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POPATHS AND HOLINKS

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Abstract

In the study of stratified spaces it is useful to examine spaces of popaths (paths which travel from lower strata to higher strata) and holinks (those spaces of popaths which immediately leave a lower stratum for their final stratum destination). It is not immediately clear that for adjacent strata these two path spaces are homotopically equivalent, and even less clear that this equivalence can be constructed in a useful way (with a deformation of the space of popaths which fixes start and end points and where popaths instantly become members of the holink). The advantage of such an equivalence is that it allows a stratified space to be viewed categorically because popaths, unlike holink paths (which are easier to study), can be composed. This paper proves the aforementioned equivalence in the case of Quinn's homotopically stratified spaces [1].

1. Introduction

There are many different notions of a stratified space. One such notion is that of a homotopically stratified space. These were introduced by Frank Quinn. Here the strata are related by "homotopy rather than geometric conditions" [1]. This makes them ideal for studying the topology of stratified spaces. Two such tools for studying that topology are holinks and popath spaces.

The popaths between two strata are any paths which travel from one stratum to the other passing only from "lower strata to higher strata". On the other hand the holink between two strata consists only of popaths which instantly leave one stratum for the other.

Popaths are very useful in obtaining a categorical view of stratified spaces. However for a space with many strata the space of popaths between two strata could be difficult to compute or visualize. The holink between two strata only depends on the two strata involved and with this in mind is easier to deal with, but problems

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may arises because holink paths can't be composed. Therefore a result connecting these concepts becomes desirable.

The result obtained here is that for the space of popaths and the holink between two fixed strata there exists a homotopy h : popaths $\times I \to$ popaths which fixes the start and end points of paths, image$\{h_s\} \subset$ holink when $s \in (0,1]$ and $h_0 =$ identity. It may seem strange to require a result which is stronger than just the inclusion being a homotopy equivalence, but exactly this result has already found a relevance in the work of Jon Woolf [2] and seems to be required to construct a particular map to prove that a stratified space can be reconstructed from its popath category (the author hopes to show this in the near future).

2. Homotopically Stratified Spaces

Definition 2.1. A topological space X is **filtered** if there are designated closed subspaces X^i indexed by a finite poset S_X such that $X = \bigcup_{i \in S_X} X^i$ and $X^j \subseteq X^i \Leftrightarrow j \leqslant i$. The **strata** are defined as path connected components of $X_i = X^i - \bigcup_{j<i} X^j$. If $j < i$ we may say X_i is a higher stratum than X_j or equivalently X_j is a lower stratum than X_i.

Definition 2.2. A closed subspace K of a filtered space is said to be **pure** if it is a union of strata.

Definition 2.3. Assume W is a subspace of a filtered space and W contains the distinct strata X_b and X_a. Let the space of **popaths** from X_a through W to X_b be denoted by $\mathrm{pop}(X_b, W, X_a)$ and defined as the space (with the compact open topology) of all order preserving paths $\omega : [0, T_\omega] \to W$ such that $\omega(0) \in X_a$, $\omega(T_\omega) \in X_b$. Here order preserving means if $t_1 \leqslant t_2$ and $\omega(t_1) \in X_j$ while $\omega(t_2) \in X_i$ then $j \leqslant i$ (meaning ω cannot flow from higher strata into lower strata).

Definition 2.4. For a space S and subspace $Y \subset S$ the **holink** (also called **homotopy link**) between Y and $S - Y$ denoted $\mathrm{hol}(S, Y)$ is defined as the space (with the compact open topology) of paths $\omega : [0, T_\omega] \to S$ where $\omega(0) \in Y$ and $\omega(t) \in S - Y$ for $t \in (0, T_\omega]$.

Remark 2.5. It should be clear that $\mathrm{hol}(X_b \cup X_a, X_a)$ is the subspace of $\mathrm{pop}(X_b, X, X_a)$ consisting of popaths which immediately leave X_a and travel straight into X_b.

Definition 2.6. Suppose K, L are unions of strata in X and $L \subseteq K$. Then L is said to be **tame** in K if there is a neighborhood N of L in K and a nearly stratum preserving strong deformation retraction r of N onto L. Here **nearly stratum preserving** means points of $K - L$ remain in the same stratum until the last moment when they get pushed into L.

