The first almost free Whitehead group

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Abstract

Assume GCH and that κ is the first uncountable cardinal such that there is a non-free κ-free Abelian Whitehead group of cardinality κ. We prove that if all κ-free Abelian groups of cardinality κ are Whitehead then κ is necessarily an inaccessible cardinal.

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An Abelian group G is called *Whitehead* if it satisfies the following condition: if H is an Abelian group, $\mathbb{Z} \subseteq H$ and $H/\mathbb{Z} \cong G$, then $\mathbb{Z}$ is a direct summand of H. The book by Eklof and Mekler [2] provides a good overview of the subject of Whitehead groups.

We call an Abelian group G *free* if it is the direct sum of copies of $(\mathbb{Z}, +)$. All free Abelian groups are Whitehead and every Whitehead group is "somewhat free" ($\aleph_1$-free, see below). Possibly (i.e., consistently with ZFC) every Whitehead group is free and possibly not (in fact, it seems that almost any behaviour is possible).

Recall that GCH, the generalized continuum hypothesis, says that $2^\lambda = \lambda^+$ for every infinite λ. A cardinal λ is called *inaccessible* if it is a strong limit (i.e., if $\mu < \lambda$, then $2^\mu < \lambda$) and regular. It is well known that inaccessible cardinals are large, e.g., their existence is not provable in ZFC, the usual axioms of set theory.

A central notion for the present paper is the notion of being "κ-free of cardinality κ": an Abelian group G is κ-free when the pure closure inside G of any subgroup generated by less than κ many elements is free. Our main result restricts the behaviour of groups like this: assume κ is the first cardinal λ such that there is a non-free λ-free Abelian Whitehead group of cardinality λ. The conclusion is, assuming GCH and κ not strongly inaccessible, that not all such Abelian groups are Whitehead.

We recall that in [8, §1] it was proved that if μ is strong limit singular, $\lambda = \mu^+ = 2^\mu$ and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\mu)\}$ is stationary, then, even though $\diamondsuit_S$ consistently fails, we can still prove a relative of $\diamondsuit_S$ sufficient for constructing Abelian groups G of cardinality λ related to satisfying $\mathrm{Hom}(G, \mathbb{Z}) \neq \{0\}$. We prove here in Claim 8 somewhat more and use it in the proof of the Theorem 4. This claim is complementary to [7] which shows that consistently there is such regular (in fact strongly inaccessible) λ.

In this paper, we deal exclusively with Abelian groups, and whenever we say "group" we mean "Abelian group". In [7], we answered the following question:

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Question 1. (Göbel) Assuming GCH, can there be a regular κ such that

$\boxdot_\kappa$ (a) $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and there are κ-free not free groups of cardinality κ, and

(b) every κ-free group of cardinality κ is a Whitehead group?

Moreover, by [7] it is consistent that: GCH + for some strongly inaccessible κ we have $\circledast_\kappa$ where the statement $\circledast_\kappa$ is defined by:

$\circledast_\kappa$ (a) κ is regular uncountable and there are κ-free non-free (Abelian) groups of cardinality κ,

(b) every κ-free (Abelian) group of cardinality κ is a Whitehead group, and

(c) every Whitehead group of cardinality $< \kappa$ is free.

[For clause (c), the following sufficient condition is used there: for every regular uncountable $\lambda < \kappa$ we have $\diamondsuit^*_\lambda$. The proof starts with κ weakly compact and adds no new sequence of length $< \kappa$, so, e.g., starting there with $\mathbf{L}$ this holds.]

The following question is natural:

Question 2. Assume GCH, is it consistent that there is an accessible κ such that $\circledast_\kappa$ holds? In other words, there is a cardinal κ satisfying the following and the first such κ is accessible:

$(*)$ there is a κ-free non-free Whitehead group of cardinality κ.

Now Theorem 4 says that the answer is "No". But another natural question is:

Question 3. Assume GCH Can there be a κ such that $\boxdot_\kappa$ but is the first such κ accessible?

This is [2, Problem F4] and it remains open.

For a cardinal $\kappa \geq \aleph_1$, an Abelian group G is called *κ-free* when every subgroup of G of cardinality $< \kappa$ is free. We say that $\bar{G} = \langle G_\alpha : \alpha < \lambda \rangle$ is a *filtration* of the Abelian group G of cardinality λ if G_α is a subgroup of G of cardinality $< \lambda$, increasing continuous with α and $G = \bigcup \{G_\alpha : \alpha < \lambda\}$. Finally, an Abelian group G of cardinality κ, is called *strongly κ-free* when it is κ-free and for every subgroup G' of G of cardinality $< \kappa$ there is a subgroup G'' of cardinality $< \kappa$ such that $G' \subseteq G''$ and G/G'' is κ-free.

Theorem 4 (GCH)**.** Let κ be the first $\lambda > \aleph_0$ such that there is a λ-free Abelian Whitehead group, not free, of cardinality λ (and we assume there is such λ). If κ is not (strongly) inaccessible, then there is a non-Whitehead group G of cardinality κ which is κ-free (and necessarily non-free).

Proof. The proof will proceed in six stages, called *Stage A* to *Stage F*.

Stage A. Let G be a witness for the choice of κ. Necessarily κ is regular, by singular compactness, cf., e.g., [2, Ch.IV,3.5]. Let $\bar{G} = \langle G_\alpha : \alpha < \kappa \rangle$ be a filtration of G: as G is κ-free, every G_α must be free. Without loss of generality, each G_α is a pure subgroup of G. Let $S := \Gamma(\bar{G}) = \{\alpha < \kappa : \alpha$ be a limit ordinal and G/G_α is not κ-free$\}$. Now since G is not free, $\Gamma(\bar{G})$ must be stationary.

Stage B. G is strongly κ-free. Towards a contradiction, assume that it is not. Without loss of generality κ is a successor cardinal. [Why? This follows from the assumption and the fact that κ is regular. Also, if κ a limit cardinal, κ-free implies strongly κ-free.]

