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REPARAMETRIZATIONS WITH GIVEN STOP DATA

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Abstract

In [1], we performed a systematic investigation of reparametrizations of continuous paths in a Hausdorff space that relies crucially on a proper understanding of stop data of a (weakly increasing) reparametrization of the unit interval. I am grateful to Marco Grandis (Genova) for pointing out to me that the proof of Proposition 3.7 in [1] is wrong. Fortunately, the statement of that Proposition and the results depending on it stay correct. It is the purpose of this note to provide correct proofs.

1. Reparametrizations with given stop maps

To make this note self-contained, we need to include some of the basic definitions from [1]. The set of all (nondegenerate) closed subintervals of the unit interval $I = [0, 1]$ will be denoted by $\mathcal{P}_{[\,]}(I) = \{[a, b] \mid 0 \leqslant a < b \leqslant 1\}$.

Definition 1.1.
- A *reparametrization* of the unit interval I is a weakly increasing continuous self-map $\varphi : I \to I$ preserving the end points.
- A *non-degenerate* interval $J \subset I$ is a *φ-stop interval* if there exists a value $t \in I$ such that $\varphi^{-1}(t) = J$. The value $t = \varphi(J) \in I$ is called a *φ-stop value*.
- The set of all φ-stop intervals will be denoted as $\Delta_\varphi \subseteq \mathcal{P}_{[\,]}(I)$. Remark that the intervals in Δ_φ are disjoint and that Δ_φ carries a natural total order. We let $D_\varphi := \bigcup_{J \in \Delta_\varphi} J \subset I$ denote the *stop set* of φ; and $C_\varphi \subset I$ the set of all stop values.
- The *φ-stop map* $F_\varphi : \Delta_\varphi \to C_\varphi$ corresponding to a reparametrization φ is given by $F_\varphi(J) = \varphi(J)$.

It is shown in [1] that F_φ is an *order-preserving bijection* between (at most) *countable sets*. It is natural to ask (and important for some of the results in [1]) which order-preserving bijections between such sets arise from some reparametrization:

To this end, let

- $\Delta \subseteq \mathcal{P}_{[\,]}(I)$ denote a subset of *disjoint closed* sub-intervals – equipped with the natural total order;

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- $C \subseteq I$ denote a subset with the same cardinality as Δ;
- $F : \Delta \to C$ denote an order-preserving bijection.

I am grateful to the referee for pointing out the following lemma and its proof:

Lemma 1.2. *A subset* $\Delta \subseteq \mathcal{P}_{[\,]}(I)$ *of disjoint closed intervals is countable.*

Proof. Given a set Δ of disjoint non-degenerate closed sub-intervals of the unit interval I, each will contain rational numbers by density. By the axiom of choice, choose for each disjoint sub-interval a specific rational number contained in that sub-interval. The chosen set $\Delta' \subset \mathbf{Q}$ of rationals is countable as a subset of $\mathbf{Q}$. Combining an enumeration of Δ' with the bijection between Δ' and Δ mapping each interval to its chosen rational yields an enumeration of Δ. □

Proposition 1.3. *There exists a reparametrization* φ *with* $F_\varphi = F$ *if and only if conditions* (1) - (8) *below are satisfied for intervals contained in* Δ *and for all* $0 < z < 1$*:*

1. $\min J = \sup_{J'<J} \max J' \Rightarrow F(J) = \sup_{J'<J} F(J')$;
2. $\max J = \inf_{J<J'} \min J' \Rightarrow F(J) = \inf_{J<J'} F(J')$;
3. $\sup_{J'<z} \max J' = \inf_{z<J''} \min J'' \Rightarrow \sup_{J'<z} F(J') = \inf_{z<J''} F(J'')$;
4. $\sup_{J'<z} \max J' < \inf_{z<J''} \min J'' \Rightarrow \sup_{J'<z} F(J') < \inf_{z<J''} F(J'')$;
5. $\inf_{0<J} \min J = 0 \Rightarrow \inf_{0<J} F(J) = 0$;
6. $\inf_{0<J} \min J > 0 \Rightarrow \inf_{0<J} F(J) > 0$;
7. $\sup_{J<1} \max J = 1 \Rightarrow \sup_{J<1} F(J) = 1$;
8. $\sup_{J<1} \max J < 1 \Rightarrow \sup_{J<1} F(J) < 1$.

