

Spectrally compact operators

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Abstract

We define the concept of a spectrally compact operator, and study the basic properties of these operators. We show that the class of spectrally compact operators is strictly contained in the class of compact operators and in the class of spectrally bounded operators. It is also proved that the set of spectrally compact operators on a spectrally normed space E is a right ideal of $SB(E)$ and in certain cases it is a two sided ideal. We will also study the spectral adjoint of a spectrally compact operator.

2000 Mathematics Subject Classification. **47B48**. 46B99, 47L10.

Keywords. spectrally normed space, spectrally bounded operator, spectrally compact operator.

1 Introduction

Let E be a normed space endowed with a spectral structure in the sense that there exists a linear topological isomorphism τ from E into a unital Banach algebra A . We consider E as a normed subspace of A , and we write $E \subseteq A$. Such a normed space E is said to be a *spectrally normed* space. It should be emphasized that the spectral structure on E depends on the embedding, up to topological isomorphisms. For $x \in E$, $\text{sp}(x)$ and $r(x)$ denote the spectrum and the spectral radius of x with respect to the Banach algebra A , respectively. A spectrally normed space is said to be commutative (semisimple) whenever A is commutative (semisimple).

Every normed space E carries at least one spectral structure via the isometric embedding $j_E : E \rightarrow C(E_1^*)$, the complex-valued continuous functions on the dual closed unit ball E_1^* , endowed with the weak*-topology. This is a commutative semisimple structure and $\|x\| = r(x)$ for $x \in E$.

Let E, F be spectrally normed spaces. A linear mapping $T : E \rightarrow F$ is called *spectrally bounded*, if there exists $M \geq 0$, such that $r(Tx) \leq Mr(x)$, for all $x \in E$. In general, a spectrally bounded operator need not be bounded and conversely, a bounded operator between spectrally normed spaces may not be spectrally bounded, see [3, Examples 2.7, 2.8].

*The authors would like to express their deepest thanks to the referees for valuable comments and suggesting the shorter proof of Theorem 2.10.

Mathieu and Schick initiated a systematic study of spectrally bounded operators between spectrally normed spaces in [3]. Spectrally bounded operators on von Neumann algebras and simple C^* -algebras were studied in [4, 2].

For each pair of spectrally normed spaces E and F , we denote by $B(E, F)$, $K(E, F)$ and $SB(E, F)$, the space of all bounded operators, all compact operators and all spectrally bounded operators from E to F , respectively. The closed unit ball of E is denoted by E_1 . For $T \in SB(E, F)$ the value

$$\|T\|_\sigma = \inf\{M \geq 0 : r(Tx) \leq Mr(x), x \in E\},$$

is called the *spectral operator norm* of T . We recall the following results from [3].

Proposition 1.1. [3, Proposition 2.4] Let E, F and G be spectrally normed spaces and $S, T \in SB(E, F)$ and $R \in SB(F, G)$ then

1. $\|T\|_\sigma = \sup\{r(Tx) : x \in E, r(x) \leq 1\} = \sup\{r(Tx) : x \in E, r(x) = 1\}$;
2. $\|\lambda T\|_\sigma = |\lambda| \|T\|_\sigma$ for all $\lambda \in \mathbb{C}$;
3. $\|RT\|_\sigma \leq \|R\|_\sigma \|T\|_\sigma$;
4. $\|S + T\|_\sigma \leq \|S\|_\sigma + \|T\|_\sigma$, if F is commutative.

Proposition 1.2. [3, Proposition 2.5] Suppose that F is a commutative semisimple spectrally normed space. For every spectrally normed space E , $(SB(E, F), \|\cdot\|_\sigma)$ is a normed space. If $E = F$ then $SB(E) = SB(E, E)$ is a unital normed algebra.

In Section 2, we define spectrally compact operators and study some of their basic properties. We show that the class of spectrally compact operators is strictly contained in the class of compact operators and in the class of spectrally bounded operators. We will also show that the set of spectrally compact operators on a spectrally normed space E is a right ideal of the algebra $SB(E)$ which, in certain cases, is a two sided ideal. Section 3 is devoted to the study of the spectral adjoint of a spectrally compact operator. We will show that the spectral adjoint of every spectrally compact operator is spectrally compact, but it remains open if this is in fact an equivalence.

2 Basic properties of spectrally compact operators

From now on, throughout the paper, E and F are assumed to be complex spectrally normed spaces.

Definition 2.1. A linear mapping $T : E \rightarrow F$ is said to be *spectrally compact* if $\overline{T(U_E)}$ is compact in F , where $U_E = \{x \in E : r(x) \leq 1\}$ and

E, F are spectrally normed spaces. The set of spectrally compact operators from E to F is denoted by $\text{KSB}(E, F)$.

Proposition 2.2. If $T : E \rightarrow F$ is a spectrally compact operator, then T is spectrally bounded.

