Long-time behaviour for Hirota equation

Xianyun Guo, Chaosheng Zhu*

School of Mathematics and Statistics, Southwest University Chongqing, 400715, P. R. China
*Corresponding author
E-mail: xianyun860720@163.com (X. Y. Guo), zcs@swu.edu.cn (C. S. Zhu)

Abstract

In this paper, we investigate the long-time behavior of the solutions for the Hirota equation with the periodic boundary condition. At first, by time uniform priori estimates of solutions, we obtain the existence of global solutions. Furthermore, we prove the existence of a global attractor. Finally, by squeezing property and Lipschitz continuity, we prove the existence of an exponential attractor of finite fractal dimension which contains the global attractor.

2000 Mathematics Subject Classification. **35B40**. 35B41, 35B45.
Keywords. Hirota equation, global attractor, exponential attractor.

1 Introduction

In this paper, we consider the long time behaviour of solutions to the following Hirota equation with the periodic boundary condition

$$iu_t + (\alpha - i\mu)u_{xx} + i\beta u_{xxx} + i\gamma(|u|^2u)_x + \delta|u|^2u + i\lambda u = 0, \quad t \in \mathbb{R}^1_+, x \in \mathbb{R}^1, \tag{1.1}$$

$$u(x,t) = u(x+L,t), \quad t \in \mathbb{R}^1_+, x \in \mathbb{R}^1, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^1, \tag{1.3}$$

where μ, λ, L are positive constants, α, β, γ, δ are real constants, L is a period.

Physical, chemical, biological, and many other areas put forward a lot of infinite dimension dynamical system problem. The Hirota equation (1.1) is Ginzburd-Landau-KdV type. Nonlinear Schrödinger (NLS) and Korteweg-de Vries equations are asymptotical models for the waterwave propagation. These models supplemented with a damping and an external force provide examples of infinite-dimensional dynamical systems, in the framework described in [16]. There is an extensive literature on the study of Korteweg-de Vries equations [8, 9, 10, 11]. On the other hand, there is an extensive literature on the study of Schrödinger equations, and many of the studies were concerned with the existence of the absorbing sets and the global attractor. For example, in [1] the authors studied the long time behaviour of the solutions to a nonlinear Schrodinger equation, in presence of a damping term, and a forcing term, when the space variable x varies over $\mathbb{R}$ and shown that the long time behaviour is described by an attractor which captures all the trajectories in $H^1(\mathbb{R})$. Their main result is concerned with the asymptotic smoothing effect for the equations. In other words, they prove that the attractor is included and compact in $H^2(\mathbb{R})$. In [5] the authors discussed that the weakly damped nonlinear Schrödinger flow in $L^2(\mathbb{R})$ provides a dynamical system which possess a global attractor. In [17] proved the existence of the global attractor $\mathcal{A}$ for the non-local equation in the strong topology of H^1 and that the global attractor is regular, i.e., $\mathcal{A} \subset H^2$ assuming that $f(x)$ is of class C^2. In [15] studied the existence, uniqueness, continuity and the asymptotic compactness of the solutions on the initial data for the Klein-Gordon-Schrödinger type equations and obtained the existence of a global compact attractor. In [6], the existence of global attractor for a generalized Ginzburd-Landau equation has

Tbilisi Mathematical Journal 5(1) (2012), pp. 51–64.
Tbilisi Centre for Mathematical Sciences & College Publications.

Received by the editors: 21 December 2011; 10 July 2012.
Accepted for publication: 13 September 2012.

been studied. In [12], the author considered the initial boundary value problems of dissipative Schrödinger-Boussinesq equations and proved the existence of global attractors.

On the other hand, there is an extensive literature on the study of exponential attractor. In [13] considered a complex Ginzburd-Landau type equation with periodic initial value condition in three spatial dimensions and finally proved the existence of exponential attractor. In [14] was constructed quasiperiodic non-autonomous evolution equations and presented two methods to prove the squeezing property, the first one is well adapted to dissipative equations and the second one to partially dissipative equations. In [4] the existence of exponential attractor for Ginzburd-Landau equation has been proved. In [7] was shown the squeezing property and the existence of finite dimensional exponential attractor.

In this paper, we investigate the long-time behavior of the solutions for (1.1)-(1.3). At first, by time-uniform a priori estimates of solutions, we obtain the existence of global solutions. Furthermore, we prove the existence of a global attractor. Finally, by squeezing property and Lipschitz continuity, we prove the existence of an exponential attractor of finite fractal dimension which contains the global attractor.

We introduce the following standard notations.

$H = L^2_{per}[0, L] = \{u \in L^2[0, L], u(x + L) = u(x)\}$,

$V = H^1_{per}[0, L] = \{u : u \in H, u_x \in H\}$.

Let $(u, v) = \int_0^L u\bar{v}dx$; $\|u\| = \sqrt{(u, u)}$ denote the inner product and norm in H. Respectively, the norm in V is defined: $\|u\|^2_V = \|u\|^2 + \|u_x\|^2$.

The main results of this paper is stated as follows.

Theorem 1.1. Let u_0 given in V, then there exists a unique solution u of (1.1)-(1.3) satisfying $u(t) \in C([0, \infty); V) \cap C^1((0, \infty); H)$.

