each finite field \mathbb{F} , where $q = |\mathbb{F}|$.

The Crabb-Hubbuck morphism $\phi_{\underline{s}}$ arises in the construction of the ring of lines (developed independently in the dual situation by Repka and Selick [8]). Namely, the divided power functors form a commutative graded algebra in \mathcal{F} and the ring of lines is the graded sub-functor generated by the images of the morphisms $\phi_{\underline{s}}$, which forms a sub-algebra of Γ^* .

The ring of lines is of interest in relation to the study of the primitives under the action of the Steenrod algebra on the singular homology $H_*(BV; \mathbb{F}_2)$ of the

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classifying space of V ; the primitives arise in a number of questions in algebraic topology. This relation can be explained from the point of view of the functor category \mathcal{F} as follows, by identifying $H_*(BV; \mathbb{F}_2)$ with the graded vector space $\Gamma^*(V)$. Steenrod reduced power operations correspond to natural transformations of the form $\Gamma^a \rightarrow \Gamma^b$, $a \geq b$ (see [6]). Define the Steenrod kernel functors K^a by:

$$K^a := \text{Ker}\{\Gamma^a \rightarrow \bigoplus_{f \in \text{Hom}(\Gamma^a, \Gamma^b), a > b} \Gamma^b\}.$$

These functors form a commutative graded algebra in \mathcal{F} . The primitives in $H_*(BV; \mathbb{F}_2)$ are obtained by evaluating on V . The analysis of the primitives is dual to the study of the indecomposables for the action of the Steenrod reduced powers on $H^*(BV; \mathbb{F}_2)$; this is a difficult problem which has attracted much interest. For the field \mathbb{F}_2 , the complete structure is known only for spaces of dimension at most four; the case of dimension three is due to Kameko and a published account is available in Boardman [5, 2]; Kameko also announced the case of dimension four, which has since been calculated by Sum, a student of Nguyen Hung.

The morphism $\phi_{\underline{s}}$ of equation (2) maps to $K^{[\underline{s}]_2}$ for elementary reasons and the ring of lines is a graded sub-algebra in \mathcal{F} of K^* . A fundamental question is to determine in which degrees the ring of lines coincides with the Steenrod kernel. Motivated by this question, Crabb and Hubbuck [3, Proposition 3.10] gave an explicit criterion upon the sequence (s_i) with respect to the dimension of V for the morphism $\phi_{\underline{s}}$ to be a monomorphism.

The purpose of this note is two-fold; to present a proof exploiting the category \mathcal{F} and to generalize the result to an arbitrary finite field \mathbb{F} , with $q = |\mathbb{F}|$. The main result of the paper is the following:

Theorem 1. *Let r be a natural number and $\underline{s} = (s_1 > \dots > s_r > s_{r+1} = 0)$ be a sequence of integers which satisfies the condition $[s_i - s_{i+1}]_q \geq (q - 1)(\dim V - i + 1)$, for $1 \leq i \leq r$; then the morphism $\phi_{\underline{s}}$ induces a monomorphism*

$$\mathbb{F}[\mathfrak{F}[\mathbf{ag}_r]](V) \hookrightarrow \Gamma^{[\underline{s}]_q}(V).$$

The proof sheds light upon the method proposed by Crabb and Hubbuck; namely, the proof establishes the stronger result that the composite with a morphism induced by the iterated diagonal on divided power algebras and the Verschiebung morphism is a monomorphism. This is of interest in the light of recent work over the field \mathbb{F}_2 by Grant Walker, Reg Wood [9] and Tran Ngoc Nam generalizing the result of Crabb and Hubbuck, relaxing the required hypothesis on the sequence (s_i) .

These techniques can be used to provide further information on the nature of the embedding results; for instance:

Proposition 2. *Let \underline{s} be a sequence of integers $(s_1 > \dots > s_r > s_{r+1} = 0)$ and V be a finite-dimensional vector space for which the morphism $\phi_{\underline{s}}(V) : \mathbb{F}[\mathfrak{F}[\mathbf{ag}_r]](V) \rightarrow \Gamma^{[\underline{s}]_q}(V)$ is a monomorphism.*

Let \underline{s}^+ denote the sequence given by $s_i^+ = s_i + 1$, for $1 \leq i \leq r$, and $s_{r+1}^+ = 0$. Then the morphism

$$\phi_{\underline{s}^+}(V) : \mathbb{F}[\mathfrak{F}[\mathbf{ag}_r]](V) \rightarrow \Gamma^{[\underline{s}^+]_q}(V)$$

is a monomorphism.

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2. Preliminaries

Fix a finite field $\mathbb{F} = \mathbb{F}_q$, and write $q = p^m$, where p is the characteristic of \mathbb{F} . Let \mathcal{F} be the category of functors from finite-dimensional \mathbb{F} -vector spaces to \mathbb{F} -vector spaces. The group of units \mathbb{F}^\times is isomorphic to the cyclic group $\mathbb{Z}/(q-1)\mathbb{Z}$ via $i \mapsto \lambda^i$, for a generator λ . In particular, the group has order prime to p , hence the category of $\mathbb{F}[\mathbb{F}^\times]$ -modules is semi-simple. This gives rise to the weight splitting of \mathcal{F} (as in [6]):

$$\mathcal{F} \cong \prod_{i \in \mathbb{Z}/(q-1)\mathbb{Z}} \mathcal{F}^i,$$

where \mathcal{F}^i is the full subcategory of functors such that $F(\lambda 1_V) = \lambda^i F(1_V)$ for all finite-dimensional spaces V and $\lambda \in \mathbb{F}^\times$. By reduction mod $q-1$, the weight category \mathcal{F}^k can be taken to be defined for k an integer.