By our present assumption towards a contradiction, for some $\alpha < \kappa$ for every $\beta \in [\alpha, \kappa)$, the Abelian group G/G_β is not κ-free. Hence, without loss of generality, $\alpha < \lambda \Rightarrow G_{\alpha+1}/G_\alpha$ is not free and if $G_{\alpha+1}/G_\alpha$ is uncountable then for some κ_α, it is κ_α-free but not free. By the assumption of the theorem (and since countable Whitehead groups are free), $\alpha < \kappa \Rightarrow G_{\alpha+1}/G_\alpha$ is not a Whitehead group, i.e., $\Gamma(\bar{G}) = \kappa$. But κ is a successor cardinal, so let $\kappa = \mu^+$; also $2^\mu < 2^\kappa$ as we are assuming GCH so the weak diamond holds for κ (see [1]) so by the previous sentence and, e.g., [2, 1.10, p. 369], we know that G is not a Whitehead group, in contradiction to the assumption on G.

Stage C. Hence, without loss of generality, we have

$(*)_1$ if α is a non-limit ordinal then G/G_α is κ-free

and also (obviously), without loss of generality, we have

$(*)_2$ (a) if $\alpha \in S$, then $G_{\alpha+1}/G_\alpha$ is not free, and
(b) if $\alpha < \kappa$, then G_α is a pure subgroup of G and is free (hence G/G_α is torsion free).

Also we can choose $\bar{H} = \langle H_\alpha : \alpha \in S\rangle$ such that

$(*)_3$ (a) H_α is a subgroup of G,
(b) $H_\alpha/(H_\alpha \cap G_\alpha)$ is not free,
(c) the rank θ_α of $H_\alpha/(H_\alpha \cap G_\alpha)$ is minimal (hence θ_α is $< \aleph_0$ or is regular uncountable $< \kappa$), and
(d) the cardinality of H_α is $\leq \theta_\alpha + \aleph_0$ (in fact, equality holds).

Note that $|H_\alpha|$ may be $< |G_\alpha|$ so it is unreasonable to ask for $G_\alpha \subseteq H_\alpha$.

$(*)_4$ Without loss of generality, we also have

(a) $H_\alpha \subseteq G_{\alpha+1}$,
(b) $G_\alpha/(H_\alpha \cap G_\alpha)$ is free, and
(c) $G_\alpha + H_\alpha$ is a pure subgroup of $G_{\alpha+1}$.

[Why? For clause (a) we can restrict $\bar{G}$ to a club. For clause (c) let H''_α be the pure closure of $H_\alpha + G_\alpha$ inside $G_{\alpha+1}$ so $H''_\alpha/G_\alpha, (H_\alpha + G_\alpha)/G_\alpha, H_\alpha/(H_\alpha \cap G_\alpha)$ has the same rank, which is θ_α. As $G_{\alpha+1}/G_\alpha$ is torsion free, also H''_α/G_α is torsion free. Hence there is $H'_\alpha \subseteq H''_\alpha$ of cardinality $\leq \theta_\alpha + \aleph_0$ such that $G_\alpha + H'_\alpha$ is a pure subgroup of H''_α hence of $G_{\alpha+1}$ and $H'_\alpha + G_\alpha = H''_\alpha$ and, e.g., the rank of $H'_\alpha/(H'_\alpha \cap G_\alpha)$ is the same as the rank of H''_α/G_α which is θ_α, so replacing H_α by H'_α also clause (c) holds.

For clause (b) note that G_α is free; so there is $G'_\alpha \subseteq G_\alpha$ of cardinality $\theta_\alpha + \aleph_0$ such that G_α/G'_α is free and $H_\alpha \cap G_\alpha \subseteq G'_\alpha$ and replace H_α by $H^*_\alpha = H_\alpha + G'_\alpha$ noting that $H^*_\alpha + G_\alpha = H_\alpha + G_\alpha$.]

By the hypothesis on κ, if K is a λ-free but not free group of cardinality λ and $\aleph_0 < \lambda < \kappa$ then K is not a Whitehead group. Therefore, clearly (recall that countable Whitehead groups are free):

$(*)_5$ $H_\alpha/(G_\alpha \cap H_\alpha)$ is not Whitehead for $\alpha \in S$.

Hence by [3] or [2, Ch.VI, 1.13] we know that

$(*)_6$ $\neg\diamondsuit_S$.

Recall that κ is regular uncountable, so towards a contradiction assume that $\kappa = \mu^+$. Let $\sigma = \mathrm{cf}(\mu)$. But, by [4] or [2, Ch.VI, 1.13], GCH implies that $\diamondsuit_{S'}$ for every stationary $S' \subseteq \kappa \backslash S^\kappa_\sigma$ where $S^\kappa_\sigma := \{\delta < \kappa : \mathrm{cf}(\delta) = \sigma\}$. Consequently, we know that

$(*)_7$ for some club E of κ, we have $S \cap E \subseteq S^\kappa_\sigma$ so, without loss of generality, $S \subseteq S^\kappa_\sigma$, i.e., $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \sigma$.

Also, without loss of generality, (from the definition of "strongly κ-free"), we obtain that

$(*)_8$ $\delta \in S$ implies $\theta_\delta + \aleph_0 \geq \sigma$.