Theorem 1.2. The semigroup of operator $(S(t))_{t\geq 0}$ defined by the problem (1.1)-(1.3) possesses a compact global attractor $\mathcal{A}$ in V.

Theorem 1.3. There exists an exponential attractor $\mathscr{M}$ for $(\{S(t)\}_{t\geq 0}, B)$ such that

$$d_F(\mathscr{M}) \leq N_0 \max\{1, \frac{\log(16L + 1)}{2\log 2}\},$$

and

$$dist_V(S(t)u_0, \mathscr{M}) \leq c_1 \exp\{(-\frac{c_2}{t_*})t\},$$

for all $u_0 \in B$, c_1, c_2 are the constants independent of u_0, t, where $d_F(\mathscr{M})$ denotes the fractal dimension of $\mathscr{M}$.

This paper is organized as follows. In next section, we establish some time-uniform a priori estimates and the existence of absorbing set in V, then we prove the existence of the global attractor for the problem (1.1)-(1.3). In section 3, we prove the squeezing property on B and Lipschitz continuity of the semi-group $\{S(t)\}_{t>0}$, therefore we obtain the the existence of exponential attractor for the problem (1.1)-(1.3).

2 Existence of Global attractor

In this section, we derive the time uniform a priori estimates, which enable us to show the existence of the global attractor.

Lemma 2.1. If $u_0(x) \in V$, the problem (1.1)-(1.3) possesses a unique solution

$$u(t) \in C((0,\tau);V) \cap C^1((0,\tau);H), \tag{2.1}$$

for some $\tau > 0$, which depends on u_0.

Proof. By the standard methods, as A. Henry and A. Pazy, it can be proved.

To extend local solution to global solution, we must derive the uniform a priori estimates for the problem (1.1)-(1.3).

Lemma 2.2. Let $\mu > 0$, $\lambda > 0$, we have

$$\|u(t)\|^2 \le \|u_0\|^2 e^{-2\lambda t}, \quad t \in [0,+\infty). \tag{2.2}$$

Proof. (1.1) is equivalent to

$$u_t = (i\alpha+\mu)u_{xx} - \beta u_{xxx} - 2\gamma|u|^2 u_x - \gamma u^2 \overline{u}_x + i\delta(|u|^2)u - \lambda u. \tag{2.3}$$

Taking the real part of the inner product of (2.3) with u in L^2, we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L |u|^2 dx \quad &= \quad -\mu\int_0^L |u_x|^2 dx - 2\gamma \text{Re}\int_0^L |u|^2 u_x \overline{u} dx \\ &\quad -\gamma\text{Re}\int_0^L u^2\overline{u}_x\overline{u}dx - \lambda\int_0^L |u|^2 dx.\end{aligned} \tag{2.4}$$

Since

$$\text{Re}\int_0^L u_x\overline{u}dx = 0, \quad \text{Re}\int_0^L |u|^2 u_x\overline{u}dx = 0,$$
$$\text{Re}\int_0^L \overline{u}_x\overline{u}dx = 0, \quad \text{Re}\int_0^L u^2\overline{u}_x\overline{u}dx = 0,$$

we have

$$\frac{d}{dt}\|u\|^2 + 2\mu\|u_x\|^2 + 2\lambda\|u\|^2 = 0. \tag{2.5}$$

(2.2) follows by Gronwall Lemma [16].

Lemma 2.3. On the assumption of Lemma 2.1 and Lemma 2.2, we have

$$2\mu\int_s^t \|u_x(\cdot,\tau)\|^2 d\tau \le \|u(s)\|^2, \tag{2.6}$$

$$2\mu\int_0^t \|u_x(\cdot,\tau)\|^2 d\tau \le \|u(0)\|^2. \tag{2.7}$$

Proof. We integrate (2.5) on $[s,t]$, $[0,t]$ separately, and we obtain (2.6) and (2.7).

Lemma 2.4. On the assumption of Lemma 2.1 and Lemma 2.2, we have

$$\frac{d}{dt}\|u_x\|^2+\mu\|u_{xx}\|^2\leq K_1(1+\|u_x\|^2)^2, \tag{2.8}$$

where K_1 depends on α, β, γ, δ, λ, but not irrelevant with L and $\|u_0\|$.

Proof. Taking the real part of the inner product of (2.3) with u_{xx}, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L|u_x|^2dx+\mu\int_0^L|u_{xx}|^2dx=2\gamma\text{Re}\int_0^L|u|^2u_x\overline{u_{xx}}dx\\ -\gamma\text{Re}\int_0^L|u_x|^2u\overline{u_x}dx-\delta\text{Im}\int_0^L u^2\overline{u_x}^2dx-\lambda\int_0^L|u_x|^2dx,\end{aligned} \tag{2.9}$$