The duality functor $D : \mathcal{F}^{\text{op}} \rightarrow \mathcal{F}$ is defined by $DF(V) := F(V^*)^*$ and D restricts to a functor $D : (\mathcal{F}^k)^{\text{op}} \rightarrow \mathcal{F}^k$

2.1. Divided powers and the Verschiebung

Recall that Γ^k denotes the k th divided power functor, defined as the invariants $\Gamma^k := (T^k)^{\mathfrak{S}_k}$ under the action of the symmetric group permuting the factors of T^k , the k th tensor power functor. (By convention, the divided power Γ^0 functor is the constant functor \mathbb{F} and $\Gamma^i = 0$ for $i < 0$). The divided power functor Γ^k is dual to the k th symmetric power functor S^k and the functors Γ^k, T^k, S^k all belong to the weight category \mathcal{F}^k .

The divided power functors Γ^* form a graded exponential functor; namely for finite-dimensional vector spaces U, V and a natural number n , there is a binatural isomorphism

$$\Gamma^n(U \oplus V) \cong \bigoplus_{i+j=n} \Gamma^i(U) \otimes \Gamma^j(V).$$

This has important consequences (see [4], for example); in particular, for pairs of natural numbers (a, b) , there are cocommutative coproduct morphisms $\Gamma^{a+b} \xrightarrow{\Delta} \Gamma^a \otimes \Gamma^b$

Γ^b and commutative product morphisms $\Gamma^a \otimes \Gamma^b \xrightarrow{\mu} \Gamma^{a+b}$, which are coassociative (respectively associative) in the appropriate graded sense.

The Verschiebung is a natural surjection $\mathcal{V} : \Gamma^{qn} \twoheadrightarrow \Gamma^n$, for integers $n \geq 0$, dual to the Frobenius q th power map on symmetric powers. More generally there is a Verschiebung morphism $\mathcal{V}_p : \Gamma^{np} \twoheadrightarrow \Gamma^{(1)}$, dual to the Frobenius p th power map, where $(-)^{(1)}$ denotes the Frobenius twist functor (see [4]). The Verschiebung \mathcal{V} is obtained by iterating \mathcal{V}_p m times, where $q = p^m$.

The p th truncated symmetric power functor \overline{S}^n is defined by imposing the relation $v^p = 0$; similarly the q th truncated symmetric power functor \tilde{S}^n is given by forming the quotient by the relation $v^q = 0$. There is a natural surjection $\tilde{S}^n \twoheadrightarrow \overline{S}^n$; over a prime field the functors coincide.

Dualizing gives the following:

Definition 2.1. For n a natural number,

1. let $\tilde{\Gamma}^n$ denote the kernel of the composite

$$\Gamma^n \xrightarrow{\Delta} \Gamma^{n-q} \otimes \Gamma^q \xrightarrow{1 \otimes \mathcal{V}} \Gamma^{n-q} \otimes \Gamma^1;$$

2. let $\overline{\Gamma}^n$ be the kernel of the composite

$$\Gamma^n \xrightarrow{\Delta} \Gamma^{n-p} \otimes \Gamma^p \xrightarrow{1 \otimes \mathcal{V}_p} \Gamma^{n-p} \otimes (\Gamma^1)^{(1)}.$$

Lemma 2.2. Let n be a natural number.

1. The functor $\tilde{\Gamma}^n$ is dual to \tilde{S}^n .
2. The functor $\overline{\Gamma}^n$ is isomorphic to \overline{S}^n and is simple.
3. $\tilde{\Gamma}^n(\mathbb{F}^d) = 0$ if $d \leq (n-1)/(q-1)$.

Proof. The first statement follows from the definitions and the identification $DS^n = \Gamma^n$. The simplicity of \overline{S}^n is a standard fact [6]; the isomorphism follows from the fact that the simple functors of \mathcal{F} are self-dual [7].

The final statement is an elementary verification, which is a consequence of the observation that the maximal degree of a free q -truncated symmetric algebra on d variables of degree one is $d(q-1)$. \square

2.2. Further properties of divided powers

Notation 2.3. For s a natural number, let $[s]_q$ denote the integer $q^s - 1$ and, for \underline{s} a sequence of integers, $s_1 \geq \dots \geq s_r > s_{r+1} = 0$, let $[\underline{s}]_q$ denote $\sum_i [s_i]_q$.

Notation 2.4. The element of $\Gamma^k(V)$ corresponding to the symmetric tensor $x^{\otimes k} \in T^k(V)$, for x an element of V , will be denoted simply by $x^{\otimes k}$.

Lemma 2.5. Let s be a positive integer. For any $x \in V$, the class $x^{\otimes [s]_q} \in \Gamma^{[s]_q}(V)$ is equal to the product

$$\prod_{j=0}^{sm-1} x^{\otimes (p-1)p^j},$$

where $|\mathbb{F}| = q = p^m$.

This Lemma is a consequence of the following well-known general property of the divided power functors.

Lemma 2.6. *Let a_0, \dots, a_t and $0 = r_0 < r_1 < \dots < r_t$ be sequences of natural numbers such that, for $0 \leq i < t$, $a_i p^{r_i} < p^{r_{i+1}}$. Then the composite*

$$\Gamma^{\Sigma_{i=0}^t a_i p^{r_i}} \rightarrow \bigotimes_{i=0}^t \Gamma^{a_i p^{r_i}} \rightarrow \Gamma^{\Sigma_{i=0}^t a_i p^{r_i}}$$

is an isomorphism, where the first morphism is the coproduct and the second the product.