[Why? Let $S^1 = \{\delta \in S : \theta_\delta + \aleph_0 < \sigma\}$; first assume S^1 is stationary. For each $\delta \in S^1$, $H_\delta \cap G_\delta$ must have cardinality $\leq \theta_\delta + \aleph_0 < \sigma = \mathrm{cf}(\delta)$. Therefore, for some $\alpha_\delta < \delta$, we have $H_\delta \cap G_\delta \subseteq G_{\alpha_\delta}$. By Fodor's lemma, there is some $\alpha(*) < \kappa$ such that the set $S^2 := \{\delta \in S^1 : \alpha_\delta = \alpha(*)\}$ is stationary. As $\sigma = \mathrm{cf}(\mu), \kappa = \mu^+$, clearly $\mu^{<\sigma} = \mu$ (keep in mind that we are working under GCH). Therefore, $\{H_\delta \cap G_{\alpha(*)} : \delta \in S^2\}$ has cardinality $\leq \mu$, and hence $S^3 = \{\delta \in S^2 : H_\delta \cap G_{\alpha(*)} = H_*, \theta_\delta = \theta_*\}$ is a stationary subset of κ for some θ_* and $H_* \subseteq H_{\alpha(*)}$. Let α_ε be the εth member of S^3 and $H^\varepsilon = H_* + \sum\{H_{\alpha_\zeta} : \zeta < \varepsilon\}$ for $\varepsilon \leq (\theta_* + \aleph_0)^+$. Clearly, for $\varepsilon = (\theta_* + \aleph_0)^+$, the group H^ε is not free and has cardinality $\leq \sigma = \mathrm{cf}(\mu) < \lambda$. This is a contradiction to the fact that G is κ-free. Thus, S^1 cannot be stationary, and as we can restrict $\bar{G}$ to a club, without loss of generality, $S^1 = \varnothing$ so $(*)_8$ holds indeed.]

Stage D. If $\alpha \in S$, then $\theta_\alpha < \mu$.

If this is not the case, then for some $\alpha \in S$, we have that $\theta_\alpha = \mu$. In particular, θ_α is infinite and thus regular. Moreover, by $(*)_3$, θ_α is uncountable and there is a θ_α-free but not free group of cardinality θ_α. From this, we get that μ must be regular, and there is a μ-free but not free Abelian group of cardinality $\mu < \kappa$, i.e., $H_\alpha/(G_\alpha \cap H_\alpha)$, hence this group is not Whitehead.

We obtain a purely increasing and continuous sequence $\langle H^*_\varepsilon : \varepsilon \leq \mu+1\rangle$ of free groups such that $\varepsilon < \mu$ implies that $H^*_{\mu+1}/H^*_\varepsilon$ is free, but $H^*_{\mu+1}/H^*_\mu$ is not free and is isomorphic to $H_\alpha/(G_\alpha \cap H_\alpha)$.

[Why? Let $\langle y_\varepsilon : \varepsilon < \mu\rangle$ be a free basis of G_α, so $G_\alpha = \bigoplus\{\mathbb{Z}y_\varepsilon : \varepsilon < \mu\}$. As $\{y_\varepsilon : \varepsilon < \mu\} \subseteq G_\alpha$ and $\mathrm{cf}(\alpha) = \mu$, there is an increasing continuous sequence $\langle \gamma_\varepsilon : \varepsilon < \mu\rangle$ of ordinals $< \alpha$ with limit α such that $y_\varepsilon \in G_{\gamma_{\varepsilon+1}}$.

Let $H^*_\varepsilon = \bigoplus\{\mathbb{Z}y_\zeta : \zeta < \varepsilon\}$ for $\varepsilon < \mu$, $H^*_\mu = G_\alpha$, $H^*_{\mu+1} = G_\alpha + H_\alpha$ as required. E.g., why is $H^*_{\mu+1}/H_\varepsilon$ free? As $G_{\alpha+1}/G_{\alpha_\varepsilon+1}$ is free by $(*)_1$, also G_α/H^*_ε is free since $\{y_\zeta + H^*_\varepsilon : \zeta \in [\varepsilon, \mu)\}$ is a free basis. But $H^*_\varepsilon \subseteq G_{\alpha_{\varepsilon+1}} \subseteq G_\alpha$ hence $G_{\alpha_\varepsilon+1}/H^*_\varepsilon$ is free. Together, $G_{\alpha+1}/H^*_\varepsilon$ is free which implies that its subgroup $H^*_{\mu+1}/H_\varepsilon$ is free as promised.]

We can find H_* and $\langle H_\eta, h_\eta : \eta \in {}^{\mu\geq}2\rangle$ such that:

1. $H_* = \bigoplus\{\mathbb{Z}x_t : t \in I\}$ and $|I| = \mu$,
2. I is the disjoint union of $\{I_\eta : \eta \in {}^{\mu>}2\}$,
3. $|I_\eta| = \mathrm{rk}(H^*_{\mathrm{lh}(\eta)+1}/H^*_{\mathrm{lh}(\eta)})$ for $\eta \in {}^{\mu>}2$,
4. $H_\eta = \bigoplus\{\mathbb{Z}x_t : t \in \bigcup\{I_{\eta\restriction\varepsilon} : \varepsilon < \mathrm{lh}(\eta)\}\} \subseteq H_*$ for $\eta \in {}^{\mu>}2$,

5. h_η is an isomorphism from $H_{\mathrm{lh}(\eta)}$ onto H_η, and

6. $h_{\eta\restriction\varepsilon} \subseteq h_\eta$ if $\varepsilon < \mathrm{lh}(\eta), \eta \in {}^{\mu>}2$.

Now we can find $\langle H^+_\eta, h^+_\eta : \eta \in {}^\mu 2\rangle$ such that

7. $H_\eta \subseteq H^+_\eta$, and

8. h^+_η is an isomorphism from $H^*_{\mu+1}$ onto H^+_η extending h_η.

Without loss of generality

9. $H^+_\eta \cap H_* = H_\eta$,

so there is

10. H^*_η which extends H^+_η and H_* such that $H^+_\eta \cup H_*$ generates H^*_η and H^*_η/H_* is isomorphic to H^*_η/H_η.