We now majorize the right hand side of (2.9). Firstly,

$$-\delta\text{Im}\int_0^L u^2\overline{u_x}^2dx\leq\delta\int_0^L|u|^2|u_x|^2dx\leq\delta\|u\|_{L^\infty}^2\|u_x\|^2,$$

by Agmon inequality

$$\|u\|_{L^\infty}^4\leq C_1(\|u\|^2+\|u_x\|^2)\|u\|^2,$$

we have

$$\begin{aligned}&-\delta\text{Im}\int_0^L u^2\overline{u_x}^2dx\leq\delta\sqrt{C_1}(\|u\|^2+\|u\|\|u_x\|)\|u_x\|^2\\ &\leq\delta C_2(\|u_x\|^2+\|u_x\|^3)\leq 2\delta C_2(\|u_x\|^2+\|u_x\|^4).\end{aligned} \tag{2.10}$$

Secondly,

$$\begin{aligned}&2\gamma\text{Re}\int_0^L|u|^2u_x\overline{u_{xx}}dx\leq 2\gamma\int_0^L|u|^2|u_x||u_{xx}|dx\\ \leq\;&\varepsilon_1\|u_{xx}\|^2+\frac{\gamma^2}{\varepsilon_1}\int_0^L|u|^4|u_x|^2dx\\ \leq\;&\varepsilon_1\|u_{xx}\|^2+\frac{\gamma^2}{\varepsilon_1}\|u\|_{L^\infty}^2\int_0^L|u|^2|u_x|^2dx\\ \leq\;&\varepsilon_1\|u_{xx}\|^2+\frac{\gamma^2}{\varepsilon_1}\sqrt{C_1}(\|u\|^2+\|u\|\|u_x\|)C_2(\|u_x\|^2+\|u_x\|^3)\\ \leq\;&\varepsilon_1\|u_{xx}\|^2+\frac{C_3}{\varepsilon_1}(\|u_x\|^2+\|u_x\|^4).\end{aligned} \tag{2.11}$$

Thirdly,

$$-\gamma\text{Re}\int_0^L|u_x|^2u\overline{u_x}dx\leq\gamma\int_0^L|u||u_x|^3dx\leq\gamma\left(\int_0^L|u|^2|u_x|^2dx\right)^{\frac{1}{2}}\left(\int_0^L|u_x|^4dx\right)^{\frac{1}{2}},$$

here by Nirenberg inequality and $\int_0^L u_x dx = 0$, we have

$$\int_0^L |u_x|^4 dx \leq C\|u_{xx}\|\|u_x\|^3,$$

so,

$$-\gamma \text{Re} \int_0^L |u_x|^2 u\overline{u_x} dx \leq C_4^2 \int_0^L |u|^2|u_x|^2 dx + \frac{1}{4}\|u_x\|^3\|u_{xx}\|,$$

by (2.10) and Young inequality, we have

$$-\gamma \text{Re} \int_0^L |u_x|^2 u\overline{u_x} dx \leq C_5(\|u_x\|^2 + \|u_x\|^4) + \varepsilon_2\|u_{xx}\|^2. \tag{2.12}$$

Put (2.10)-(2.12) into (2.9), let $\varepsilon_1 = \varepsilon_2 = \frac{\mu}{4}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L |u_x|^2 dx + \mu \int_0^L |u_{xx}|^2 dx \\
&\leq \lambda \int_0^L |u_x|^2 dx + \varepsilon_1\|u_{xx}\|^2 + \frac{C_3}{\varepsilon_1}(\|u_x\|^2 + \|u_x\|^4) \\
&\quad + C_5(\|u_x\|^2 + \|u_x\|^4) + \varepsilon_2\|u_{xx}\|^2 + 2\delta C_2(\|u_x\|^2 + \|u_x\|^4).
\end{aligned}$$

The lemma is proved.

Lemma 2.5. Under the assumption of Lemma 2.1 and Lemma 2.2, we have

$$\|u(t)\|_V \leq K_2, \tag{2.13}$$

where K_2 depends on K_1 and $\|u_0\|_V$.

Proof. By Lemma 2.1 and Lemma 2.2, we have

$$\frac{d}{dt}(1 + \|u_x\|^2) \leq K_1(1 + \|u_x\|^2)^2, \tag{2.14}$$

let $y = 1 + \|u_x\|^2$, when $\tau < (K_1C_1 + \|u_{0x}\|^2)^{-1}$, By (2.14) we have

$$1 + \|u_x\|^2 \leq (\frac{1}{1 + \|u_{0x}\|^2} - K_1 t)^{-1}, t < \tau,$$

let $t \leq (2K_1(1 + \|u_{0x}\|^2))^{-1} = \overline{\tau}$, we have $\|u_x\|^2 \leq 1 + 2\|u_{0x}\|^2$, $t \leq \overline{\tau}$. when $t > \tau$, $t - \overline{\tau} \leq s \leq t$, let $z(t) = y(t)exp(-\int_s^t K_1 y(\tau)d\tau)$, Then $z_t(t) \leq 0$.

Thus,

$$z(t) \leq z(s) = y(s) = 1 + \|u_{0x}\|^2,$$

$$y(t) \leq (1 + \|u_x\|^2)\exp(K_1 \int_s^t y(\tau)d\tau),$$

by (2.6), we have

$$\exp(K_1 \int_s^t y(\tau)d\tau) \le \exp(\frac{1}{\alpha}K_1 c\|u(s)\|^2 + K_1\overline{\tau}) \le \exp(K_1 C_6),$$

therefore,

$$y(s) \le (1 + \|u_x\|^2)\exp(K_1 C_6). \tag{2.15}$$

We integrate (2.15), $s \in [t-\overline{\tau}, t]$, we have

$$y(t) \le \frac{C_6 \exp(K_1 C_6)}{\overline{\tau}}.$$

This concludes Lemma2.5.