Proof. Conditions (1) – (3), (5) and (7) are necessary for the stop data of a *continuous* reparametrization φ; (4), (6) and (8) are necessary to avoid further stop intervals.

Given a stop map satisfying conditions (1) – (8), we construct a reparametrization φ_F with $F(\varphi_F) = F$ as follows: For $t \in D = \bigcup_{J\in\Delta} J$, one has to define: $\varphi(t) = F(J)$ with $t \in J$. This defines a weakly increasing function φ_F on D. Conditions (1) and (2) make sure that this function is continuous (on D). Condition (3) makes it possible to extend φ_F uniquely to a weakly increasing continuous function on the closure $\bar{D}$: $\varphi_F(z)$ is defined as $\sup_{J'<z} F(J')$ for $z = \sup_{J'<z} \max J'$ and/or as $\inf_{z<J''} F(J'')$ for $z = \inf_{z<J''} \min J$. By (5) and (7), $\varphi_F(0) = 0$ and $\varphi_F(1) = 1$ if $0, 1 \in \bar{D}$; if not, we have to take these as a definition.

The complement $O = I \setminus \bar{D}$ is an open (possibly empty) subspace of I, hence a union of at most countably many open subintervals $J = [a_-^J, a_+^J]$ with boundary in $\partial D \cup \{0, 1\}$. Condition (4), (6) and (8) make sure, that $\varphi_F(a_-^J) < \varphi_F(a_+^J)$. Hence, every collection of strictly increasing homeomorphisms between $[a_-^J, a_+^J]$ and $[\varphi_F(a_-^J), \varphi_F(a_+^J)]$ – preserving endpoints – extends φ_F to a continuous increasing map $\varphi_F : I \to I$ with $\Delta_{\varphi_F} = \Delta, C_{\varphi_F} = C$ and $F_{\varphi_F} = F$. □

It is natural to ask, whether

- every at most countable subset $C \subset I$ occurs as set of stop values of some reparametrization: This is answered affirmatively in [1], Lemma 2.10;
- every set $\{I\} \neq \Delta \subset \mathcal{P}_{[\,]}(I)$ of closed disjoint intervals arises as set of stop intervals of a reparametrization:

Proposition 1.4. *For every $\{I\} \neq \Delta$ of closed disjoint sub-intervals in the unit interval I, there exists a reparametrization φ with $\Delta_\varphi = \Delta$.*

Proof. We use Lemma 1.2 to provide us with an enumeration j of the totally ordered set Δ (defined either on $\mathbf{N}$ or on a finite integer interval $[1, n]$). Using j, we are going to construct a reparametrization φ with stop value set C_φ included in the set $I[\frac{1}{2}] = \{0 \leqslant \frac{l}{2^k} \leqslant 1\}$ of rational numbers with denominators a power of 2. To this end, we will associate to every number $z \in I[\frac{1}{2}]$ either an interval in Δ or a degenerate one point interval; we end up with an ordered bijection beween $I[\frac{1}{2}]$ and a superset of Δ; all excess intervals will be degenerate one-point sets.

To get started, let I_0 denote either *the* interval in Δ containing 0 or, if no such interval exists, the degenerate interval $[0, 0] = \{0\}$; likewise define I_1. Every number $z \in I[\frac{1}{2}]$ apart from 0 and 1 has a unique representation $z = \frac{l}{2^k}$ with l *odd*, $0 < l < 2^k$. The construction proceeds by induction on k using the enumeration j.