Lastly, without loss of generality,

11. the sets $\langle H^*_\eta \backslash H_* : \eta \in {}^\mu 2\rangle$ are pairwise disjoint,

so there is an Abelian group H such that

12. $H^*_\eta \subseteq H$ for $\eta \in {}^\mu 2$,

13. $H/H_* = \bigoplus\{H^*_\eta/H_* : \eta \in {}^\mu 2\}$, and

14. H has cardinality $2^\mu = \kappa$.

Next we note

15. H is κ-free.

[Why? Note that $\bigcup\{H^+_\eta : \eta \in {}^\mu 2\}$ includes $\{x_t : t \in I\}$. Therefore, the subgroup it generates includes H. Let $H' \subseteq H$ be a sub-group of cardinality $< \kappa \leq \mu$. Thenthere are $\eta_i \in {}^\mu 2$ for $i < \mu$ such that $H' \subseteq \sum\{H^+_{\eta_i} : i < \mu\}$ and $j < i \Rightarrow \eta_i \neq \eta_j$. We can choose $\zeta(i) < \mu$ by induction on $i < \mu$ such that $\langle\{\eta_i\restriction\varepsilon : \varepsilon \in [\zeta(i), \mu)\} : i < \mu\}$ is a sequence of pairwise disjoint sets.

For each i, let $H^{**}_i \subseteq H^+_{\eta_i}$ be such that $H^+_{\eta_i} = H^{**}_i \oplus H_{\eta_i\restriction\zeta(i)}$. Let us define H'_i for $i \leq \mu$ by: $H'_0 \subseteq H$ is generated by $\{x_t : t \in I$ but $t \notin \bigcup\{I_{\eta_i\restriction\varepsilon} : i < \mu$ and $\varepsilon \in [\zeta(i), \mu)\}\}$ and H'_i is $\bigoplus\{H^{**}_j : j < i\} \oplus H'_0$. Clearly $H' \subseteq H'_\mu \subseteq H, \langle H'_i : i \leq \mu\rangle$ is increasing continuous, H'_0 is free and $H'_{i+1}/H'_i \cong H^{**}_i$ is free. It follows that H'_μ is free, hence $H' \subseteq H'_\mu$ is free. So clause 15. holds.]

Let $\langle \eta_\alpha : \alpha < \kappa\rangle$ be a list of ${}^\mu 2$, let $H^{**}_\alpha \subseteq H$ be $\sum\{H^+_{\eta_\beta} : \beta < \alpha\} + H_*$ such that $\bar{H}^{**} = \langle H^{**}_\alpha : \alpha < \kappa\rangle$ is a filtration of H, and $\Gamma(\bar{H}^{**}) = \kappa$ and $H^{**}_{\alpha+1}/H^{**}_{\alpha+1}$ is isomorphic to $H^*_{\mu+1}/H^*_\mu$. Thus, it is not free and not Whitehead. Hence, by [2, Ch.XII, 1.10, p. 369], H is not Whitehead. So H is κ-free (see clause 15., considering that the cardinality is κ and H is not Whitehead, so not free) the desired conclusion of the theorem. So indeed, without loss of generality, the desired conclusion of *Stage D* holds.

Stage E. The cardinal μ is singular: By the earlier stages, we know that $\mathrm{cf}(\mu) = \sigma$, that $\alpha \in S$ implies $\mathrm{cf}(\alpha) \leq \theta_\alpha < \mu$, and that $\alpha \in S$ implies that $\sigma = \mathrm{cf}(\alpha)$.

Stage F. Let $\sigma = \mathrm{cf}(\mu)$, so σ is regular $< \mu$; also choose $\theta = \mathrm{cf}(\theta) < \mu$ such that $S_1^* := \{\delta \in S : \theta_\delta = \theta\}$ is stationary. Note that $\mu = \mu^{<\sigma}$ as GCH holds (keep in mind that $\sigma = \mathrm{cf}(\delta)$ and if $\delta \in S$, then $\mathrm{cf}(\delta) = \sigma$ by $(*)_7$). We shall now use [6, §3] and its notation, cf. also Theorem 5 and Claims 6 and 8 below.

By Theorem 5, we can find a κ-witness $\mathbf{x}$, so it consists of $n \geq 1$, $\mathbf{S}$, $\langle B_\eta : \eta \in \mathbf{S}_c\rangle$, $\langle s_\eta^\ell : \eta \in \mathbf{S}_f, \ell < n\rangle$, as in [6, 3.6] (the (λ, κ^+, S) of [6, 3.6] will have to be instantiated by $(\kappa, \aleph_1, \mathbf{S})$ here), such that

(∗) $\langle\alpha\rangle \in \mathbf{S}$ is equivalent to $\alpha \in S_1^*$, i.e., $W(\langle\,\rangle, \mathbf{S}) = S_1^*$.

We continue to use the notation of [6, §3]. Clearly, if $\alpha \in S_1^*$, we get that $\lambda(\langle\alpha\rangle, \mathbf{S}) \leq \mu$. Thus, since it is regular, it must be strictly smaller than μ, and so, without loss of generality, constant (in fact, it is θ by the proof). Now we apply Claim 8 with $\lambda = \kappa$.

Why does clause (d) of Claim 8 hold? If $\theta > \aleph_0$, by [6, 1.2], the group $G_{\mathbf{x}(\langle\alpha\rangle)}$, derived from $\mathbf{x}(\langle\alpha\rangle)$ is a $\lambda(<\infty, \mathbf{S}_\mathbf{x})$-free non-free Abelian group of cardinality $\lambda(\langle\alpha\rangle, \mathbf{S}_\mathbf{x}) = \theta$, so is not Whitehead by the assumption on κ from the statement of the theorem; note that $G_{\mathbf{x}(\langle\alpha\rangle)}$ was derived in some way from $H_\alpha/(H_\alpha \cap G_\alpha)$, but it is not necessarily equal to it. If $\theta \leq \aleph_0$, then $G_{\mathbf{x}(<\alpha>)}$ is a non-free (Abelian) countable group hence is not Whitehead.

So the assumption of Claim 8 says that there is a strongly κ-free Abelian group G of cardinality κ by the theorem's assumption which is not Whitehead, so we are done. Q.E.D. (Theorem 4)

Recall from [5, §5], [6, §3] or [2] the following theorem:

Theorem 5. For any $\lambda > \aleph_0$ the following conditions are equivalent:

(a) There is a λ-free not free Abelian group.