Definition 2.7. Let $R : N - L \to \mathrm{hol}(N, L)$ be defined by $R(x)(t) = r(x, 1-t)$ for all $x \in N - L$.

Definition 2.8. A filtered metric space X is a **homotopically stratified space** if for every $j < i$, X_j is tame in $X_j \cup X_i$ and the map from $\mathrm{hol}(X_i \cup X_j, X_j)$ to X_j given by evaluation at the start point is a fibration.

3. Popaths and Holinks in the 2 Strata Case

Definition 3.1. Let $\kappa : \mathrm{pop}(X_b, X_b \cup X_a, X_a) \to \mathbb{R}$ be the map which sends $\omega \in \mathrm{pop}(X_b, X_b \cup X_a, X_a)$ to the unique point, $\kappa(\omega)$, in $\mathbb{R}$ such that $\omega(t) \in X_a$ for $t \leqslant \kappa(\omega)$ and $\omega(t) \in X_b$ for $t > \kappa(\omega)$. Note κ is not a continuous map, but it is upper-continuous. We define upper-continuity as meaning for any $\omega \in \mathrm{pop}(X_b, X_b \cup X_a, X_a)$ and any neighborhood V of $\kappa(\omega)$ having the form $[0, r)$ there exists a neighborhood U of ω such that $f(U) \subset V$.

Lemma 3.2. Let N be a neighborhood of tameness for X_a in $X_a \cup X_b$ where X_a, X_b are distinct. There exists a continuous map $\lambda : \mathrm{pop}(X_b, X_b \cup X_a, X_a) \to \mathbb{R}$ that for any popath $\omega : [0, T_\omega] \to X$ satisfies:

1. $\lambda(\omega) \in (\kappa(\omega), T_\omega)$
2. $\omega(t) \in N - X_a$ for $t \in (\kappa(\omega), \lambda(\omega)]$

Remark 3.3. This will be useful when we wish to use the κ map but cannot because it is not continuous.

Proof. Since X is metric, $\mathrm{pop}(X_b, X_b \cup X_a, X_a)$ is also metric and so paracompact. Therefore by a partition of unity type argument it suffices to show it is true locally. Fix $\sigma \in \mathrm{pop}(X_b, X_b \cup X_a, X_a)$, clearly we can choose a point Γ in $[0, T_\sigma]$ which satisfies the conditions of $\lambda(\sigma)$. Now since κ is upper continuous the same value Γ satisfies the conditions to be $\lambda(\omega)$ for all ω within a small enough neighborhood of σ. Hence the lemma holds locally and therefore holds globally. $\square$

Definition 3.4. Let E denote the space

$$\{(\omega, t) \in \mathrm{pop}(X_b, X_b \cup X_a, X_a) \times \mathbb{R} \,:\, t \in (\kappa(\omega), \lambda(\omega)]\}$$

and p denote the canonical map $p : E \to \mathrm{pop}(X_b, X_b \cup X_a, X_a)$. In fact p is a fiber bundle homeomorphic to the trivial bundle with total space $\mathrm{pop}(X_b, X_b \cup X_a, X_a) \times (0, 1]$. A trivialisation is given by $(\omega, t) \mapsto (\omega, \beta_\omega(t))$, where $\beta_\omega(t)$ is the unique member of $(0, 1]$ such that

$$\int_{\kappa(\omega)}^{t} \mathrm{dist}_{X_a}(\omega(t'))dt' = \beta_\omega(t) \int_{\kappa(\omega)}^{\lambda(\omega)} \mathrm{dist}_{X_a}(\omega(t'))dt'.$$

