Rationality and Brauer group of a moduli space of framed bundles

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Abstract

We prove that the moduli spaces of framed bundles over a smooth projective curve are rational. We compute the Brauer group of these moduli spaces to be zero under some assumption on the stability parameter.

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1 Introduction

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. A framed bundle on X is a pair of the form (E, φ), where E is a vector bundle on X, and

$$\varphi : E_{x_0} \longrightarrow \mathbb{C}^r$$

is a non–zero $\mathbb{C}$–linear homomorphism, where r is the rank of E. The notion of a (semi)stable vector bundle extends to that for a framed bundle. But the (semi)stability condition depends on a parameter $\tau \in \mathbb{R}_{>0}$. Fix a positive integer r, and also fix a holomorphic line bundle L over X. Also, fix a positive number $\tau \in \mathbb{R}$. Let $\mathcal{M}_L^\tau(r)$ be the moduli space of τ–stable framed bundles of rank r and determinant L.

In [BGM], we investigated the geometric structure of the variety $\mathcal{M}_L^\tau(r)$. The following theorem was proved in [BGM]:

Assume that $\tau \in (0, \frac{1}{(r-1)!(r-1)})$. Then the isomorphism class of the Riemann surface X is uniquely determined by the isomorphism class of the variety $\mathcal{M}_L^\tau(r)$.

Our aim here is to investigate the rationality properties of the variety $\mathcal{M}_L^\tau(r)$. We prove the following (see Theorem 2.3 and Corollary 3.2):

The variety $\mathcal{M}_L^\tau(r)$ is rational.

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If $\tau \in (0, \frac{1}{(r-1)!(r-1)})$, *then*

$$\mathrm{Br}(\mathcal{M}_L^\tau(r)) = 0,$$

where $\mathrm{Br}(\mathcal{M}_L^\tau(r))$ *is the Brauer group of* $\mathcal{M}_L^\tau(r)$.

The rationality of $\mathcal{M}_L^\tau(r)$ is proved by showing that $\mathcal{M}_L^\tau(r)$ is birational to the total space of a vector bundle over the moduli space of stable vector bundles E on X together with a line in the fiber of E over a fixed point. The rationality of these moduli spaces can also be derived from [Ho2] by taking D in Example 6.9 to be the point x_0; we thank N. Hoffmann for pointing this out. The Brauer group of $\mathcal{M}_L^\tau(r)$ is computed by considering the morphism to the usual moduli space that forgets the framing.

2 Rationality of moduli space

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. Fix a holomorphic line bundle L over X, and take an integer $r > 0$. Fix a point $x_0 \in X$. A framed coherent sheaf over X is a pair of the form (E, φ), where E is a coherent sheaf on X of rank r, and

$$\varphi : E_{x_0} \longrightarrow \mathbb{C}^r$$

is a non–zero $\mathbb{C}$–linear homomorphism. Let $\tau > 0$ be a real number. A framed coherent sheaf is called *τ–stable* (respectively, *τ–semistable*) if for all proper subsheaves $E' \subset E$, we have

$$\deg E' - \varepsilon(E',\varphi)\tau < \mathrm{rk}\, E' \, \frac{\deg E - \tau}{\mathrm{rk}\, E} \tag{2.1}$$

(respectively, $\deg E' - \varepsilon(E',\varphi)\tau \leq \mathrm{rk}\, E'(\deg E - \tau)/\mathrm{rk}\, E$), where

$$\varepsilon(E',\varphi) = \begin{cases} 1 & \text{if} \quad \varphi|_{E'_{x_0}} \neq 0, \\ 0 & \text{if} \quad \varphi|_{E'_{x_0}} = 0. \end{cases}$$

A framed bundle is a framed coherent sheaf (E, φ) such that E is locally free.

We remark that the framed coherent sheaves considered here are special cases of the objects considered in [HL], and hence from [HL] we conclude that the moduli space $\mathcal{M}_L^\tau(r)$ of τ–stable framed bundles of rank r and determinant L is a smooth quasi–projective variety.

Let (E, φ) be a τ–semistable framed coherent sheaf. We note that if $\tau < 1$, then E is necessarily torsion–free, because a torsion subsheaf of E will contradict τ–semistability, hence in this case E is locally free. But if τ is large, then E can have torsion. In particular, the natural compactification of $\mathcal{M}_L^\tau(r)$ using τ–semistable framed coherent sheaves could have points which are not framed bundles.