By Lemma 2.1 and Lemma 2.5, we can extend $\tau = \infty$. Thus, we have proved Theorem 1.1.

Now let us prove the existence of global attractor.

Proof of Theorem 1.2. At first, let us prove the existence of the bounded absorbing set in V.

By (2.14) we have $\frac{dy}{dt} \le K_1 y^2 = K_1 y \cdot y$. On the other hand, by (2.15) we replace interval (s, t) with $(t, t+1)$, and we have

$$\int_t^{t+1} (1 + \|u_x\|^2)d\tau \le 1 + \frac{1}{2\mu}\|u(t)\|^2,$$

let $\|u_0\| \le R$, when $t \ge 0$, by (2.2), we have $\|u(t)\|^2 \le R^2$.

Now K_1 is irrelevant with u_0, and $K_1 \int_t^{t+1} y(\tau)d\tau \le K_1 + K_1 R^2 = \rho_1$. Thus, by the Uniform Gronwall inequality, we have $y(t+1) \le \rho_1 \exp(\rho_1)$, $t \ge 0$. So, when $t \ge 1$, we have $\|u_x\|^2 \le \rho_1 \exp(\rho_1) - 1 = \rho_2^2$.

Thus, we get the existence of the bounded absorbing set in V. That is, if u is the solution of the problem (1.1)-(1.3), and $\|u_0\| \le R$, when $t \ge 1$, we have $\|u\|_V^2 \le R^2 + \rho_2^2$.

Next we prove the semigroup $(S(t))_{t\ge 0}$ defined by (1.1)-(1.3) is compact. To this end, taking the inner product of (1.1) with u_{xxxx}, use Gagliardo-Nirenberg inequality, Sobolev imbedding theorem and some basic inequalities, we have

$$\frac{d}{dt}\|u_{xx}\|^2 + \mu\|u_{xx}\|^2 \le C_7 + C_8\|u_{xx}\|^2,$$

thus,

$$\frac{d}{dt}(t\|u_{xx}\|^2) + \mu(t\|u_{xx}\|^2) \le C_7 + C_8(t\|u_{xx}\|^2) + \|u_{xx}\|^2,$$

by Lemma 2.4, we have

$$\int_0^1 \|u_{xx}\|^2 dt \le C(R)t.$$

By Gronwall lemma, we have

$$t\|u_{xx}\|^2 dt \leq C(R,t), \quad t \in (0,+\infty).$$

Thus, when $t \in (0,+\infty)$, $S(t)$ is compact in V. Therefore, if $u(t)$ is the solution of the problem (1.1)-(1.3), $u(t) = S(t)u_0$, let $B = \{u \in V, \|u\|_V^2 \leq R^2 + \rho_2^2\}$, then the omega limit set of B, $\mathscr{A} = \omega(B)$, is the compact global attractor of $S(t)$ in V, the proof of Theorem 1.2 is completed.

3 Existence of exponential attractor

Let $\mathscr{H}$ be a separable Hilbert space and B be a compact subset of $\mathscr{H}$. Let $\{S(t)\}_{t>0}$ be a nonlinear continuous semi-group that leaves the set B invariant and set

$$\mathscr{A} = \bigcap_{t>0} S(t)B,$$

that is, $\mathscr{H}$ is the global attractor for $\{S(t)\}_{t>0}$ on B.

Definition 3.1. A compact set $\mathscr{M}$ is an exponential attractor for $S(t)$ if
(1) it has finite fractal dimension, $\dim_F \mathscr{M} < +\infty$;
(2) it is positively invariant, $S(t)\mathscr{M} \subset \mathscr{M}$, for every $t \in [0,+\infty)$;
(3) it attracts exponentially the bounded subsets of E in the following sense:

$$\forall B \subset E \; bounded, \; dist(S(t)B, \mathscr{M}) \leq Q(\|B\|_E)e^{-\alpha t}, \; t \in [0,+\infty),$$

where the positive constant α and the monotonic function Q are independent of B.

We have given a sufficient condition for the existence of exponential attractors in [2]. The key idea to prove this is the so-called squeezing property; we recall this property as follows:

Definition 3.2. A mapping S: $X \to X$, where X is a compact subset of a Hilbert space E, enjoys the squeezing property on X if, for some $\delta \in (0, \frac{1}{4})$, there exists an orthogonal projector $P = P(\delta)$ with finite rank such that, for every $u, v \in X$, either

$$\|(I-P)(Su - Sv)\|_E \leq \|P(Su - Sv)\|_E,$$

or

$$\|Su - Sv\|_E \leq \delta\|u - v\|_E.$$

We can note that this property makes an essential use of orthogonal projectors with finite rank, so the corresponding construction is valid in Hilbert spaces.