Lemma 3.5. The inclusion map i induces a homotopy equivalence

$$\mathrm{hol}(X_b \cup X_a, X_a) \simeq \mathrm{pop}(X_b, X_b \cup X_a, X_a).$$

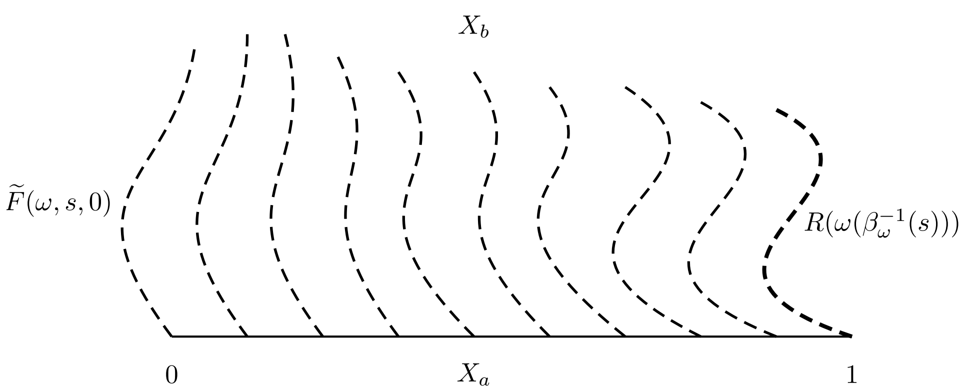
Furthermore it has a homotopy inverse $\varphi : \mathrm{pop}(X_b, X_b \cup X_a, X_a) \to \mathrm{hol}(X_b \cup X_a, X_a)$ where there exists a homotopy h from the identity map to $i \circ \varphi$ which fixes the start and end points of paths and $h_s(\omega) \in \mathrm{hol}(X_b \cup X_a, X_a)$ for all popaths ω when $s \in (0, 1]$.

Remark 3.6. This lemma means in the two strata case we have a continuous way of deforming the space of popaths into the holink so that popaths instantly become holink paths.

Proof. We will in fact directly construct the maps $h_s : \mathrm{pop}(X_b, X_b \cup X_a, X_a) \to \mathrm{pop}(X_b, X_b \cup X_a, X_a)$ which fix start and end points, have image in $\mathrm{hol}(X_b \cup X_a, X_a)$ when $s \in (0, 1]$ and where h_0 is the identity map. Then by setting $\varphi = h_1$ we have proved the lemma.

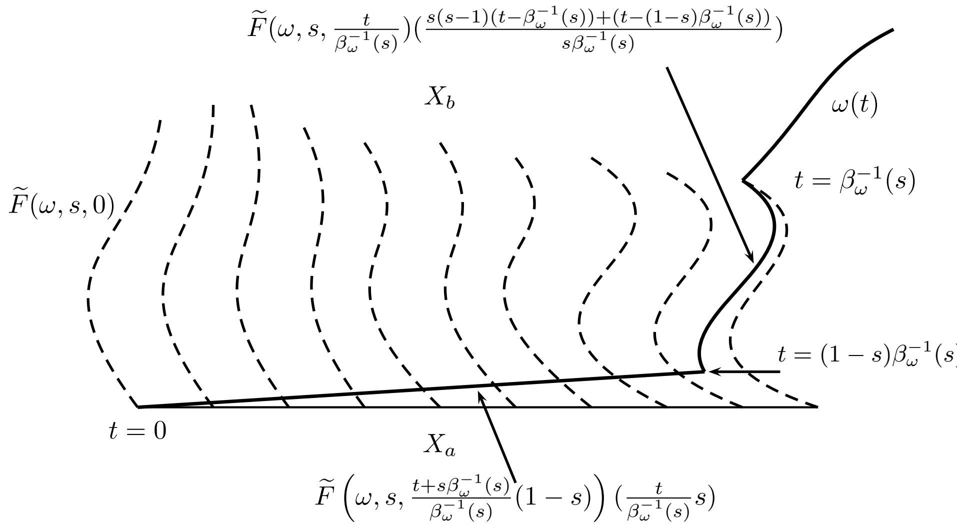
Let N be a neighborhood of tameness for X_a in $X_a \cup X_b$ and r be the corresponding strong deformation retraction. Define a map $F : \mathrm{pop}(X_b, X_b \cup X_a, X_a) \times (0, 1] \times I \to X_a$ by for all $\omega \in \mathrm{pop}(X_b, X_b \cup X_a, X_a)$ sending (ω, s, t) to $r\left(\omega(t \cdot \beta_\omega^{-1}(s)), 1\right)$. Define another map $G : \mathrm{pop}(X_b, X_b \cup X_a, X_a) \times (0, 1] \to \mathrm{hol}(X_b \cup X_a, X_a)$ by $\omega \mapsto R\left(\omega(\beta_\omega^{-1}(s))\right)$ (see Definition 1.7). The definition of Quinn stratified spaces tells us the map E_0, evaluating a holink path at its start point, is a fibration. So there is a lift $\widetilde{F}$,