(b) $\mathrm{PT}(\lambda, \aleph_1)$.[1]

(c) There is a λ-witness $\mathbf{x}$ as in [6, 3.6, 3.7]; so $\mathbf{x}$ consists of

- (α) a natural number n,
- (β) a so-called λ-set $\mathbf{S} \subseteq \{\eta \in {}^{n\geq}\lambda : \eta \text{ decreasing}\}$ closed under initial segments (cf. [6, 3.1]),
- (γ) a disjoint λ-system $\bar{B} = \langle B_\eta : \eta \in \mathbf{S}_c\rangle$ (cf. [6, 3.4]),
- (δ) $\bar{s} = \langle s_\eta^\ell : \eta \in \mathbf{S}_f, \ell < n\rangle$, etc.

(d) Without loss of generality, $\bigcup\{B_\eta : \eta \in \mathbf{S}_c\} = \lambda$, so $\mathbf{S} = \mathbf{S}_\mathbf{x}$, etc.

(e) Let $\langle a_{\eta,m}^\ell : m < \omega\rangle$ be a listing of $s_\eta^{\ell,\mathbf{x}}$ with no repetition for $\eta \in \mathbf{S}_f^\mathbf{x}$ and $\ell < n$ such that

- $a_{\eta(1),m(1)}^{\ell(1)} = a_{\eta(2),m(2)}^{\ell(2)}$ implies $\ell(1) = \ell(2)$ and $m(1) = m(2)$, and
- $m < m(1)$ implies $a_{\eta(\ell),m}^{\ell(1)} = a_{\eta(2),m}^{\ell(2)}$,

[1] This means: "There is a family $\mathscr{P}$ of countable sets of cardinality λ with no transversal (i.e., a one-to-one choice function) but any subfamily of cardinality $< \lambda$ has a transversal."

so, without loss of generality, $a^\ell_{\eta,m} < a^\ell_{\eta,m+1}$ for every relevant η, ℓ, and m; also $\delta = \sup\bigcup\{s^{0,\mathbf{x}}_\eta : \langle\delta\rangle \trianglelefteq \eta \in \mathbf{S}_f\}$ when $\langle\delta\rangle \in \mathbf{S}_f$ and $\eta \mapsto \lambda(\eta, \mathbf{S}_\mathbf{x})$ and $\eta \mapsto W(\eta, S_\mathbf{x})$ are well defined.

The following definition and claim were used but not named in [6]:

Claim 6. For a λ-witness $\mathbf{x}$ and $\nu \in \mathbf{S}_\mathbf{x}$, we define $\mathbf{x}_\nu = \mathbf{x}(\nu)$ by

$$\begin{aligned} n_{\mathbf{x}(\nu)} &= n_\mathbf{x} - \mathrm{lh}(\nu),\\ \mathbf{S}_{\mathbf{x}(\nu)} &= \{\eta : \nu^\frown\eta \in \mathbf{S}_\mathbf{x}\},\\ B^{\mathbf{x}(\nu)}_\eta &= B^\mathbf{x}_{\eta^\frown\nu}, \text{ and}\\ s^{\ell,\mathbf{x}(\nu)}_\eta &= s^{\mathrm{lh}(\nu)+\ell,\mathbf{x}}_{\nu^\frown\eta}. \end{aligned}$$

For a λ-witness $\mathbf{x}$ let $G_\mathbf{x}$ be the Abelian group $G_{\{\langle\alpha\rangle:\alpha\in W(\langle\,\rangle,S_\mathbf{x})\}}$ defined inside the proof of [6, 1.2].

Remark 7. We may use the following:

1. For a λ-witness $\mathbf{x}$ let $K_\mathbf{x} := \{I \subseteq \mathbf{S}_\mathbf{x} : I$ is a set of pairwise $\triangleleft$-incomparable sequences such that $\{\beta : \eta^\frown\langle\beta\rangle \in I\}$ is an initial segment of $W(\eta, S)$ for any $\eta \in \mathbf{S}\}$.
2. For a λ-witness $\mathbf{x}$ and $I \in K_\mathbf{x}$, let $Y[I]$ and G_I be defined as in the proof of [6, 1.2] (before Fact A). We may write η instead of $I = \{\eta\}$.

Claim 8. Assume that

(*a*) μ is singular strong limit cardinal, $\lambda = \mu^+ = 2^\mu$, and $\sigma = \mathrm{cf}(\mu)$,

(*b*) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \sigma\}$ is stationary,

(*c*) $\mathbf{x}$ is a λ-witness (cf. Theorem 5) with $W(\langle\,\rangle, \mathbf{S}) \subseteq S$,

(*d*) for each $\alpha \in W(\langle\,\rangle, \mathbf{S})$, the Abelian group $G_{\mathbf{x}(\langle\alpha\rangle)}$ is not Whitehead (where $G_{\mathbf{x}(\langle\alpha\rangle)}$ is defined as in the proof of [6, 1.2]).

Then

1. there is a strongly λ-free Abelian group G of cardinality λ which is not Whitehead, in fact $\Gamma(G) \subseteq S$, and
2. there is a strongly λ-free Abelian group G^* of cardinality λ satisfying $\mathrm{HOM}(G^*, \mathbb{Z}) = \{0\}$, in fact $\Gamma(G^*) \subseteq S$ (even the same Abelian group suffices).

In our notation, we rely here on [6, §3]. This means that in clause (d), $G_{\mathbf{x}(\langle\alpha\rangle)}$ is the Abelian group defined from $\mathbf{x}(\langle\alpha\rangle)$. If the reader does not like clause (d) of Claim 8, he or she can replace it by "λ is as in Theorem 4".

Proof. We start by proving 1.: let $\mathbf{S} = \mathbf{S}_\mathbf{x}$, etc. Without loss of generality, we have

$(*)_0$ $\bigcup\{B^\mathbf{x}_\eta : \eta \in \mathbf{S}_c\} = \lambda$.

Let $\mathscr{H}(\lambda) = \bigcup_{\alpha<\lambda} M_\alpha$ where $M_\alpha \prec (\mathscr{H}(\lambda), \in)$ has cardinality μ, is increasing continuous with α such that $\mu + 1 \subseteq M_0$ and $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$.