Theorem 3.3. ([3]) Let $\mathscr{H}$ be a separable Hilbert space and B be a nonempty closed bounded subset of E. Assume that
(1) S is a Lipschitz continuous map with Lipschitz constant L on B;
(2) S is asymptotically compact on B;
(3) S satisfies the discrete squeezing property on B (with rank N_0), then S has an exponential attractor $\mathscr{A} \subseteq \mathscr{M}$ on B, where $\mathscr{A}$ is a global attractor for S on B. Moreover, the fractal dimension of $\mathscr{M}$ satisfies

$$d_F(\mathscr{M}) \leq N_0 \max\{1, \frac{\log(16L+1)}{2\log 2}\},$$

and

$$dist_{\mathscr{H}}(S(t)u_0, \mathscr{M}) \leq c_1 \exp\{-\frac{c_2}{t_*}t\}.$$

Where c_1, c_2 are the constants which are irrelevant of u_0, t.

Clearly, we need only to find the time t_* and the projection P of rank N_0 to evaluate the right-hand sides of the above inequality. That is what we proceed to do.

Now, we take $\mathscr{H} = V = H^1_{per}[0, L]$ and choose B_0 as follows,

$$B_0 = \{u \in H^1_{per}[0, L], \|u\| \leq \rho_0, \|u_x\| \leq \rho_1\},$$

where ρ_0, ρ_1 are constants which depend on the coefficients of (1.1). If $u_0 \in B_0$, then by the Sobolev embedding theorem we have $\|u\|_{L^\infty} \leq \rho$, $\|u\|_{W^{1,\infty}} \leq \rho_3$. By the second section we know B_0 is an absorbing set of $S(t)$ in V, and there is $T_1 > 0$, when $t \geq T_1$, and $\|u\| + \|u_x\| + \|u_{xx}(t)\| \leq \rho_2$, ρ_2 only depends on ρ_0, ρ_1. Let

$$B = \bigcup_{t > T_1} S(t)B_0,$$

then B is a compact subset in V_1, and is invariant under the action of $S(t)$. Let $\{\lambda_n\}_{n\in N}$ denote the eigenvalues of $-\partial_{xx}$ with perdioc boundary condition, $\{\omega_n\}_{n\in N}$ are the eigenfunctions with respect to $\{\lambda_n\}_{n\in N}$. Let

$$H_N = span\{\omega_1, \cdots, \omega_N\},$$

then we let $P_N : H \to H_N$ be the orthogonal projection onto H_N, and $Q_N = 1 - P_N$. Note that the projection P_N and Q_N are the orthogonal both in H and V. It follows easily from the definition of the projection Q_N that

$$\|u\|^2 \leq \frac{1}{\lambda_{N_H}}\|u_x\|^2, \quad \forall u \text{ in } Q_N V_1. \tag{3.1}$$

We intend to show the sequeezing property (defined by Definition 3.2) by these orthogonal projection; let us emphasize once again that B is a bound set in $H^2_{per}[0, L]$. Once t_* and N_0 are specified, we can obtain the existence of an exponential attractor directly from Theorem 3.1. Moreover, both the fractal dimension of $\mathscr{M}$ and the exponential rate of convergence to $\mathscr{M}$ can be estimated explicitly.

Firstly, we give the estimate of the difference of two solutions of (1.1)-(1.3). Let u, v be the solutions of (1.1) with initial value u_0, v_0 respectively, we have:

$$u, v \in C((0,\infty); V) \cap C^1((0,\infty); V) \cap C((0,\infty); H^n_{per}[0, L]),$$

for $n \leq 2$, let $\omega = u - v$, then ω satisfies

$$\begin{aligned}\omega_t &= (i\alpha + \mu)\omega_{xx} - \beta\omega_{xxx} - 2\gamma(|u|^2 u_x - |v|^2 v_x) \\ &\quad -\gamma(u^2\overline{u}_x - v^2\overline{v}_x) + i\delta(|u|^2 u - |v|^2 v) - \lambda\omega.\end{aligned}$$

Since

$$\begin{aligned} |u|^2u_x-|v|^2v_x &= |u|^2u_x-|v|^2u_x+|v|^2u_x-|v|^2v_x \\ &= (|u|^2-|v|^2)u_x+|v|^2\omega_x \\ &= (u\overline{u}-u\overline{v}+u\overline{v}-v\overline{v})u_x+|v|^2\omega_x \\ &= (u\overline{\omega}+\omega\overline{v})u_x+|v|^2\omega_x, \end{aligned}$$

similarly

$$u^2\overline{u}_x-v^2\overline{v}_x=v^2\overline{\omega}_x+(u+v)\overline{u}_x\omega,$$
$$|u|^2u-|v|^2v=u^2\overline{\omega}+\omega\overline{v}u+|v|^2\omega,$$

then

$$\omega_t=(i\alpha+\mu)\omega_{xx}-\beta\omega_{xxx}-2\gamma(|v|^2\omega_x+\omega\overline{v}u_x+\overline{\omega}uu_x)$$
$$-\gamma(v^2\overline{\omega}_x+(u+v)\overline{u}_x\omega)+i\delta(u^2\overline{\omega}+\omega\overline{v}u+|v|^2\omega)-\lambda\omega,\quad x\in\mathbb{R}^1,t\in\mathbb{R}^1_+, \tag{3.2}$$
$$\omega(x,t)=\omega(x+L,t),\quad x\in\mathbb{R}^1,t\in\mathbb{R}^1_+, \tag{3.3}$$
$$\omega(x,0)=\omega_0(x),\qquad x\in\mathbb{R}^1. \tag{3.4}$$