$$\begin{tikzcd}
\mathrm{pop}(X_b, X_b \cup X_a, X_a) \times (0,1] \times \{1\} \arrow[r, "G"] \arrow[d] & \mathrm{hol}(X_b \cup X_a, X_a) \arrow[d, "E_0"] \\
\mathrm{pop}(X_b, X_b \cup X_a, X_a) \times (0,1] \times I \arrow[r, "F"] \arrow[ur, dashed, "\widetilde{F}"] & X_a
\end{tikzcd}$$



Now we can use $\widetilde{F}$ to construct to h. When $s = 0$, h_s is the identity and when $s \in (0, 1]$

$$h_s(\omega)(t) = \begin{cases} \widetilde{F}\left(\omega, s, \frac{t + s\beta_\omega^{-1}(s)}{\beta_\omega^{-1}(s)}(1-s)\right)\left(\frac{t}{\beta_\omega^{-1}(s)} s\right) & 0 \leqslant t \leqslant (1-s)\beta_\omega^{-1}(s) \\ \widetilde{F}\left(\omega, s, \frac{t}{\beta_\omega^{-1}(s)}\right)\left(\frac{s(s-1)(t-\beta_\omega^{-1}(s)) + (t-(1-s)\beta_\omega^{-1}(s))}{s\beta_\omega^{-1}(s)}\right) & (1-s)\beta_\omega^{-1}(s) < t \leqslant \beta_\omega^{-1}(s) \\ \omega(t) & \beta_\omega^{-1}(s) \leqslant t \leqslant T_\omega \end{cases}$$



Clearly this is continuous within the three intervals for $s \in (0,1]$. To see it is continuous at $t = (1-s)\beta_\omega^{-1}(s)$ just substitute for t and get $\widetilde{F}(\omega, s, 1-s)(s(1-s))$ for both expressions. To see it is continuous at $t = \beta_\omega^{-1}(s)$ substitute into the second expression to get $\widetilde{F}(\omega, s, 1)(1)$, now

$$\widetilde{F}(\omega, s, 1)(1) = G(\omega, s)(1) = R\left(\omega(\beta_\omega^{-1}(s))\right)(1) = \omega(\beta_\omega^{-1}(s))$$

so it is continuous at $t = \beta_\omega^{-1}(s)$.
To prove $h_s \to Id$ as $s \to 0$ we will prove it is true in each of the three intervals in the definition of h_s (in the last interval there is nothing to prove).
In the first interval

$$\begin{aligned}
&\lim_{s\to 0} \widetilde{F}\left(\omega, s, \frac{t+s\beta_\omega^{-1}(s)}{\beta_\omega^{-1}(s)}(1-s)\right)(\frac{t}{\beta_\omega^{-1}(s)}s) \\
&= \lim_{s\to 0} \widetilde{F}\left(\omega, s, \frac{t}{\beta_\omega^{-1}(s)}\right)(0) \\
&= \lim_{s\to 0} F\left(\omega, s, \frac{t}{\beta_\omega^{-1}(s)}\right) \\
&= \lim_{s\to 0} r\left(\omega(\frac{t}{\beta_\omega^{-1}(s)}\beta_\omega^{-1}(s))\right), 1) \\
&= \lim_{s\to 0} r(\omega(t)), 1),
\end{aligned}$$

and since we are within $0 \leqslant t \leqslant (1-s)\beta_\omega^{-1}(s)$ which tends towards $0 \leqslant t \leqslant \kappa(\omega)$ as s tends to 0 then $r(\omega(t)), 1)$ tends to $\omega(t)$ as s tends to 0 for $0 \leqslant t \leqslant (1-s)\beta_\omega^{-1}(s)$.