Proof. Using the (co)associativity of the product (respectively the coproduct), it suffices to prove the result for $t = 1$. The composite morphism in this case is multiplication by the scalar $\binom{a_0 + a_1 p^{r_1}}{a_0}$, which is equal to one modulo p , since $a_0 < p^{r_1}$, by hypothesis. \square

The following Lemma is the key to the construction of the Crabb-Hubbuck morphism, ϕ_s , in Section 4.

Lemma 2.7. *Let x be an element of V , s be a natural number and $0 < i \leq [s]_q$ be an integer. The product $x^{\otimes [s]_q} x^{\otimes i}$ is zero in $\Gamma^{[s]_q+i}(V)$.*

Proof. The element $x^{\otimes [s]_q}$ is equal to the product $\prod_{j=0}^{sm-1} x^{\otimes (p-1)p^j}$, by Lemma 2.5. Similarly, considering the p -adic expansion $i = \sum_{j=0}^{sm-1} i_j p^j$, where $0 \leq i_j < p$ and at least one i_j is non-zero, there is an equality $x^{\otimes i} = \prod_{j=0}^{sm-1} x^{\otimes i_j p^j}$. The product is associative and commutative, hence it suffices to show that, if $i_j \neq 0$, then $x^{\otimes (p-1)p^j} x^{\otimes i_j p^j}$ is zero. Up to non-zero scalar in \mathbb{F}_p^\times , the element $x^{\otimes i_j p^j}$ is the i_j -fold product of $x^{\otimes p^j}$ (since $0 < i_j < p$, by hypothesis), hence it suffices to show that $x^{\otimes (p-1)p^j} x^{\otimes p^j}$ is zero. This element identifies with $\binom{p^{j+1}}{p^j} x^{\otimes p^{j+1}}$ and the scalar is zero in \mathbb{F} . \square

The behaviour of the Verschiebung morphism with respect to products is important.

Lemma 2.8. *Let β_1, \dots, β_k be positive integers such that $\sum_{i=1}^k \beta_i = pN$ for some integer N . The composite*

$$\bigotimes_{i=1}^k \Gamma^{\beta_i} \xrightarrow{\mu} \Gamma^{pN} \xrightarrow{\mathcal{V}_p} (\Gamma^N)^{(1)},$$

where \mathcal{V}_p is the Verschiebung and μ is the product, is trivial unless for each i , $\beta_i = p\beta'_i$, $\beta'_i \in \mathbb{N}$. In this case, there is a commutative diagram

$$\begin{array}{ccc} \bigotimes_{i=1}^k \Gamma^{p\beta'_i} & \xrightarrow{\mu} & \Gamma^{pN} \\ \otimes \mathcal{V}_p \downarrow & & \downarrow \mathcal{V}_p \\ \bigotimes_{i=1}^k (\Gamma^{\beta'_i})^{(1)} & \xrightarrow{\mu^{(1)}} & (\Gamma^N)^{(1)}. \end{array}$$

Proof. The statement is more familiar in the dual situation, where it corresponds to the fact that the diagonal of the symmetric power algebra commutes with the Frobenius. \square

Remark 2.9. An analogous statement holds for iterates of \mathcal{V}_p and for the Verschiebung $\mathcal{V} : \Gamma^{qN} \rightarrow \Gamma^N$.

3. Projectives and flags

This section introduces the flag functors and relates them to the standard projective generators of the category \mathcal{F} .

3.1. The projective and flag functors

For r a natural number, the standard projective object $P_{\mathbb{F}^r}$ in \mathcal{F} is given by $P_{\mathbb{F}^r}(V) = \mathbb{F}[\text{Hom}(\mathbb{F}^r, V)]$ and is determined up to isomorphism by $\text{Hom}_{\mathcal{F}}(P_{\mathbb{F}^r}, G) = G(\mathbb{F}^r)$. In particular, the projective $P_{\mathbb{F}}$ is the functor $V \mapsto \mathbb{F}[V]$.

The weight splitting determines a direct sum decomposition

$$P_{\mathbb{F}} \cong \bigoplus_{i \in \mathbb{Z}/(q-1)\mathbb{Z}} P_{\mathbb{F}}^i$$

in which $P_{\mathbb{F}}^i$ is indecomposable for $i \neq 0$ and $P_{\mathbb{F}}^0$ admits a decomposition $P_{\mathbb{F}}^0 = \mathbb{F} \oplus \overline{P_{\mathbb{F}}^0}$ (Cf [6, Lemma 5.3], noting that Kuhn uses a splitting associated to the multiplicative semigroup \mathbb{F}).

There is a Knneth isomorphism for projectives so that, for r a positive integer, $P_{\mathbb{F}}^{\otimes r}$ is projective and identifies with the projective functor.

Definition 3.1. For r a positive integer, let $\mathbb{F}[\mathfrak{F}\mathbf{ag}_r]$ be the functor which is defined in terms of complete flags of length r as follows.

As a vector space, $\mathbb{F}[\mathfrak{F}\mathbf{ag}_r](V)$ has basis the set of complete flags of length r . A morphism $V \rightarrow W$ sends a complete flag to its image, if this is a complete flag, and to zero otherwise.

Recall that a functor F of \mathcal{F} is said to be constant-free if $F(0) = 0$.