Taking the real part of the inner product of (3.2) with ω, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L|\omega|^2dx+\mu\int_0^L|\omega_x|^2dx \\ =\ &-2\gamma\mathrm{Re}\int_0^L(|v|^2\overline{\omega}\omega_x+uu_x\overline{\omega}^2+\overline{v}u_x|\omega|^2)dx \\ &-\gamma\mathrm{Re}\int_0^L(v^2\overline{\omega}_x\overline{\omega}+\overline{u}_x(u+v)|\omega|^2)dx \\ &-\delta\mathrm{Im}\int_0^L(u^2\overline{\omega}^2+u\overline{v}|\omega|^2)dx-\lambda\int_0^L|\omega|^2dx \\ \leq\ &2\gamma\int_0^L(|v|^2|\omega||\omega_x|+|u||u_x||\omega|^2+|v||u_x||\omega|^2)dx \\ &+\gamma\int_0^L(|v|^2|\omega_x||\omega|+|u_x|(|u|+|v|)|\omega|^2)dx \\ &+\delta\int_0^L(|u|^2|\omega|^2+|u||v||\omega|^2)dx+\lambda\int_0^L|\omega|^2dx \\ \leq\ &\int_0^L 3\gamma|v|^2|\omega||\omega_x|+(3\gamma|u||u_x|+3\gamma|v||u_x|+\delta|u|^2+\delta|u||v|+\lambda)|\omega|^2dx \end{aligned}$$

since $u_0, v_0 \in B$, by the invariant of B, we know $u, v \in B$, then

$$\begin{aligned}
&\frac{d}{dt}\int_0^L |\omega|^2 dx + 2\mu \int_0^L |\omega_x|^2 dx \\
\leq\ & \int_0^L 6\rho_0^2\gamma|\omega||\omega_x|dx + \int_0^L (12\rho_0\rho_1\gamma + 4\rho_0^2\delta + 2\lambda)|\omega|^2 dx \\
\leq\ & \int_0^L \mu|\omega_x|^2 dx + \int_0^L \frac{9\rho_0^4\gamma^2}{\mu}|\omega|^2 dx + \int_0^L (12\rho_0\rho_1\gamma + 4\rho_0^2\delta + 2\lambda)|\omega|^2 dx,
\end{aligned}$$

thus, we get

$$\frac{d}{dt}\int_0^L |\omega|^2 dx + \mu \int_0^L |\omega_x|^2 dx \leq M_0 \int_0^L |\omega|^2 dx, \tag{3.5}$$

where $M_0 = \frac{9\rho_0^4\gamma^2}{\mu} + 12\rho_0\rho_1\gamma + 4\rho_0^2\delta + 2\lambda$, that is,

$$\frac{d}{dt}\|\omega\|^2 \leq M_0\|\omega\|^2. \tag{3.6}$$

Taking the real part of the inner product of (3.2) with ω_{xx}, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L |\omega_x|^2 dx + \mu \int_0^L |\omega_{xx}|^2 dx \\
=\ & 2\gamma \mathrm{Re}\int_0^L (|v|^2\omega_x + \omega\overline{v}u_x + \overline{\omega}uu_x)\overline{\omega_{xx}}dx \\
& + \gamma \mathrm{Re}\int_0^L (v^2\overline{\omega}_x + (u+v)\overline{u}_x\omega)\overline{\omega_{xx}}dx \\
& + \mathrm{Im}\delta(u^2\overline{\omega} + \omega\overline{v}u + |v|^2\omega)\overline{\omega_{xx}}dx - \lambda\int_0^L |\omega_x|^2 dx \\
\leq\ & 3\gamma \int_0^L (\rho_0^2|\omega_x| + 2\rho_0\rho_1|\omega|)|\omega_{xx}|dx \\
& + 3\delta \int_0^L \rho_0^2|\omega||\omega_{xx}|dx + \lambda\int_0^L |\omega_x|^2 dx \\
=\ & 3\rho_0^2\gamma \int_0^L |\omega_x||\omega_{xx}|dx + \lambda\int_0^L |\omega_x|^2 dx \\
& + (6\rho_0\rho_1\gamma + 3\rho_0^2\delta)\int_0^L |\omega||\omega_{xx}|dx,
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{d}{dt}\int_0^L |\omega_x|^2 dx + 2\mu \int_0^L |\omega_{xx}|^2 dx \\
\leq\ & 6\rho_0^2\gamma\int_0^L |\omega_x||\omega_{xx}|dx + 2\lambda\int_0^L |\omega_x|^2 dx + (12\rho_0\rho_1\gamma + 6\rho_0^2\delta)\int_0^L |\omega||\omega_{xx}|dx,
\end{aligned}$$