The second interval tends to $t = \lim_{s\to 0} \beta_\omega^{-1}(s) = \kappa(\omega)$ as s tends to 0 and

$$\begin{aligned}
&\lim_{s\to 0} \widetilde{F}\left(\omega, s, \frac{t}{\beta_\omega^{-1}(s)}\right)\left(\frac{s(s-1)(t-\beta_\omega^{-1}(s)) + (t-(1-s)\beta_\omega^{-1}(s))}{s\beta_\omega^{-1}(s)}\right)\\
&= \lim_{s\to 0} \widetilde{F}(\omega, s, 1)(q)\\
&= \lim_{s\to 0} G(\omega, s)(q)\\
&= \lim_{s\to 0} R\left(\omega(\beta_\omega^{-1}(s))\right)(q),
\end{aligned}$$

where

$$q = \frac{s(s-1)(t-\beta_\omega^{-1}(s)) + (t-(1-s)\beta_\omega^{-1}(s))}{s\beta_\omega^{-1}(s)}.$$

Now $\beta_\omega^{-1}(s) \to \kappa(\omega)$ as $s \to 0$ so $r\left(\omega(\beta_\omega^{-1}(s)), -\right)$ tends to the constant path $\omega(\kappa(\omega))$. Therefore $\lim_{s\to 0} R\left(\omega(\beta_\omega^{-1}(s))\right)(q) = \omega(\kappa(\omega))$.
This h satisfies our requirements as detailed at the start of the proof. □

4. Popaths and Holinks in the General Case

In a slight abuse of notation we will throughout this section redefine and use symbols like κ, N, λ, E and β in a more general context.

Definition 4.1. Consider X_a, X_b to be distinct strata in a homotopically stratified space X. Let $\kappa : \mathrm{pop}(X_b, X, X_a) \to \mathbb{R}$ be the map which sends $\omega \in \mathrm{pop}(X_b, X, X_a)$ to the unique point, $\kappa(\omega)$, in $\mathbb{R}$ such that $\omega(t) \in X_a$ for $t \leqslant \kappa(\omega)$ and $\omega(t) \notin X_a$ for $t > \kappa(\omega)$. Note κ is not a continuous map, but as in Definition 2.1 it is upper-continuous.

Lemma 4.2. Suppose X is a homotopically stratified space and $K \subset X$ is pure. Then there is a nearly stratum preserving strong deformation retract r of a neighborhood N of K in X to K. The neighborhood may be referred to as a neighborhood of tameness.

Proof. This is the first part of Proposition 3.2 of "Homotopically Stratified Sets" by Frank Quinn [1]. □

Lemma 4.3. Consider X_a as a lowest possible stratum in X by if necessary discarding any lower strata (which are of no consequence when considering $\mathrm{pop}(X_b, X, X_a)$). Let N be a neighborhood of tameness for X_a in X. There exists a continuous map $\lambda : \mathrm{pop}(X_b, X, X_a) \to \mathbb{R}$ that for all popaths $\omega : [0, T_\omega] \to X$ satisfies:

1. $\lambda(\omega) \in (\kappa(\omega), T_\omega)$
2. $\omega(t) \in N - X_a$ for $t \in (\kappa(\omega), \lambda(\omega)]$

Remark 4.4. Again this will be useful when we wish to use the κ map but cannot because it is not continuous.

Proof. This is proved in exactly the same way as Lemma 2.2. □

Definition 4.5. Let E denote the space $\{(\omega, t) \in \text{pop}(X_b, X, X_a) \times \mathbb{R} : t \in (\kappa(\omega), \lambda(\omega)]\}$ and p denote the canonical map $p : E \to \text{pop}(X_b, X, X_a)$. In fact p is a fiber bundle homeomorphic to the trivial bundle with total space $\text{pop}(X_b, X, X_a) \times (0, 1]$. Let β_ω denote a reparametrization of $(\kappa(\omega), \lambda(\omega)]$ to $(0, 1]$ giving a trivialization. The trivialisation can be obtained in the same way as in Definition 2.4.

Definition 4.6. Let B be a path space and A a subspace of B. We will say the inclusion $A \subset B$ is a special path inclusion if there exists a homotopy $h : B \times I \to B$ which fixes the start and end points of paths, image$\{h_s\} \subset A$ when $s \in (0, 1]$ and $h_0 =$ identity. Note this implies the inclusion map is a homotopy equivalence. Also note the composition (not concatenation) of two special path inclusions is again a special path inclusion.