Assume for a given $k \geqslant 1$, I_z and thus the map $I : z \mapsto I_z$ defined for all $z = \frac{l}{2^{k-1}}$, $0 \leqslant l \leqslant 2^{k-1}$ as an ordered map. For $0 < z = \frac{l}{2^k} < 1$ and l odd, both $z_\pm = z \pm \frac{1}{2^k}$ have a representation as fraction with denominator 2^{k-1} and thus $I_{z_-} < I_{z_+}$ are already defined. Let $I_z = j(m)$ with m minimal such that $I_{z_-} < j(m) < I_{z_+}$ if such an m exists; if not, then I_z is defined as the degenerate interval containing the single element $\frac{1}{2}(\max I_{z_-} + \min I_{z_+})$. The map $I : z \mapsto I_z$ thus constructed on $I[\frac{1}{2}]$ is order-preserving. Moreover, this map is onto, since – by an induction over k – $I_{j(k)}$ occurs as I_z with some z of the form $\frac{l}{2^k}$. Hence, there is an order-preserving inverse map $I^{-1} : I_z \mapsto z$.

For $k \geqslant 0$, let φ_k denote the piecewise linear reparametrization that has constant value z on I_z for $z = \frac{l}{2^k}$, $0 \leqslant l \leqslant 2^k$ and that is linear inbetween these intervals. Remark that $\varphi_{k+1} = \varphi_k$ on all I_z with $z = \frac{l}{2^k}$ including all occuring degenerate intervals. As a consequence, $\| \varphi_k - \varphi_{k+1} \| < \frac{1}{2^k}$, and hence for all $l > k$, $\| \varphi_k - \varphi_l \| < \frac{1}{2^{k-1}}$. Hence, the sequence $(\varphi_k)_{k\in\mathbf{N}}$ converges uniformly to a continuous reparametrization φ.

By construction, the resulting reparametrization φ is constant on all intervals in Δ; on every open interval between these stop intervals, it is linear and strictly increasing. In particular, $\Delta_\varphi = \Delta$. □

Remark 1.5. I was first tempted to prove Proposition 1.4 by taking some integral of the characteristic function of the complement of D and to normalize the resulting function. But in general, this does not work out since, as already remarked in [1], it may well be that $\bar{D} = I$!

2. Concluding remarks

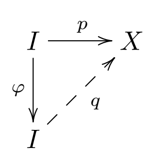
Remark 2.1. 1. Instead of constructing the reparametrization φ in Proposition 1.4, it is also possible to apply the criteria in Proposition 1.3 to the restriction $I_{|\Delta}$ of the map I from the proof above.

2. Proposition 1.3 replaces Proposition 2.13 in [1]. To get sufficiency, requirements (1) and (2) had to be added to those mentioned in [1] in order to make sure that the map φ_F is continuous on D. Moreover, (6) and (8) had to be added to avoid stop intervals containing 0, resp. 1 in case Δ does not contain such intervals.
   In particular, the midpoint map m that associates to every interval in Δ its midpoint satisfies the criteria given in [1], Proposition 2.13, but if fails in general to satisfy conditions (1) and (2) in Proposition 1.3 in this note; in particular, the map φ_m will in general not be continuous, as remarked by M. Grandis. The midpoint map m was used in the flawed proof of [1], Proposition 3.7 – stated as Proposition 2.2 below.

The main focus in [1] is on reparametrizations of continuous paths $p : I \to X$ into a Hausdorff space X. A continuous path q is called *regular* if it is constant or if the restriction $q|_J$ to every non-degenerate sub-interval $J \subseteq I$ is *non-constant*.

Proposition 2.2. (Proposition 3.7 in [1])
For every path $p : I \to X$, there exists a regular path q and a reparametrization such that $p = q \circ \varphi$.

Proof. A non-constant path p gives rise to the set of all (closed disjoint) *stop intervals* $\Delta_p \subset \mathcal{P}_{[\,]}(I)$, consisting of the maximal subintervals $J \subset I$ on which p is constant. Proposition 1.4 yields a reparametrization φ with $\Delta_\varphi = \Delta_p$ and thus a set-theoretic factorization



through a map $q : I \to X$ that is not constant on any non-degenerate subinterval $J \subseteq I$. The continuity of q follows as in the remaining lines of the proof in [1]. $\square$

References

[**1**] U. Fahrenberg and M. Raussen *Reparametrizations of continuous paths*, J. Homotopy Relat. Struct. **2** (2007), no.2, 93 – 117.

See also the references in [**1**].

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