since

$$\int_0^L |\omega_x||\omega_{xx}|dx \le \frac{\varepsilon_1}{2}\int_0^L |\omega_{xx}|^2dx + \frac{1}{2\varepsilon_1}\int_0^L |\omega_x|^2dx,$$
$$\int_0^L |\omega||\omega_{xx}|dx \le \frac{\varepsilon_2}{2}\int_0^L |\omega_{xx}|^2dx + \frac{1}{2\varepsilon_2}\int_0^L |\omega|^2dx,$$

we have

$$\begin{aligned} &\frac{d}{dt}\int_0^L |\omega_x|^2dx + 2\mu\int_0^L |\omega_{xx}|^2dx \\ \le\ & \big(\varepsilon_1 3\rho_0^2\gamma + \varepsilon_2(6\rho_0\rho_1\gamma + 3\rho_0^2\delta)\big)\int_0^L |\omega_{xx}|^2dx \\ &+(\frac{3\rho_0^2\gamma}{\varepsilon_1} + 2\lambda)\int_0^L |\omega_x|^2dx + \frac{6\rho_0\rho_1\gamma + 3\rho_0^2\delta}{\varepsilon_2}\int_0^L |\omega|^2dx \end{aligned}$$

Let $\varepsilon_1 = \frac{\mu}{6\rho_0^2\gamma}$, $\varepsilon_2 = \frac{\mu}{12\rho_0\rho_1\gamma+6\rho_0^2\delta}$, we can find a suitable M_1, such that:

$$\frac{d}{dt}\int_0^L |\omega_x|^2dx + \mu\int_0^L |\omega_{xx}|^2dx \le M_1(\int_0^L |\omega|^2dx + \int_0^L |\omega_x|^2dx), \tag{3.7}$$

that is

$$\frac{d}{dt}\|\omega_x\|^2 \le M_1(\|\omega\|^2 + \|\omega_x\|^2). \tag{3.8}$$

Thus, by (3.6) and (3.8), we have

$$\frac{d}{dt}(\|\omega\|^2 + \|\omega_x\|^2) \le (M_0 + M_1)(\|\omega\|^2 + \|\omega_x\|^2),$$

then

$$\|\omega\|_V^2 = \|\omega\|^2 + \|\omega_x\|^2 \le \exp[(M_0 + M_1)t](\|\omega(0)\|^2 + \|\omega_x(0)\|^2). \tag{3.9}$$

We apply Q_N to (3.2) and note that Q_N commutes ∂_{xx}, hence $\varphi = Q_N\omega$ satisfies

$$\varphi_t - (i\alpha + \mu)\varphi_{xx} = Q_N(F(u) - F(v)), \tag{3.10}$$

where $F(u)$ is the right hand of (2.3) subtract $(i\alpha + \mu)u_{xx}$.

Take the real part of the inner product of (3.10) with φ and proceed with the similar estimate for (3.5) , then we have

$$\frac{d}{dt}\|\varphi\|^2 + \mu\|\varphi_x\|^2 \le (M_0 + \mu)\|\varphi\|^2 \le \frac{M_0 + \mu}{\lambda_{N+1}}\|\varphi_x\|^2.$$

Similarly,

$$\frac{d}{dt}\|\varphi_x\|^2 + \mu\|\varphi_{xx}\|^2 \le (M_1 + \mu)\|\varphi_x\|^2 + M_1\|\varphi\|^2.$$

Because

$$\|\varphi_x\|^2 \le \frac{1}{\lambda_{N+1}}\|\varphi_{xx}\|^2,$$

and $\lambda_{N+1} \to +\infty$ when $N \to +\infty$, we can choose $\overline{N}_0$ such that when $N \ge \overline{N}_0$,

$$\mu\lambda_{N+1} \ge M_1 + \mu,$$

therefore

$$\begin{cases} \dfrac{d}{dt}\|\varphi_x\|^2 \le M_1\|\varphi\|^2 \le \dfrac{M_1}{\lambda_{N+1}}\|\omega_x\|^2, \\ \dfrac{d}{dt}\|\varphi_x\|^2 + \mu\|\varphi_x\|^2 \le \dfrac{1}{\lambda_{N+1}}(M_0 + M_1 + \mu)\|\omega_x\|^2. \end{cases} \tag{3.11}$$

If we give t_* and N_0, such that if

$$\|(P_{N_0}\omega(t_*))_x\| \le \|(Q_{N_0}\omega(t_*))_x\|,$$

satisfies, then

$$\|\omega_x(t_*)\| \le \frac{1}{6}\|\omega_x(0)\|,$$

thus we complete our proof.

Combine (3.9) with (3.11), we have

$$\begin{aligned} \frac{d}{dt}\|\varphi_x\|^2 + \mu\|\varphi_x\|^2 &\le \frac{1}{\lambda_{N+1}}(M_0 + M_1 + \mu)\|\omega_x\|^2 \\ \le\ & \frac{1}{\lambda_{N+1}}(M_0 + M_1 + \mu)\exp[(M_0 + M_1)t]\|\omega_x(0)\|^2. \end{aligned}$$