Let $S_0 = \{\delta \in W(\langle\,\rangle, \mathbf{S}) : M_\delta \cap \lambda = \delta\}$. Since $W(\langle\,\rangle, \mathbf{S})$ is stationary and $\{\delta < \lambda : M_\delta \cap \lambda = \delta\}$ is a club of λ, clearly also S_0 is a stationary subset of λ. Since as μ is strong limit (and so $\mu = \mu^{<\sigma}$), for each $\delta \in S_0$, we get

$(*)_1$ if $Y \subseteq M_\delta, |Y| < \mu$ and $\mathrm{cf}(\delta) = \sigma$, then there is an increasing continuous sequence $\langle X_i : i < \sigma \rangle$ of subsets of Y with union Y such that $M_\delta \supseteq \{X_i : i < \sigma\}$.

If $\alpha \in W(\langle\,\rangle, \mathbf{S})$, then $\lambda(\langle \alpha \rangle, S) \leq \mu < \lambda$; therefore, for some θ, we have that

$(*)_2$ θ is regular $< \mu$ and the set $S_1 = \{\delta \in S_0 : \lambda(\langle \alpha \rangle, S) = \theta\}$ is stationary.

For each $\delta \in S_1$, clearly $\mathbf{x}(\langle \delta \rangle)$ is a θ-witness. Now define

$(*)_3$ $X_\delta := \{(\rho, s) : \rho \in \mathbf{S}_f^{\mathbf{x}(\langle\delta\rangle)}$ (i.e., is $\triangleleft$-maximal in $\mathbf{S}^{\mathbf{x}(\langle\delta\rangle)}$) and s is a finite initial segment of $s^0_{\langle\delta\rangle^\frown\rho}$ which is a set of members of $B_{\langle\delta\rangle}$ of order type $\omega\}$.

Then clearly,

$(*)_4$ $X_\delta \subseteq M_\delta$ has cardinality θ.

We can find $\bar{X}_\delta$ such that

$(*)_5$ $\bar{X}_\delta = \langle X_{\delta,i} : i < \sigma \rangle$ is as in $(*)_1$ above, i.e., is $\subseteq$-increasing continuous. Thus, $X_{\delta,i} \in M_\delta, |X_{\delta,i}| \leq \theta$. If we define $X_{\delta,\sigma} := \bigcup\{X_{\delta,i} : i < \sigma\}$ we have $X_{\delta,\sigma} = X_\delta$, we obtain that

$(*)_6$ $Z_{\delta,i} := \{(\rho, |s|) : (\rho, s) \in X_{\delta,i}\}$ for $i \leq \sigma$.

We define an equivalence relation E on S_1 by:

$(*)_7$ $\delta_1 \mathrel{E} \delta_2$ iff

- (a) $\mathbf{S}^{\langle\delta_1\rangle} = \mathbf{S}^{\langle\delta_2\rangle}$ (equivalently, $\mathbf{S}_{\mathbf{x}(\langle\delta_1\rangle)} = \mathbf{S}_{\mathbf{x}(\langle\delta_2\rangle)}$), and
- (b) for each $i < \sigma$ we have $Z_{\delta_1,i} = Z_{\delta_2,i}$.

Clearly, E is an equivalence relation. We have that $\theta < \mu$ and μ is strong limit, and thus E has $\leq 2^\theta < \mu < \lambda$ equivalence classes. Therefore, for some $\delta^* \in S_1, S_2 := \delta^*/E$ is a stationary subset of λ. Clearly, there is $F \in M_0$ which is a one-to-one function with range $\subseteq \{\delta : \delta < \lambda, \delta = \mu\delta$ is $< \lambda$ but $> 0\}$ and with domain $\mathscr{H}(\lambda)$. Thus, for every $\delta \in S_2$, the function F maps M_δ onto $\mathrm{Rang}(F) \cap \delta$. Furthermore, without loss of generality, if $\varrho_1 \triangleleft \varrho_2$ are from ${}^{\lambda>}\mathscr{H}(\lambda)$ then $F(\varrho_1) < F(\varrho_2)$ and $F(\langle X_{\delta,j} : j \leq i \rangle)$ is $> \sup\{s \cup \mathrm{Rang}(\rho) : (\rho, s) \in X_{\delta,i}\}$.

Let $\alpha^*_{\delta,i} = F(\langle X_{\delta,j} : j \leq i \rangle)$ for $\delta \in S_2$ and $i < \sigma$. Then clearly $\langle \alpha^*_{\delta,i} : i < \sigma \rangle$ is increasing with limit δ by the choice of F and of S_2, as $\delta = \sup \bigcup \{s^{0,\mathbf{x}}_\eta : \langle \delta \rangle \trianglelefteq \eta \in \mathbf{S}_f\}$ (cf. Theorem 5). By [8, 4.2], we can find $\langle (\nu_{\delta,1}, \nu_{\delta,2}) : \delta \in S_2 \rangle$ such that

$\circledast$ (a) $\nu_{\delta,\ell} \in {}^\sigma\mu$,

(b) $\nu_{\delta,1}(i) < \nu_{\delta,2}(i)$ for $i < \sigma$, and

(c) if $\mathbf{c}:\lambda\to 2^\theta+\sigma$, then for stationarily many $\delta\in S_2$ we have: if $i<\sigma$ and $\varepsilon<\theta$, then $\mathbf{c}(\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,1}(i))=\mathbf{c}(\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,2}(i))$.

Let $\mathbf{y}=\mathbf{x}\restriction\mathbf{S}'$ where $\mathbf{S}'=\{\langle\,\rangle\}\cup\{\rho\in\mathbf{S}:\rho\neq\langle\,\rangle \text{ and } \rho(0)\in S_2\}$.