Lemma 3.2. *Let $r \geq s > 0$ be integers.*

1. *The functor $\mathbb{F}[\mathfrak{F}\mathbf{ag}_r]$ is constant-free and belongs to \mathcal{F}^0 .*
2. *There is a diagonal morphism in \mathcal{F} :*

$$\mathbb{F}[\mathfrak{F}\mathbf{ag}_r] \rightarrow \mathbb{F}[\mathfrak{F}\mathbf{ag}_r] \otimes \mathbb{F}[\mathfrak{F}\mathbf{ag}_r].$$

3. *There is a surjection*

$$\pi_{r,s} : \mathbb{F}[\mathfrak{F}\mathbf{ag}_r] \twoheadrightarrow \mathbb{F}[\mathfrak{F}\mathbf{ag}_s]$$

which forgets the subspaces of dimension greater than s .

3.2. Structure of the projectives

Lemma 3.3. *Let $n \geq 1$ be an integer, then $\text{Hom}_{\mathcal{F}}(\overline{P}_{\mathbb{F}}^0, \Gamma^{n(q-1)}) = \mathbb{F}$.*

Proof. The result follows from the Yoneda lemma, the weight splitting and the fact that $n \geq 1$ allows passage to the constant-free part, $\overline{P}_{\mathbb{F}}^0$. \square

Recall that a functor is said to be finite if it has a finite composition series.

Proposition 3.4. [6, 7]

1. *The surjection $P_{\mathbb{F}} \rightarrow \mathbb{F}[\mathfrak{F}|\mathfrak{ag}_1]$ induces an isomorphism $\overline{P}_{\mathbb{F}}^0 \cong \mathbb{F}[\mathfrak{F}|\mathfrak{ag}_1]$.*
2. *There is an inverse system of finite functors $\dots \rightarrow \mathfrak{q}_k \overline{P}_{\mathbb{F}}^0 \rightarrow \mathfrak{q}_{k-1} \overline{P}_{\mathbb{F}}^0 \rightarrow \dots$ such that*
 - (a) $\overline{P}_{\mathbb{F}}^0 \cong \lim_{\leftarrow} \mathfrak{q}_k \overline{P}_{\mathbb{F}}^0$;
 - (b) *for $k \geq 1$, $\mathfrak{q}_k \overline{P}_{\mathbb{F}}^0$ is isomorphic to the image of any non-trivial morphism $\overline{P}_{\mathbb{F}}^0 \rightarrow \Gamma^{k(q-1)}$;*
 - (c) *for $k \geq 2$, there is a non-split short exact sequence*

$$0 \rightarrow \tilde{\Gamma}^{k(q-1)} \rightarrow \mathfrak{q}_k \overline{P}_{\mathbb{F}}^0 \rightarrow \mathfrak{q}_{k-1} \overline{P}_{\mathbb{F}}^0 \rightarrow 0.$$

3. *The functor $\overline{P}_{\mathbb{F}}^0$ is dual to a locally-finite functor.*
4. *If \mathbb{F} is the prime field \mathbb{F}_p , then the functor $\overline{P}_{\mathbb{F}}^0$ is uniserial with composition factors $\{\tilde{\Gamma}^{k(p-1)} | k \geq 1\}$, each occurring with multiplicity one.*

Proof. It is more straightforward to deduce the result from the description of the dual $D\overline{P}_{\mathbb{F}}^0$; this is isomorphic to the functor $(\bigoplus_{k=0}^{\infty} S^{k(q-1)})/\langle v^q - v \rangle$, where the relation is induced by the weight zero part of the ideal $\langle v^q - v \rangle$ (this is deduced from [6, Lemma 4.12] by applying the evident weight splitting). \square

It is important to have a measure of how good an approximation $\mathfrak{q}_k \overline{P}_{\mathbb{F}}^0$ is to $\overline{P}_{\mathbb{F}}^0$.

Lemma 3.5. *Let $k \geq 1$ be an integer. Up to scalar in \mathbb{F}^{\times} , there is a unique non-trivial morphism $\overline{P}_{\mathbb{F}}^0 \rightarrow \Gamma^{k(q-1)}$. Any non-trivial morphism $\overline{P}_{\mathbb{F}}^0 \rightarrow \Gamma^{k(q-1)}$*

1. *factors as*

$$\overline{P}_{\mathbb{F}}^0 \twoheadrightarrow \mathfrak{q}_k \overline{P}_{\mathbb{F}}^0 \hookrightarrow \Gamma^{k(q-1)}$$

and

2. *induces a monomorphism*

$$\overline{P}_{\mathbb{F}}^0(V) \hookrightarrow \Gamma^{k(q-1)}(V)$$

if $\dim V \leq k$.

Proof. The unicity follows from Lemma 3.3 and the factorization follows from this unicity together with the identification of $\mathfrak{q}_k \overline{P}_{\mathbb{F}}^0$ which is given in Proposition 3.4.

The kernel of the surjection $\overline{P}_{\mathbb{F}}^0 \twoheadrightarrow \mathfrak{q}_k \overline{P}_{\mathbb{F}}^0$ has a filtration with subquotients of the form $\tilde{\Gamma}^{l(q-1)}$ with $l > k$. The functor $\tilde{\Gamma}^{l(q-1)}$ is zero when evaluated on spaces with $\dim V \leq k$, by Lemma 2.2 (3). It follows that the kernel is zero when evaluated on such spaces. Thus $\overline{P}_{\mathbb{F}}^0(V) \rightarrow \mathfrak{q}_k \overline{P}_{\mathbb{F}}^0(V)$ is an isomorphism when $\dim V \leq k$ and the result follows from the first part of the Lemma. \square

4. The flag morphism of Crabb and Hubbuck

Throughout this section, let r denote a fixed positive integer and \underline{s} denote a fixed decreasing sequence of positive integers, $s_1 \geq s_2 \geq \dots \geq s_r > s_{r+1} = 0$.