Lemma 2.1. There is a dense Zariski open subset

$$\mathcal{M}_L^\tau(r)^0 \subset \mathcal{M}_L^\tau(r) \tag{2.2}$$

corresponding to pairs $(E\,,\varphi)$ such that E is a stable vector bundle of rank r, and φ is an isomorphism.

The moduli space $\mathcal{M}_L^\tau(r)$ is irreducible.

Proof. From the openness of the stability condition it follows immediately that the locus of framed bundles $(E\,,\varphi)$ such that E is not stable is a closed subset of the moduli space $\mathcal{M}_L^\tau(r)$ (see [Ma, p. 635, Theorem 2.8(B)] for the openness of the stability condition). It is easy to check that the locus of framed bundles $(E\,,\varphi)$ such that φ is not an isomorphism is a closed subset of $\mathcal{M}_L^\tau(r)$. Therefore, $\mathcal{M}_L^\tau(r)^0$ is a Zariski open subset of $\mathcal{M}_L^\tau(r)$.

We will now show that this open subset $\mathcal{M}_L^\tau(r)^0$ is dense. Let (E,φ) be a τ–stable framed bundle. The moduli stack of stable vector bundles is dense in the moduli stack of coherent sheaves, and both stacks are irreducible (see, for instance, [Ho, Appendix]). Therefore we can construct a family $\{E_t\}_{t\in T}$ of vector bundles parametrized by an irreducible smooth curve T with a base point $0 \in T$ such that the following two conditions hold:

1. $E_0 \cong E$, and
2. the vector bundle E_t is stable for all $t \neq 0$.

Shrinking T if necessary (by taking a nonempty Zariski open subset of T), we get a family of frames $\{\varphi_t\}_{t\in T}$ such that φ_0 is the given frame φ, and $\varphi_t : E_{t,x_0} \longrightarrow \mathbb{C}^r$ is an isomorphism for all $t \neq 0$. Since E_t is stable, and φ_t is an isomorphism, it is easy to check that (E_t,φ_t) is τ–stable. Therefore, $\mathcal{M}_L^\tau(r)^0$ is dense in $\mathcal{M}_L^\tau(r)$.

To prove that $\mathcal{M}_L^\tau(r)$ is irreducible, first note that $\mathcal{M}_L^\tau(r)^0$ is irreducible because the moduli stack of stable vector bundles of fixed rank and determinant is irreducible. Since $\mathcal{M}_L^\tau(r)^0 \subset \mathcal{M}_L^\tau(r)$ is dense, it follows that $\mathcal{M}_L^\tau(r)$ is irreducible. Q.E.D.

Let $\mathcal{N}_P$ be the moduli space of pairs of the form $(E\,,\ell)$, where E is a stable vector bundle on X of rank r with determinant L, and $\ell \subset E_{x_0}$ is a line. Consider $\mathcal{M}_L^\tau(r)^0$ defined in (2.2). Let

$$\beta : \mathcal{M}_L^\tau(r)^0 \longrightarrow \mathcal{N}_P \tag{2.3}$$

be the morphism defined by $(E\,,\varphi) \longmapsto (E\,,\varphi^{-1}(\mathbb{C}\cdot e_1))$, where the standard basis of $\mathbb{C}^r$ is denoted by $\{e_1\,,\ldots\,,e_r\}$.

Proposition 2.2. The variety $\mathcal{M}_L^\tau(r)^0$ is birational to the total space of a vector bundle over $\mathcal{N}_P$.