Now at last we shall define the group. Essentially it will be similar to the group $G_{\mathbf{x}}$ (cf. Claim 6) as defined in the proof of [6, 1.2] from the system $\mathbf{x}$, restricted to $\mathbf{S}'$. However, in the proof in [6, 1.2] (before Fact A) in $(*)^a_{I,\eta}$, we used $2y_{\eta,m+1}=y_{\eta,m}+\sum\{x[a^\ell_{\eta,m}]):\ell<n \text{ and } a^\ell_{\eta,m}\in Y[I]\}$; in this setting, we replace $x[a^\ell_{\eta,m}]$ by the difference of two, related to $\circledast$; this may become clearer after reading the proof. The λ-freeness will be inherited from $\mathbf{S}$ being λ-free. The fact that the group is non-Whitehead will come from $\circledast$.

For $\delta\in S_1$ let $\mathbf{g}_\delta:\mathbf{S}_{\mathbf{x}(\langle\delta\rangle)}\to\theta$ be a one-to-one function, so by the choice of S_2, there is some $\mathbf{g}$ such that if $\delta\in S_2$, we have $\mathbf{g}_\delta=\mathbf{g}$.

We continue as in [6, §1]: for each $\eta\in\mathbf{S}_f$ and $\ell<n$, we have that $\langle a^\ell_{\eta,m}:m<\omega\rangle$ is a listing of $s^\ell_\eta\subseteq B_{\eta\restriction(\ell+1)}$. We let $Y=\bigcup\{B_\nu:\nu\in\mathbf{S}_c\}\backslash B^{\mathbf{x}}_{\langle\lambda\rangle}$. For $\eta\in\mathbf{S}_f$ and $m<\omega$, we let $\mathbf{i}_m(\eta)$ be the minimal i such that $(\eta\restriction[1,n),\{a^0_{\eta,\ell}:\ell\le m\})\in X_{\eta(0),i}$. Finally, we define G as the Abelian group generated by

$$\Xi=\{y_{\eta,m}:m<\omega \text{ and } \eta\in\mathbf{S}'_f\}\cup\{x[a]:a\in Y\}\cup\{z_\beta:\beta<\lambda\}$$

freely except for the following equations: Let $\eta\in\mathbf{S}'_f$ and $m<\omega$, and define $\delta:=\eta(0)\in S_2$ as well as $i:=\mathbf{i}_m(\eta)$. Recall the function $\mathbf{g}:\mathbf{S}_f\to\mu$ such that $\mathbf{g}\restriction\{\eta\in\mathbf{S}_f:\eta(0)=\delta\}$ is one to one, and $\alpha^*_{\delta,i}=F(\langle X_{\delta,j}:j\le i\rangle)$. Then our equations are:

$$\begin{aligned}2y_{\eta,m+1}=y_{\eta,m}\quad &+\textstyle\sum\{x[a^\ell_{\eta,m}]:0<\ell<n\}\\ &+z_{\alpha^*_{\delta,i}+\mu\mathbf{g}(\eta)+\nu_{\delta,2}(i)}\\ &-z_{\alpha^*_{\delta,i}+\mu\mathbf{g}(\eta)+\nu_{\delta,1}(i)}\end{aligned}$$

For $\alpha\le\lambda$ let G_α be the subgroup of G generated by

$$\begin{aligned}\{y_{\eta,m}:\quad &m<\omega \text{ and } \eta\in\mathbf{S}'_f \text{ and } \eta(0)<\alpha\}\\ &\textstyle\bigcup\{x[a]:a\in Y\cap\alpha \text{ and } a+1<\alpha\}\\ &\textstyle\bigcup\{z_\beta:\beta<\alpha \text{ and } \beta+1<\alpha\}.\end{aligned}$$

We get easily that

$\oplus_1$ G_α is a pure subgroup of G, increasing continuous with α, and

$\oplus_2$ if $\delta\in S_2$, then $G_{\delta+1}/G_\delta$ is isomorphic to $G_{\mathbf{x}(\langle\delta\rangle)}$ (which is not Whitehead).

[Why? By clause (d) of the assumption; at first glance, it seems that the set of generators and equations in the proof of [6, 1.2] and in this proof are different. But note that only $y_\eta,\langle\alpha\rangle\triangleleft\eta\in\mathbf{S}_f$ and $x[a],a\in\bigcup\{s^\ell_\eta:\ell\in[1,n),\langle\alpha\rangle\trianglelefteq\eta\in\mathbf{S}_f\}$ appear in the equation. Alternatively, use Remark 7 and prove $G_{\alpha+1}/G_\alpha$ is isomorphic to $G_{\mathbf{x},\alpha+1}/G_{\mathbf{x},\alpha}$ in the notation of [6, 1.2]; again note that the z_α here and $x[a]$ for $a\in\bigcup\{s^{0,\mathbf{x}}_\eta:\eta\in\mathbf{S}_f\}$ disappear.]

Now recall that

$\oplus_3$ G is strongly λ-free, moreover if $\alpha\in\lambda\backslash S_2$ and $\beta\in(\alpha,\lambda)$ then G_β/G_α is free.

[Why? As in the proof of Fact A in the proof of [6, 1.2].]

$\oplus_4$ G is not Whitehead.

[Why? We choose $(H_\alpha, h_\alpha, g_\alpha)$ by induction on $\alpha \le \lambda$ such that

(a) H_α is an Abelian group extending $\mathbb{Z}$,

(b) h_α is a homomorphism from H_α onto G_α with kernel $\mathbb{Z}$,

(c) g_α is a function from G_α to H_α inverting h_α (but in general not a homomorphism),

(d) H_α is increasing continuous with α,

(e) h_α is increasing continuous with α,

(f) g_α is increasing continuous with α, and

(g) if $\alpha = \delta + 1$ and $\delta \in S_2$, then there is no homomorphism g^* from G_α into H_α inverting h_α such that:

$\odot$ if $i < \sigma$ and $\varepsilon < \theta$, then

$$g^*(z_{\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,1}(i)}) - g_{\delta+1}(z_{\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,1}}) = g^*(z_{\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,2}(i)}) - g_{\delta+1}(z_{\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,2}(i)}).$$

(Note that the subtraction is in $\mathbb{Z}$, the kernel of h_α.)