Notation 4.1. For each positive integer s , let ϕ_s be the element of $\text{Hom}_{\mathcal{F}}(\overline{P_{\mathbb{F}}^0}, \Gamma^{[s]_q})$ which sends the canonical generator of $\overline{P_{\mathbb{F}}^0}(\mathbb{F}) \cong \mathbb{F}[\mathfrak{F}\mathbf{lag}_1](\mathbb{F})$ to $\iota^{\otimes [s]_q}$, where ι is any generator of \mathbb{F} . (The morphism is independent of the choice of ι).

Definition 4.2. For \underline{s} a sequence of positive integers, let $\tilde{\phi}_{\underline{s}}$ denote the morphism

$$\tilde{\phi}_{\underline{s}} : P_{\mathbb{F}^r} \cong P_{\mathbb{F}}^{\otimes r} \xrightarrow{\otimes \phi_{s_i}} \bigotimes_{i=1}^r \Gamma^{[s_i]_q} \rightarrow \Gamma^{[\underline{s}]_q}$$

in which the second morphism is induced by the product.

The following Proposition is proved in the case $q = 2$ in [3].

Proposition 4.3. *The morphism $\tilde{\phi}_{\underline{s}}$ factorizes as*

$$P_{\mathbb{F}}^{\otimes r} \rightarrow \mathbb{F}[\mathfrak{F}\mathbf{lag}_r] \xrightarrow{\phi_{\underline{s}}} \Gamma^{[\underline{s}]_q}.$$

Proof. Fixing a basis of \mathbb{F}^r , a canonical basis element of $P_{\mathbb{F}^r}(V)$ is an ordered sequence (v_i) of r elements of V . The morphism $\tilde{\phi}_{\underline{s}}$ sends this generator to $\prod_{i=1}^r v_i^{\otimes [s_i]_q} = \prod_{i=1}^r \prod_{j=0}^{s_i-1} v_i^{\otimes (q-1)q^j}$.

Using this notation, define a natural surjection $P_{\mathbb{F}^r} \rightarrow \mathbb{F}[\mathfrak{F}\mathbf{lag}_r]$ by sending (v_i) to the flag $\langle v_1 \rangle < \langle v_1, v_2 \rangle < \dots < \langle v_1, \dots, v_r \rangle$ if the elements are linearly independent and zero otherwise. The proposition asserts that $\tilde{\phi}_{\underline{s}}$ factorizes across this surjection.

The result follows as in the proof of [3, Lemma 3.1], by applying Lemma 2.7. \square

5. The embedding theorem

The purpose of this section is to prove the main result of the paper, stated here as Theorem 5.17, which gives a criterion for

$$\phi_{\underline{s}}(V) : \mathbb{F}[\mathfrak{F}\mathbf{lag}_r](V) \rightarrow \Gamma^{[\underline{s}]_q}(V)$$

to be a monomorphism, where r is a positive integer and $\underline{s} = s_1 > s_2 > \dots > s_r > s_{r+1} = 0$ is a strictly decreasing sequence of integers.

Remark 5.1. The morphism $\phi_{\underline{s}}$ is clearly not a monomorphism of functors, since the functor $\Gamma^{[\underline{s}]_q}$ is finite whereas $\mathbb{F}[\mathfrak{F}\mathbf{lag}_r]$ is highly infinite.

However, Lemma 3.5 (2) provides the key calculational input, which is restated as the following:

Lemma 5.2. *Let s be a natural number. The morphism $\phi_s : \mathbb{F}[\mathfrak{F}\mathbf{lag}_1] \cong \overline{P_{\mathbb{F}}^0} \rightarrow \Gamma^{[s]_q}$ induces a monomorphism $\phi_s(V)$ if $[s]_q \geq (q - 1) \dim V$.*

The theorem is proved by an induction using Lemma 5.2 to provide the inductive step. The strategy involves composing $\phi_{\underline{s}}$ with a morphism $\delta_{\underline{s}}$ (defined below) to

give a morphism $\psi_{\underline{s}}$ which is amenable to induction. The key to setting up the induction is Lemma 5.6.

Write $[\underline{s}]_q = r[s_r]_q + \sum_{i=1}^{r-1} ([s_i]_q - [s_r]_q)$; thus the coproduct gives a morphism $\Delta : \Gamma^{[\underline{s}]_q} \rightarrow \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma([s_i]_q - [s_r]_q)}$. For each i , $[s_i]_q - [s_r]_q = q^{s_r} [s_i - s_r]_q$, hence there is an iterated Verschiebung morphism $\mathcal{V}^{s_r} : \Gamma^{\Sigma([s_i]_q - [s_r]_q)} \rightarrow \Gamma^{\Sigma[s_i - s_r]_q}$.

Definition 5.3.

1. Let $\delta_{\underline{s}} : \Gamma^{[\underline{s}]_q} \rightarrow \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{i=1}^{r-1} [s_i - s_r]_q}$ be the composite morphism

$$\Gamma^{[\underline{s}]_q} \xrightarrow{\Delta} \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma([s_i]_q - [s_r]_q)} \xrightarrow{1 \otimes \mathcal{V}^{s_r}} \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{i=1}^{r-1} [s_i - s_r]_q}.$$

2. Let $\psi_{\underline{s}} : \mathbb{F}[\mathfrak{F}\mathbf{lag}_r] \rightarrow \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{j=1}^{r-1} [s_j - s_r]_q}$ be the composite morphism $\delta_{\underline{s}} \circ \phi_{\underline{s}}$.