Remark 4.7. Lemma 2.5 proves that $\text{hol}(X_b \cup X_a, X_a) \subset \text{pop}(X_b, X_b \cup X_a, X_a)$ is a special path inclusion. The aim of this paper is to show that $\text{hol}(X_b \cup X_a, X_a) \subset \text{pop}(X_b, X, X_a)$ is a special path inclusion for any two strata X_a and X_b of a homotopically stratified space X.

Lemma 4.8. Consider X_b as any stratum in X and X_a as a lowest possible stratum in X by if necessary discarding any lower strata. If $\text{hol}(X_b \cup X_a, X_a) \subset \text{pop}(X_b, X, X_a)$ is a special path inclusion then

$$\{\sigma \in \text{pop}(X_b, X, X) : \sigma(\delta) \notin X_a \text{ for all } \delta > 0\} \subset \text{pop}(X_b, X, X)$$

is also a special path inclusion.

Proof. First let us show $\{\sigma \in \text{pop}(X_b, X, N) : \sigma(\delta) \notin X_a \text{ for all } \delta > 0\} \subset \text{pop}(X_b, X, N)$ is a special path inclusion for N a neighborhood of tameness of X_a in X.

Given a path $\sigma \in \text{pop}(X_b, X, N)$ define $\sigma^+ \in \text{pop}(X_b, X, X_a)$ by

$$\sigma^+(t) = \begin{cases} r(\sigma(0), 1-t) & 0 \leqslant t \leqslant 1 \\ \sigma(t-1) & 1 \leqslant t \leqslant T_\omega + 1 \end{cases}.$$

Let f be the map from $\text{pop}(X_b, X, X_a) \times I$ to Moore paths onto I defined by

$$f(\omega, s)(t) = \begin{cases} 0 & 0 \leqslant t \leqslant 1 \\ (\min\{1, (t-1)(\lambda(\omega) - t)\})\, s & 1 \leqslant t \leqslant \lambda(\omega) \\ 0 & \lambda(\omega) \leqslant t \leqslant T_\omega \end{cases}$$

for all $\omega \in \text{pop}(X_b, X, X_a)$ where $1 < \lambda(\omega)$ and defined by $f(\omega, s)(t) = 0$ for all $t \in [0, T_\omega]$ if $\lambda(\omega) \leqslant 1$.

Now define a suitable homotopy for special path inclusion $g : \text{pop}(X_b, X, N) \times I \to \text{pop}(X_b, X, N)$ by $g_s(\sigma)(t) = \left(h_{f(\sigma^+,s)(t+1)}\sigma^+\right)(t+1)$ where h is a homotopy for the special path inclusion $\text{hol}(X_b \cup X_a, X_a) \subset \text{pop}(X_b, X, X_a)$. Intuitively this can be thought of as concatenating a path $[0, 1] \to X$ to the start of σ then manipulating the $(1, \lambda(\sigma))$ part of the new path away from X_a using h and finally removing the $[0, 1)$ part that was added.

Now since for all s, $g_s(\sigma)(t) = \sigma(t)$ when $t \geqslant t_0$ for any t_0 where $\sigma(t_0) \notin N$ we can extend g to give a special path inclusion $\{\sigma \in \text{pop}(X_b, X, X) : \sigma(\delta) \notin X_a \text{ for all } \delta > 0\} \subset \text{pop}(X_b, X, X)$ by setting $g_s(\sigma)$ as σ when $\sigma \in \text{pop}(X_b, X, X - N)$. $\square$

Theorem 4.9. $\mathrm{hol}(X_b \cup X_a, X_a) \subset \mathrm{pop}(X_b, X, X_a)$ *is a special path inclusion for any two distinct strata X_a and X_b of a homotopically stratified space X.*