For $\alpha = 0$ or a limit ordinal and $\alpha = \beta + 1$, $\beta \notin S_2$, this is obvious. For $\alpha = \delta + 1$, $\delta \in S_2$ it is known that if instead of $\odot$ in clause (g) we know $g^* \restriction G_\delta$, this is possible. But $\odot$ gives all the necessary information.

Let us give more details: let G'_δ be the subgroup of G_δ generated by $\{z_{\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,2}(i)} - z_{\alpha^*_{\delta,i}+\mu\varepsilon+\nu_{\delta,1}(i)} : \varepsilon < \theta$ and $i < \sigma\}$. Let $G'_{\delta+1}$ be the subgroup of $G_{\delta+1}$ generated by $G'_\delta \cup \{y_{\eta,m} : \langle\delta\rangle \trianglelefteq \eta \in \mathbf{S}'_f\}$. Clearly, G'_δ is a pure subgroup of $G'_{\delta+1}$ and of G_δ and

$$G_{\delta+1} = G'_{\delta+1} \bigoplus_{G'_\delta} G_\delta.$$

Let $H'_\delta = h_\delta^{-1}(G'_\delta)$, clearly $h_\delta \restriction H'_\delta$ is a homomorphism from H'_δ onto G'_δ with kernel $\mathbb{Z}$. Clearly,

$\oplus_{4.1}$ if $g', g'' \in \operatorname{Hom}(G_\delta, H_{\delta+1})$ invert h_δ and both satisfy $\odot$ of clause (g), then $g' \restriction G'_\delta = g'' \restriction G'_\delta$.

So $|\mathscr{G}_\delta| \le 1$ where

$$\mathscr{G}_\delta = \{g \restriction G'_\delta : \ g \text{ is a homomorphism from } G_\delta \text{ to } H_\delta \text{ inverting } h_\alpha \text{ and satisfying } \odot \text{ in clause (g)}\}.$$

Let g^* be the unique member of $\mathscr{G}$ if $\mathscr{G}$ is non-empty, and otherwise let it be any homomorphism from G'_δ into H'_δ inverting $h_\delta \restriction H'_\delta$ (this exists as G'_δ is free). We now choose $(H'_{\delta+1}, h'_{\delta+1})$ such that $H'_\delta \subseteq H'_{\delta+1}$, $h'_{\delta+1} \in \operatorname{Hom}(H'_{\delta+1}, G'_{\delta+1})$, $h'_{\delta+1}$ has kernel $\mathbb{Z}$, $h'_{\delta+1}$ extends $h_\delta \restriction G'_\delta$, and g^* cannot be extended.

Next, without loss of generality, $H'_{\delta+1} \cap H_\delta = H'_\delta$. Let

$$H_{\delta+1} = H'_{\delta+1} \underset{H'_\delta}{\oplus} H_\delta,$$

and let $h_{\delta+1} \in \mathrm{Hom}(H_{\delta+1}, G_{\delta+1})$ extend $h_\delta, h'_{\delta+1}$. Let $g_{\delta+1} \supseteq g_\delta$ invert $h_{\delta+1}$ but is not necessarily a homomorphism. So we have chosen $(H_\alpha, h_\alpha, g_\alpha)$ such that the following holds: if $\mathscr{G}_\delta \neq \varnothing$, then we cannot find $g' \in \mathrm{Hom}(G'_{\delta+1}, \mathbb{Z})$ inverting h_α such that $g' \restriction G'_\delta \in \mathscr{G}_\delta$. This suffices for carrying out the induction.

Having carried out the induction, clearly $h = h_\lambda$ is a homomorphism from $H = H_\lambda$ onto $G_\lambda = G$ with kernel $\mathbb{Z}$. To show that G is not a Whitehead group, it suffices to prove that h is not invertible as a homomorphism. But if $g \in \mathrm{Hom}(G, H)$ inverts h then $x \in G \Rightarrow g(x) - g_\lambda(x) \in \mathbb{Z}$.

We define a function $\mathbf{f}$ with domain λ as follows: for $\alpha < \lambda$ and $\zeta < \mu$, we let

$$\mathbf{f}(\mu\alpha + \zeta) = \langle g(z_{\mu^2\alpha+\mu\varepsilon+\zeta}) - g_\lambda(z_{\mu^2\alpha+\mu\varepsilon+\zeta}) : \varepsilon < \theta \rangle.$$

So for stationarily many $\delta \in S_2$, we have that if $i < \sigma$, then $\mathbf{f}(\alpha^*_{\delta,i} + \nu_{\delta,1}(i)) = \mathbf{f}(\alpha^*_{\delta,i} + \nu_{\delta,2}(i))$. For any such δ, we get a contradiction by clause (g) of the construction, so we have proved $\oplus_4$. This finishes the proof of part (1), as $G = G_\lambda$ is as required.

For the proof of part 2., we can use the following:

⊠ for regular uncountable λ, the following conditions are equivalent

- (a) every λ-free Abelian group of cardinality λ is Whitehead, and
- (b) for every λ-free Abelian group of cardinality λ we have that $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$.

[Why? If (a) and G as in (b), let h be a pure embedding of $\mathbb{Z}$ into G, let $G' = G/\mathrm{Rang}(h)$ and use the definition of "G' is a Whitehead group". If G, H and $h \in \mathrm{Hom}(H, G)$ form a counterexample, then we can find a purely increasing continuous sequence $\langle G_\alpha : \alpha \leq \lambda \rangle$ and $\langle x_\alpha : \alpha < \lambda \rangle$ such that $\{x_\alpha : \alpha < \lambda\} = \{x \in G_\lambda : \mathbb{Z}x$ is a pure subgroup of $G_\lambda\}$ and for each α, there is a pure embedding h_α of H into $G_{\alpha+1}$ such that $h(1_\mathbb{Z}) = x_\alpha$ and

$$G_{\alpha+1} = G_\alpha \underset{\mathbb{Z}x_\alpha}{\oplus} h_\alpha(H).$$

Easily G_λ contradicts clause (b).] Q.E.D. (Claim 8)

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