The following elementary observation is recorded as a Lemma.

Lemma 5.4. *Let V be a finite-dimensional vector space. If the morphism $\psi_{\underline{s}}(V)$ is a monomorphism, then $\phi_{\underline{s}}(V)$ is a monomorphism.*

Notation 5.5. Let \underline{s}' denote the sequence (of length $r - 1$) of positive integers ($s_1 - s_r > \dots > s_{r-1} - s_r > 0$).

The following Lemma is the key to the inductive proof, and relies upon the fact that the iterated Verschiebung is used in the definition of $\psi_{\underline{s}}$. Observe that the Crabb-Hubbuck morphism associated to the sequence of integers (s_r, \dots, s_r) of length r induces a morphism

$$\phi_{(s_r, \dots, s_r)} : \mathbb{F}[\mathfrak{F}\mathbf{lag}_r] \rightarrow \Gamma^{r[s_r]_q}.$$

Lemma 5.6. *The morphism $\psi_{\underline{s}}$ identifies with the composite morphism*

$$\begin{array}{ccc} \mathbb{F}[\mathfrak{F}\mathbf{lag}_r] \xrightarrow{\text{diag}} \mathbb{F}[\mathfrak{F}\mathbf{lag}_r] \otimes \mathbb{F}[\mathfrak{F}\mathbf{lag}_r] & \xrightarrow{1 \otimes \pi_{r,r-1}} & \mathbb{F}[\mathfrak{F}\mathbf{lag}_r] \otimes \mathbb{F}[\mathfrak{F}\mathbf{lag}_{r-1}] \\ & & \downarrow \phi_{(s_r, \dots, s_r)} \otimes \phi_{\underline{s}'} \\ & & \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{j=1}^{r-1} [s'_j]_q} \end{array}$$

Proof. We are required to prove that the following diagram is commutative.

$$\begin{array}{ccc} \mathbb{F}[\mathfrak{F}\mathbf{lag}_r] \xrightarrow{\text{diag}} \mathbb{F}[\mathfrak{F}\mathbf{lag}_r] \otimes \mathbb{F}[\mathfrak{F}\mathbf{lag}_r] & \xrightarrow{1 \otimes \pi_{r,r-1}} & \mathbb{F}[\mathfrak{F}\mathbf{lag}_r] \otimes \mathbb{F}[\mathfrak{F}\mathbf{lag}_{r-1}] \\ \phi_{\underline{s}} \downarrow & & \downarrow \phi_{(s_r, \dots, s_r)} \otimes \phi_{\underline{s}'} \\ \Gamma^{[\underline{s}]_q} & \xrightarrow{\delta_{\underline{s}}} & \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{j=1}^{r-1} [s'_j]_q}. \end{array}$$

Choose a surjection $P_{\mathbb{F}^r} \rightarrow \mathbb{F}[\mathfrak{F}\mathbf{lag}_r]$ as in the proof of Proposition 4.3; it is equivalent to prove the commutativity of the diagram obtained by composition, replacing the top left entry by $P_{\mathbb{F}^r}$. The analysis of the composite morphism

$$P_{\mathbb{F}^r} \rightarrow \Gamma^{[\underline{s}]_q} \rightarrow \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{j=1}^{r-1} [s'_j]_q}. \tag{3}$$

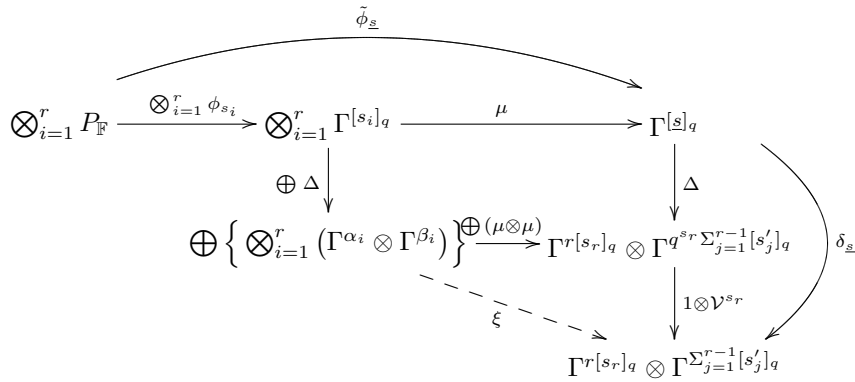
around the bottom of the diagram can then be carried out as follows.

The definition of the morphism $\delta_{\underline{s}}$ and of the morphism $\phi_{\underline{s}}$ implies that this composite factors across

$$\bigotimes_{i=1}^r \Gamma^{[s_i]_q} \xrightarrow{\mu} \Gamma^{[s]_q} \xrightarrow{\Delta} \Gamma^{r[s_r]_q} \otimes \Gamma^{q^{s_r} \Sigma_{j=1}^{r-1} [s'_j]_q}$$

where μ denotes the product on divided powers and Δ the diagonal.

The exponential algebra structure of Γ^* (essentially the fact that these functors take values in bicommutative Hopf algebras) implies that there is a commutative diagram



where the sum is labelled over sequences of pairs of natural numbers (α_i, β_i) satisfying $\alpha_i + \beta_i = [s_i]_q$ for each i and $\Sigma \alpha_i = r[s_r]_q$.