Proof. In the case when X only has two strata then $X = X_a \cup X_b$ and the proposition is proved as Lemma 2.5. Likewise if there are no other strata which popaths from X_a to X_b can pass through then $\mathrm{pop}(X_b, X, X_a) = \mathrm{pop}(X_b, X_a \cup X_b, X_a)$ and the proposition is again proved by Lemma 2.5. Therefore we can assume X_a is a lowest stratum in X and there is at least one other stratum, X_c, which popaths from X_a to X_b may pass through, let X_c denote a lowest stratum of this type.
We will assume inductively that the lemma holds for $\mathrm{hol}(X_b \cup X_c, X_c) \subset \mathrm{pop}(X_b, X, X_c)$ and using this show $\mathrm{pop}(X_b, X - X_c, X_a) \subset \mathrm{pop}(X_b, X, X_a)$ is a special path inclusion. Then the proposition can be proven by removing strata like X_c until we are in the two strata case which has been proved already as Lemma 2.5.
We will define the homotopy we require as a composition (not concatenation) of two homotopies.
Define the first homotopy j by j_0 = identity and for $s \in (0, 1]$ and all $\omega \in \mathrm{pop}(X_b, X, X_a)$

$$j_s(\omega)(t) = \begin{cases} r\left(\omega(t \cdot \frac{\beta_\omega^{-1}(s)}{\beta_\omega^{-1}(\frac{1}{2}s)}), 1\right) & 0 \leqslant t \leqslant \beta_\omega^{-1}(\frac{1}{2}s) \\ R\left(\omega(\beta_\omega^{-1}(s))\right)\left(\frac{t-\beta_\omega^{-1}(\frac{1}{2}s)}{\beta_\omega^{-1}(s)-\beta_\omega^{-1}(\frac{1}{2}s)}\right) & \beta_\omega^{-1}(\frac{1}{2}s) \leqslant t \leqslant \beta_\omega^{-1}(s) \\ \omega(t) & \beta_\omega^{-1}(s) \leqslant t \leqslant T_\omega \end{cases} .$$

Intuitively this homotopy retracts the start of the path back into X_a and then travels along the reverse of the tameness retraction of $\omega(\beta_\omega^{-1}(s))$ before continuing along the original path from then onwards.
For the second homotopy k again define k_0 = identity and for $s \in (0, 1]$ and all $\omega \in \mathrm{pop}(X_b, X, X_a)$

$$k_s(\omega)(t) = \begin{cases} \omega(t) & 0 \leqslant t \leqslant \beta_\omega^{-1}(\frac{1}{2}s) \\ g_s\left(\omega|_{[\beta_\omega^{-1}(\frac{1}{2}s), T_\omega]}\right)(t) & \beta_\omega^{-1}(\frac{1}{2}s) \leqslant t \leqslant T_\omega \end{cases} ,$$

where g is the homotopy constructed in the previous lemma with respect to $\{\sigma \in \mathrm{pop}(X_b, X - X_a, X - X_a) | \sigma(\delta) \notin X_c \text{ for all } \delta > 0\} \subset \mathrm{pop}(X_b, X - X_a, X - X_a)$ which can used because of the inductive hypothesis. This homotopy ensures that for $s > 0$, $k_s(\omega)(t) \notin X_c$ for $t \in (\beta_\omega^{-1}(\frac{1}{2}s), T_\omega]$.
Now if we define a homotopy $l : \mathrm{pop}(X_b, X, X_a) \times I \to \mathrm{pop}(X_b, X, X_a)$ by $l_s = j_s \circ k_s$ we get $\mathrm{image}\{l_s\} \subset \mathrm{pop}(X_b, X - X_c, X_a)$ for $s > 0$ because the start of the path is the retraction to X_a by a nearly stratum preserving retraction of the point $g_s\left(\omega|_{[\beta_\omega^{-1}(\frac{1}{2}s), T_\omega]}\right)(\beta_\omega^{-1}(s))$ which is not in X_c or a lower strata. Clearly l_0 = identity because $j_0 = k_0$ = identity and l fixes start and end points of paths because both j and k fix start and end points of paths.
Thus we have satisfied the requirements for $\mathrm{pop}(X_b, X - X_c, X_a) \subset \mathrm{pop}(X_b, X, X_a)$ to be a special path inclusion and so using induction to remove strata like X_c we reach the two strata case and so conclude $\mathrm{hol}(X_b \cup X_a, X_a) \subset \mathrm{pop}(X_b, X, X_a)$ is a special path inclusion by Lemma 2.5. $\square$

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