Proof. We will first construct a tautological vector bundle over $\mathcal{N}_P$. Let $\mathcal{N}_L(r)$ be the moduli space of stable vector bundles on X of rank r and determinant L. Consider the projection

$$f : \mathcal{N}_P \longrightarrow \mathcal{N}_L(r) \tag{2.4}$$

defined by $(E\,,\ell) \longrightarrow E$. Let $P_{\mathrm{PGL}} \longrightarrow \mathcal{N}_L(r)$ be the principal $\mathrm{PGL}(r,\mathbb{C})$–bundle corresponding to f; the fiber of P_{PGL} over any $E \in \mathcal{N}_L(r)$ is the space of all linear isomorphisms from $P(\mathbb{C}^r)$ (the space of lines in $\mathbb{C}^r$) to $P(E_{x_0})$ (the space of lines in E_{x_0}); since the automorphism group of E is the nonzero scalar multiplications (recall that E is stable), the projective space $P(E_{x_0})$ is canonically defined by the point E of $\mathcal{N}_L(r)$. Let

$$Q \subset \mathrm{PGL}(r,\mathbb{C})$$

be the maximal parabolic subgroup that fixes the point of $P(\mathbb{C}^r)$ representing the line $\mathbb{C}\cdot e_1$. The principal $\mathrm{PGL}(r,\mathbb{C})$–bundle

$$f^*P_{\mathrm{PGL}} \longrightarrow \mathcal{N}_P$$

has a tautological reduction of structure group

$$\widetilde{E}_Q \subset f^*P_{\mathrm{PGL}}$$

to the parabolic subgroup Q; the fiber of $\widetilde{E}_Q$ over any point $(E\,,\ell) \in \mathcal{N}_P$ is the space of all linear isomorphisms

$$\rho : P(\mathbb{C}^r) \longrightarrow P(E_{x_0})$$

such that $\rho(\mathbb{C}\cdot e_1) = \ell$. The standard action of $\mathrm{GL}(r,\mathbb{C})$ on $\mathbb{C}^r$ defines an action of Q on $(\mathbb{C}\cdot e_1)^* \bigotimes_{\mathbb{C}} \mathbb{C}^r$. Let

$$W := f^*P_{\mathrm{PGL}}((\mathbb{C}\cdot e_1)^* \otimes \mathbb{C}^r) \longrightarrow \mathcal{N}_P \tag{2.5}$$

be the vector bundle over $\mathcal{N}_P$ associated to the principal $\mathrm{PGL}(r,\mathbb{C})$–bundle f^*P_{PGL} for the above $\mathrm{PGL}(r,\mathbb{C})$–module $(\mathbb{C}\cdot e_1)^* \bigotimes_{\mathbb{C}} \mathbb{C}^r$. The action of Q on $(\mathbb{C}\cdot e_1)^* \bigotimes_{\mathbb{C}} \mathbb{C}^r$ fixes

$$\mathrm{Id}_{\mathbb{C}\cdot e_1} \in (\mathbb{C}\cdot e_1)^* \otimes_{\mathbb{C}} \mathbb{C}^r = \mathrm{Hom}(\mathbb{C}\cdot e_1\,,\mathbb{C}^r)\,.$$

Therefore, the element $\mathrm{Id}_{\mathbb{C}\cdot e_1}$ defines a nonzero section

$$\sigma \in H^0(\mathcal{N}_P,\, W)\,, \tag{2.6}$$

where W is the vector bundle in (2.5). Note that the fiber of W over (E,ℓ) is $\ell^* \otimes E_{x_0}$, and the evaluation of σ at (E,ℓ) is Id_ℓ.

The projective bundle $P(W) \longrightarrow \mathcal{N}_P$ parametrizing lines in W is identified with the pullback $f^*\mathcal{N}_P$ of the projective bundle $\mathcal{N}_P$ to the total space of $\mathcal{N}_P$, where f is constructed in (2.4). The tautological section $\mathcal{N}_P \longrightarrow f^*\mathcal{N}_P$ of the projection $f^*\mathcal{N}_P \longrightarrow \mathcal{N}_P$ coincides with the section given by σ in (2.6).

Let $U \subset \mathcal{N}_P$ be some nonempty Zariski open subset such that there exists

$$V \subset W|_U ,$$

a direct summand of the line subbundle of $W|_U$ generated by σ. Consider the vector bundle

$$\mathcal{W} := V^* \otimes_{\mathbb{C}} \mathbb{C}^r \longrightarrow U .$$

The total space of $\mathcal{W}$ will also be denoted by $\mathcal{W}$. Consider the map β defined in (2.3). Let

$$\gamma : \mathcal{M}^{\tau}_L(r)^0 \supset \beta^{-1}(U) \longrightarrow \mathcal{W}$$

be the morphism that sends any $y := (E\,, \varphi) \in \beta^{-1}(U)$ to the homomorphism

$$V_{\beta(y)} \longrightarrow \mathbb{C}^r$$

defined by $v \longmapsto \varphi(v)/\lambda$, where $\lambda \in \mathbb{C}^* - \{0\}$ satisfies the identity $\varphi(\sigma(\beta(y))) = \lambda \cdot e_1$. The morphism γ is clearly birational. Q.E.D.