Consider the composite morphism ξ in the diagram; Lemma 2.8 implies that the only components of this morphism which are non-trivial are those corresponding to sequences (α_i, β_i) for which $\beta_i = q^{s_r} \beta'_i$ for natural numbers β'_i . The condition $\alpha_i + q^{s_r} \beta'_i = [s_i]_q$ implies that α_i is non-zero; it follows that $\alpha_i \geq [s_r]_q$, for each i , since $\alpha_i \equiv [s_r]_q \pmod{(q^{s_r})}$. The condition $\Sigma_i \alpha_i = r[s_r]_q$ therefore implies that $\alpha_i = [s_r]_q$ for each i . It follows that ξ has only one non-zero component and a straightforward verification shows that the composite corresponds to the composite around the top of the diagram given in the statement of the Lemma. \square

The inductive argument is simplified using the following:

Lemma 5.7. *Let V be a finite-dimensional vector space for which the morphism $\phi_{s'}(V)$ is a monomorphism. Then the morphism $\psi_{\underline{s}}(V)$ is a monomorphism if and only if the composite morphism*

$$\mathbb{F}[\mathfrak{F}\mathfrak{lag}_r] \xrightarrow{\text{diag}} \mathbb{F}[\mathfrak{F}\mathfrak{lag}_r] \otimes \mathbb{F}[\mathfrak{F}\mathfrak{lag}_r] \xrightarrow{\phi_{s_r, \dots, s_r} \otimes \pi_{r, r-1}} \Gamma^{r[s_r]_q} \otimes \mathbb{F}[\mathfrak{F}\mathfrak{lag}_{r-1}]$$

induces a monomorphism when evaluated upon V .

Proof. This follows from the identification of $\psi_{\underline{s}}$ which is given in Lemma 5.6. \square

Using the fact that $\mathbb{F}[\mathfrak{F}\mathfrak{lag}_{r-1}](V)$ is generated by complete flags of length $r - 1$, this allows the decomposition into components.

Notation 5.8. For Φ a complete flag of length $r - 1$ in V , let

1. $\langle \Phi \rangle \leq V$ denote the $(r - 1)$ -dimensional subspace of V defined by Φ ;
2. $\mathbb{F}[\mathfrak{F}\mathfrak{lag}_r]_{\Phi}(V)$ denote the subspace generated by flags containing Φ ;
3. γ_{Φ} denote the image of $[\Phi] \in \mathbb{F}[\mathfrak{F}\mathfrak{lag}_{r-1}](V)$ under the Crabb-Hubbuck morphism $\phi_{s_r, \dots, s_r}(V) : \mathbb{F}[\mathfrak{F}\mathfrak{lag}_{r-1}](V) \rightarrow \Gamma^{(r-1)[s_r]_q}(V)$.

Remark 5.9. The space $\mathbb{F}[\mathfrak{F}\mathfrak{lag}_r]_{\Phi}(V)$ is isomorphic to $\mathbb{F}[\mathfrak{F}\mathfrak{lag}_1](V/\langle \Phi \rangle)$.

Lemma 5.10. *Let V be a finite-dimensional vector space for which the morphism $\phi_{s_r}(V)$ is a monomorphism. The morphism $\psi_{\underline{s}}(V)$ is a monomorphism if and only if, for each complete flag Φ in V of length $r - 1$, the restriction of*

$$\mathbb{F}[\mathfrak{F}\mathfrak{lag}_r](V) \xrightarrow{\phi_{s_r, \dots, s_r}} \Gamma^{r[s_r]_q}(V)$$

to $\mathbb{F}[\mathfrak{F}\mathfrak{lag}_r]_{\Phi}(V)$ is a monomorphism.

Proof. A straightforward consequence of Lemma 5.7. □

Notation 5.11. Let V be a finite-dimensional vector space and Φ be a complete flag in V of length $r - 1$. Let ρ_{Φ} denote the composite linear map

$$\mathbb{F}[\mathfrak{F}\mathfrak{lag}_r]_{\Phi}(V) \xrightarrow{\cong} \mathbb{F}[\mathfrak{F}\mathfrak{lag}_1](V/\langle \Phi \rangle) \xrightarrow{\phi_{s_r}} \Gamma^{[s_r]_q}(V/\langle \Phi \rangle)$$

induced by the projection $V \twoheadrightarrow V/\langle \Phi \rangle$ and the morphism ϕ_{s_r} .

Notation 5.12. For V a finite-dimensional vector space, Φ a complete flag in V of length $r - 1$ and σ a section of the projection $V \twoheadrightarrow V/\langle \Phi \rangle$, let

$$\gamma_{\Phi} \cap_{\sigma} : \Gamma^{[s_r]_q}(V/\langle \Phi \rangle) \rightarrow \Gamma^{r[s_r]_q}(V)$$

denote the linear morphism induced by the section σ followed by the product with γ_{Φ} with respect to the algebra structure of $\Gamma^*(V)$.

Lemma 5.13. *Let V , Φ and σ be as above. The linear morphism*

$$\gamma_{\Phi} \cap_{\sigma} : \Gamma^{[s_r]_q}(V/\langle \Phi \rangle) \rightarrow \Gamma^{r[s_r]_q}(V)$$

is a monomorphism.