Integrate above inequality, we obtain

$$\begin{aligned} \|\varphi_x\|^2 &\le \exp(-\mu t)\|\varphi_x(0)\|^2 + \frac{1}{\lambda_{N+1}}(\exp[(M_0 + M_1)t] - \exp(-\mu t))\|\omega_x(0)\|^2 \\ &\le \exp(-\mu t)\|\varphi_x(0)\|^2 + \frac{1}{\lambda_{N+1}}\exp[(M_0 + M_1)t]\|\omega_x(0)\|^2, \end{aligned}$$

that is

$$\|(Q_N\omega(t))_x\|^2 \le \exp(-\mu t)\|\omega_x(0)\|^2 + \frac{1}{\lambda_{N+1}}\exp[(M_0 + M_1)t]\|\omega_x(0)\|^2. \tag{3.12}$$

Now we are to choose the value of t_* and N_0. First, we choose t_* such that $\exp(-\mu t_*) = \frac{1}{4}\cdot(\frac{1}{8})^2$, then we choose N_1 large enough, such that

$$\frac{1}{\lambda_{N+1}}\exp[(M_0 + M_1)t_*] \le \frac{1}{4}\cdot(\frac{1}{8})^2.$$

Assume that for this particular choice of

$$\|(P_{N_1}\omega(t_*))_x\|^2 \le \|(Q_{N_1}\omega(t_*))_x\|^2,$$

then from (3.12) we deduce that

$$\begin{aligned} \|\omega_x(t_*)\|^2 &= \|(P_{N_1}\omega(t_*))_x\|^2 + \|(Q_{N_1}\omega(t_*))_x\|^2 \\ &\le 2(Q_{N_1}\omega(t_*))_x\|^2 \le \frac{1}{8^2}\|\omega_x(0)\|^2, \end{aligned}$$

so, we take $N_0 = \max(\overline{N}_0,\, N_1)$, $t_* = \frac{8}{\mu}\ln 2$, $L = \frac{\lambda N_0+1}{512}$. Thus, there exist $t_* = \frac{8}{\mu}\ln 2$, N_0 is large enough, such that

$$\lambda_{N_0+1} \ge \max(512exp[(M_0+M_1)t_*], 1+\frac{M_1}{\mu}),$$

then $S(t)$ satisfies the squeezing property on B, and $S_* = S(t_*)$ is Lipschitz constant L, by the Theorem 3.1, we get Theorem 1.3.

Acknowledgments

This work was supported by the Postdoctor Research Fund of Chongqing (yuXM201102006), the Postdoctor Research Fund of Southwest University (102060-207153) and the Fundamental Research Funds for the Central Universities(No. XDJK2009C070).

References

[1] N. Akroune, *Regularity of the attractor for a weakly damped Schrödinger equation on* $\mathbb{R}$, Applied Math. Letters, 12, (1999), 45-48

[2] A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Inertial sets for dissipative evolution equations, Part I: Construction and applications*, IMA Preprint Series 812, University Minnesota, (1991).

[3] X. M. Fan, H. Yang, *Exponential attractor and its fractal dimension for a second order lattice dynamical system*, J. Math. Anal. Appl., 367 (2010), 350-359.

[4] H. J. Gao, *Exponential attractors for a generalized Ginzburg-Laudau equation*, Applied Mathematics and Mechanics, English Edition, 16(9) (1995), 877-882.

[5] O. Goubet, L. Molinet, *Global attractor for weakly damped nonlinear Schrödinger equations in* $L^2(\mathbb{R})$, Nonlinear Analysis, 71 (2009), 317-320.

[6] B. L. Guo, H. Y. Huang, M. Y. Jiang, *Ginzburg-Laudau equation*, Beijing: Science Press, (2002), 27-34.

[7] B. L. Guo, B. X. Wang, *Exponential attractors for the generalized Ginzburg-Landau equation*, Acta Mathematica Sinica, English Series, (2000), 515-526.

[8] C. E. Kenig, G. Ponce, and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation*, J. Amer. Math. Soc. 4 (1991), no.2, 323-347.

[9] C. E. Kenig, G. Ponce, and L. Vega, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J., 71 (1993), 1-21.

[10] C. E. Kenig, G. Ponce, and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc., 9 (1996), 573-603.

[11] C. Kenig, G. Ponce and L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. 106 (2001), no. 3, 617-633.

[12] Y. S. Li, Q. Y. Chen, *Finite dimensional global attractor for dissipative Schrödinger Boussinesq equations*, J. Math. Anal. Appl., 205 (1997), 107-132.

[13] S. J. Lü, Q. S. Lu, *Exponential attractor for the 3D Ginzburg-Laudau type equation*, Nonlinear Analysis, 67 (2007), 3116-3135.

[14] A. Miranville, *Exponential attractors for non-autonomous evolution equations*, Appl. Math. Lett., 11(2), (1988), 19-22.

[15] M. Poulou, N. M. Stavrakakis, *Global attractor for a system of Klein-Gordon-Schrödinger type in all* $\mathbb{R}$, Nonlinear Analysis, 74 (2011), 2548-2562.

[16] R. Temam, *Infinite dimensional dynamical dystems in dechanics and dhysics*, New York: Springer-Verlag, (2000).

[17] C. S. Zhu, C. L. Mu, Z. L. Pu, *Attractor for the nonlinear Schrödinger equation with a non-local nonlinear term*, J. Dyna. Contr. Syst., 16 (4) (2010), 585-603.