Theorem 2.3. The moduli space $\mathcal{M}^{\tau}_L(r)$ is rational.

Proof. Since any vector bundle is Zariski locally trivial, the total space of a vector bundle of rank n over $\mathcal{N}_P$ is birational to $\mathcal{N}_P \times \mathbb{A}^n$. Therefore, from Proposition 2.2 we conclude that $\mathcal{M}^{\tau}_L(r)^0$ is birational to $\mathcal{N}_P \times \mathbb{A}^n$, where $n = \dim \mathcal{M}^{\tau}_L(r)^0 - \dim \mathcal{N}_P$.

The variety $\mathcal{N}_P$ is known to be rational [BY, p. 472, Theorem 6.2]. Hence $\mathcal{N}_P \times \mathbb{A}^n$ is rational, implying that $\mathcal{M}^{\tau}_L(r)^0$ is rational. Now from Lemma 2.1 we infer that $\mathcal{M}^{\tau}_L(r)$ is rational. Q.E.D.

3 Brauer group of moduli of framed bundles

We quickly recall the definition of Brauer group of a variety Z. Using the natural isomorphism $\mathbb{C}^r \otimes \mathbb{C}^{r'} \xrightarrow{\sim} \mathbb{C}^{rr'}$, we have a homomorphism $\mathrm{PGL}(r, \mathbb{C}) \times \mathrm{PGL}(r', \mathbb{C}) \longrightarrow \mathrm{PGL}(rr', \mathbb{C})$. So a principal $\mathrm{PGL}(r, \mathbb{C})$–bundle $\mathbb{P}$ and a principal $\mathrm{PGL}(r', \mathbb{C})$–bundle $\mathbb{P}'$ on Z together produce a principal $\mathrm{PGL}(rr', \mathbb{C})$–bundle on Z, which we will denote by $\mathbb{P} \otimes \mathbb{P}'$. The two principal bundles $\mathbb{P}$ and $\mathbb{P}'$ are called *equivalent* if there are vector bundles V and V' on Z such that the principal bundle $\mathbb{P} \otimes \mathbb{P}(V)$ is isomorphic to $\mathbb{P}' \otimes \mathbb{P}(V')$. The equivalence classes form a group which is called the *Brauer group* of Z. The addition operation is defined by the tensor product, and the inverse is defined to be the dual projective bundle. The Brauer group of Z will be denoted by $\mathrm{Br}(Z)$.

As before, fix r and L. Define

$$\tau(r) := \frac{1}{(r-1)!(r-1)}.$$

Henceforth, we assume that

$$\tau \in (0, \tau(r)),$$

where τ is the parameter in the definition of a (semi)stable framed bundle. As before, let $\mathcal{M}^\tau_L(r)$ be the moduli space of τ–stable framed bundles of rank r and determinant L.

Let $\overline{\mathcal{N}}_L(r)$ be the moduli space of semistable vector bundles on X of rank r and determinant L. As in the previous section, the moduli space of stable vector bundles on X of rank r and determinant L will be denoted by $\mathcal{N}_L(r)$.

If E is a stable vector bundle of rank r and determinant L, then for any nonzero homomorphism

$$\varphi : E_{x_0} \longrightarrow \mathbb{C}^r,$$

the framed bundle (E, φ) is τ–stable (see [BGM, Lemma 1.2(ii)]). Also, if (E, φ) is any τ–stable framed bundle, then E is semistable [BGM, Lemma 1.2(i)]. Therefore, we have a morphism

$$\delta : \mathcal{M}^\tau_L(r) \longrightarrow \overline{\mathcal{N}}_L(r) \tag{3.1}$$

defined by $(E, \varphi) \longrightarrow E$. Define

$$\mathcal{M}^\tau_L(r)' := \delta^{-1}(\mathcal{N}_L(r)) \subset \mathcal{M}^\tau_L(r), \tag{3.2}$$

where δ is the morphism in (3.1). From the openness of the stability condition (mentioned in the proof of Lemma 2.1) it follows that $\mathcal{M}^\tau_L(r)'$ is a Zariski open subset of $\mathcal{M}^\tau_L(r)$.