Proof. The result follows from the exponential structure of the divided power functors, since the element γ_{Φ} is the image of an element of $\Gamma^{(r-1)[s_r]_q}(\langle \Phi \rangle)$ under the morphism induced by the natural inclusion. □

Lemma 5.14. *Let V , Φ , σ be as above. The restriction of $\phi_{s_r, \dots, s_r}(V)$ to $\mathbb{F}[\mathfrak{F}\mathfrak{lag}_r]_{\Phi}(V)$ identifies with the linear morphism $(\gamma_{\Phi} \cap_{\sigma}) \circ \rho_{\Phi}$.*

Proof. The result follows from the definition of the morphism $\phi_{s_r, \dots, s_r}(V)$. □

Lemmas 5.13 and 5.14 together imply the following result:

Lemma 5.15. *Let V be a finite-dimensional vector space and Φ be a complete flag in V of length $r - 1$. The restriction of $\phi_{s_r, \dots, s_r}(V)$ to $\mathbb{F}[\mathfrak{F}\mathfrak{lag}_r]_{\Phi}(V)$ is a monomorphism if and only if*

$$\phi_{s_r} : \mathbb{F}[\mathfrak{F}\mathfrak{lag}_1](V/\langle \Phi \rangle) \rightarrow \Gamma^{[s_r]_q}(V/\langle \Phi \rangle)$$

is a monomorphism.

Remark 5.16. By lemma 5.2, a sufficient condition is

$$[s_r]_q \geq (q - 1) \dim(V/\langle \Phi \rangle) = (q - 1)(\dim V - r + 1).$$

When $q = 2$, this is an equivalent condition.

Putting these results together, one obtains the following generalization of [3, Proposition 3.10].

Theorem 5.17. *Suppose that the sequence \underline{s} satisfies the condition $[s_i - s_{i+1}]_q \geq (q-1)(\dim V - i + 1)$, for $1 \leq i \leq r$. Then the morphism $\phi_{\underline{s}}$ induces a monomorphism*

$$\mathbb{F}[\mathfrak{F}\mathfrak{lag}_r](V) \hookrightarrow \Gamma^{[\underline{s}]_q}(V).$$

Proof. The result is proved by induction upon r , starting with the initial case, $r = 1$, which is provided by Lemma 5.2. For the inductive step, by Lemma 5.4, it is sufficient to show that $\psi_{\underline{s}}(V)$ is a monomorphism, under the given hypotheses.

Observe that the hypotheses upon \underline{s} imply that \underline{s}' also satisfy the hypotheses with respect to V , so that the morphism $\phi_{\underline{s}'}(V)$ is injective, by induction. Hence Lemma 5.10 reduces the proof to showing that the restriction of ϕ_{s_r, \dots, s_r} to $\mathbb{F}[\mathfrak{F}\mathfrak{lag}_r]_{\Phi}(V)$ is a monomorphism, for each complete flag Φ of length $r - 1$ in V . The inductive step is completed by combining Lemma 5.15 with Lemma 5.2. \square

6. A stabilization result

The techniques employed in the proof of Theorem 5.17 can be used to provide further information on the nature of the embedding results. For instance, one has a direct proof of the following stabilization result.

Proposition 6.1. *Let \underline{s} be a sequence of integers ($s_1 > \dots > s_r > s_{r+1} = 0$) and V be a finite-dimensional vector space for which the morphism $\phi_{\underline{s}}(V) : \mathbb{F}[\mathfrak{F}\mathfrak{lag}_r](V) \rightarrow \Gamma^{[\underline{s}]_q}(V)$ is a monomorphism.*

Let \underline{s}^+ denote the sequence given by $s_i^+ = s_i + 1$, for $1 \leq i \leq r$, and $s_{r+1}^+ = 0$. Then the morphism

$$\phi_{\underline{s}^+}(V) : \mathbb{F}[\mathfrak{F}\mathfrak{lag}_r](V) \rightarrow \Gamma^{[\underline{s}^+]_q}(V)$$

is a monomorphism.

Proof. The diagonal induces a morphism $\Gamma^{[\underline{s}^+]_q} \rightarrow \Gamma^q[\underline{s}]_q \otimes \Gamma^{(q-1)r}$. Hence, composing with the Verschiebung on the first morphism gives $\eta : \Gamma^{[\underline{s}^+]_q} \rightarrow \Gamma^{[\underline{s}]_q} \otimes \Gamma^{(q-1)r}$, as in Definition 5.3.

There is a commutative diagram

$$\begin{array}{ccc} \mathbb{F}[\mathfrak{F}\mathfrak{lag}_r] & \xrightarrow{\phi_{\underline{s}^+}} & \Gamma^{[\underline{s}^+]_q} \\ \text{diag} \downarrow & & \downarrow \eta \\ \mathbb{F}[\mathfrak{F}\mathfrak{lag}_r] \otimes \mathbb{F}[\mathfrak{F}\mathfrak{lag}_r] & \xrightarrow{\phi_{\underline{s}} \otimes \phi_{1, \dots, 1}} & \Gamma^{[\underline{s}]_q} \otimes \Gamma^{(q-1)r}, \end{array}$$

the commutativity of which is established by an argument similar to that employed in the proof of Lemma 5.6.

It suffices to show that the composite

$$\mathbb{F}[\mathfrak{F}(\mathbf{ag}_r)](V) \xrightarrow{\phi_{s^+}} \Gamma^{[s^+]_q}(V) \xrightarrow{\eta} \Gamma^{[s]_q} \otimes \Gamma^{(q-1)r}(V)$$

is a monomorphism. By hypothesis, the morphism $\phi_{\underline{s}}(V)$ is a monomorphism. As in the inductive step of the proof of Theorem 5.17, the result then follows from the fact that the morphism $\mathbb{F}[\mathfrak{F}(\mathbf{ag}_r)](V) \rightarrow \Gamma^{(q-1)r}(V)$ is non-trivial. The latter follows from the fact that the hypothesis upon $\phi_{\underline{s}}(V)$ implies that V has dimension at least r . \square

Remark 6.2. This argument is related to standard techniques using the Kameko Sq^0 operation [5], which is based in an essential way upon the analysis of the Verschiebung morphism.

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