Lemma 3.1. The Brauer group of the variety $\mathcal{M}^\tau_L(r)'$ vanishes.

Proof. We noted above that (E, φ) is τ–stable if E is stable. Therefore, the morphism

$$\delta_1 := \delta|_{\mathcal{M}^\tau_L(r)'} : \mathcal{M}^\tau_L(r)' \longrightarrow \mathcal{N}_L(r)$$

defines a projective bundle over $\mathcal{N}_L(r)$, where δ is constructed in (3.1); for notational convenience, this projective bundle $\mathcal{M}^\tau_L(r)'$ will be denoted by $\mathcal{P}$. The homomorphism

$$\delta_1^* : \mathrm{Br}(\mathcal{N}_L(r)) \longrightarrow \mathrm{Br}(\mathcal{P})$$

is surjective, and the kernel of δ_1^* is generated by the Brauer class

$$\mathrm{cl}(\mathcal{P}) \in \mathrm{Br}(\mathcal{N}_L(r))$$

of the projective bundle $\mathcal{P}$ (see [Ga, p. 193]). In other words, we have an exact sequence

$$\mathbb{Z}\cdot\mathrm{cl}(\mathcal{P}) \longrightarrow \mathrm{Br}(\mathcal{N}_L(r)) \xrightarrow{\delta_1^*} \mathrm{Br}(\mathcal{M}_L^\tau(r)') \longrightarrow 0\,. \tag{3.3}$$

Let

$$\mathbb{P} := \mathcal{N}_L(r)\times P(\mathbb{C}^r) \longrightarrow \mathcal{N}_L(r)$$

be the trivial projective bundle over $\mathcal{N}_L(r)$. Consider the projective bundle

$$f : \mathcal{N}_P \longrightarrow \mathcal{N}_L(r)$$

in (2.4). Let

$$(\mathcal{N}_P)^* \longrightarrow \mathcal{N}_L(r)$$

be the dual projective bundle; so the fiber of $(\mathcal{N}_P)^*$ over any point $z \in \mathcal{N}_L(r)$ is the space of all hyperplanes in the fiber of $\mathcal{N}_P$ over z. It is easy to see that

$$\mathcal{P} = (\mathcal{N}_P)^*\otimes\mathbb{P} \tag{3.4}$$

(the tensor product of two projective bundles was defined at the beginning of this section).

Since $\mathbb{P}$ is a trivial projective bundle, from (3.4) it follows that

$$\mathrm{cl}(\mathcal{P}) = \mathrm{cl}((\mathcal{N}_P)^*) = -\mathrm{cl}(\mathcal{N}_P) \in \mathrm{Br}(\mathcal{N}_L(r))\,.$$

But the Brauer group $\mathrm{Br}(\mathcal{N}_L(r))$ is generated by $\mathrm{cl}(\mathcal{N}_P)$ [BBGN, Proposition 1.2(a)]. Hence $\mathrm{cl}(\mathcal{P})$ generates $\mathrm{Br}(\mathcal{N}_L(r))$. Now from (3.3) we conclude that $\mathrm{Br}(\mathcal{M}_L^\tau(r)') = 0$. Q.E.D.

Corollary 3.2. The Brauer group of the moduli space $\mathcal{M}_L^\tau(r)$ vanishes.

Proof. Since $\mathcal{M}_L^\tau(r)'$ is a nonempty Zariski open subset of $\mathcal{M}_L^\tau(r)$, the homomorphism

$$\mathrm{Br}(\mathcal{M}_L^\tau(r)) \longrightarrow \mathrm{Br}(\mathcal{M}_L^\tau(r)')$$

induced by the inclusion $\mathcal{M}_L^\tau(r)' \hookrightarrow \mathcal{M}_L^\tau(r)$ is injective. Therefore, from Lemma 3.1 it follows that $\mathrm{Br}(\mathcal{M}_L^\tau(r)) = 0$. Q.E.D.

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