Proof. Let $U_E = \{x \in E : r(x) \leq 1\}$. Since $\overline{T(U_E)}$ is compact, there is $M \geq 0$ such that

$$\|T(x)\| \leq M \text{ for every } x \in U_E. \quad (2.1)$$

Now suppose that $x \in E$. If $r(x) > 0$ then by (2.1), $\|T(\frac{x}{r(x)})\| \leq M$ and so $r(Tx) \leq Mr(x)$. If $r(x) = 0$, for given $\varepsilon > 0$ we have $r(\frac{x}{(\varepsilon/M)}) = 0$. Thus by (2.1)

$$\left\| T \left(\frac{x}{(\varepsilon/M)} \right) \right\| \leq M$$

and $\|T(x)\| \leq M \cdot \frac{\varepsilon}{M} = \varepsilon$. Therefore $r(Tx) \leq \|Tx\| = 0$, since $\varepsilon > 0$ is arbitrary. It follows that $r(Tx) \leq Mr(x)$ for all $x \in E$, and hence $T \in \text{SB}(E, F)$. Q.E.D.

Proposition 2.3. Each spectrally compact operator $T : E \rightarrow F$ is a compact operator.

Proof. Let E_1 be the closed unit ball of E . Since for each $x \in E_1$, $r(x) \leq 1$ we have $E_1 \subseteq U_E$, and $T(E_1) \subseteq \overline{T(U_E)}$. Therefore $\overline{T(E_1)}$ is compact. Q.E.D.

The following examples show that in Propositions 2.2 and 2.3 the reverse implications do not hold.

Example 2.4. (i) Let A be an infinite dimensional commutative unital Banach algebra and $M_2(\mathbb{C})$ the C^* -algebra of all complex 2×2 matrices. Suppose that φ is a character and f is an unbounded linear functional on A . Then the linear mapping $T : A \rightarrow M_2(\mathbb{C})$ defined by

$$T(a) = \begin{pmatrix} \varphi(a) & f(a) \\ 0 & \varphi(a) \end{pmatrix} \text{ for every } a \in A,$$

is an unbounded operator and for each $a \in A$, $r(T(a)) = |\varphi(a)| \leq r(a)$. Thus T is spectrally bounded. Clearly this mapping is not spectrally compact, otherwise, by Proposition 2.3, it should be compact and hence bounded.

(ii) Let $A = M_2(\mathbb{C})$, and $f : A \rightarrow \mathbb{C}$ a linear functional defined by $f(a_{ij}) = a_{12}$ for all $(a_{ij}) \in A$. Clearly, f is compact. We show that it is not spectrally compact. Let

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then $r(a) = 0$ and $r(f(a)) = 1$. It follows that f is not spectrally bounded and hence it is not spectrally compact.

If E and F are spectrally normed spaces and $T : E \rightarrow F$ is a linear mapping then it is easy to see that for $R \geq 0$,

- (i) $\overline{\{T(x) : r(x) \leq R\}}$ is compact if and only if $\overline{\{T(x) : r(x) \leq 1\}}$ is compact.
- (ii) $\overline{\{T(x) : r(x) < R\}}$ is compact if and only if $\overline{\{T(x) : r(x) < 1\}}$ is compact.

A subset B of a spectrally normed space E is called *spectrally bounded* if there is $M \geq 0$ such that $r(x) \leq M$, for all $x \in B$.

Theorem 2.5. Suppose that $T : E \rightarrow F$ is a linear mapping between spectrally normed spaces E and F . Then the following conditions are equivalent:

- (i) T is a spectrally compact operator,
- (ii) for every spectrally bounded subset B of E , $\overline{T(B)}$ is compact, and
- (iii) for every spectrally bounded sequence (x_n) in E , $(T(x_n))$ has a convergent subsequence in F .

Proof. “(i) \Leftrightarrow (ii)” is obvious.

“(ii) \Rightarrow (iii)”. Suppose that (x_n) is a spectrally bounded sequence in E . Then there is $M \geq 0$ such that $r(x_n) \leq M$ for all $n \in \mathbb{N}$. Let $B = \{x \in E : r(x) \leq M\}$. By (ii), $\overline{T(B)}$ is compact and $(T(x_n))$ is a sequence in the compact set $\overline{T(B)}$, so it has a convergent subsequence.

“(iii) \Rightarrow (ii)”. Suppose that B is a spectrally bounded subset of E . To show that $\overline{T(B)}$ is compact, we prove that every sequence in this set has a convergent subsequence. Let (y_n) be a sequence in $\overline{T(B)}$, then there exists a sequence (x_n) in B such that

$$\|y_n - Tx_n\| < \frac{1}{n} \text{ for every } n \in \mathbb{N}.$$

Since (x_n) is a spectrally bounded sequence, by the hypothesis there is a subsequence (Tx_{n_j}) of (Tx_n) , such that $Tx_{n_j} \rightarrow y$ for some $y \in F$, as $j \rightarrow \infty$. Let $\varepsilon > 0$ be given. There are $N_1, N_2 \in \mathbb{N}$ such that $\frac{1}{n_j} < \frac{\varepsilon}{2}$ for all $j \geq N_1$ and

$$\|Tx_{n_j} - y\| < \frac{\varepsilon}{2} \text{ for every } j \geq N_2.$$

So for all $j \geq \max(N_1, N_2)$ we have

$$\|y_{n_j} - y\| \leq \|y_{n_j} - Tx_{n_j}\| + \|Tx_{n_j} - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $y_{n_j} \rightarrow y$.

Q.E.D.

Corollary 2.6. If $S, T : E \rightarrow F$ are spectrally compact operators, and if $\lambda \in \mathbb{C}$ then $S + \lambda T$ is a spectrally compact operator.

As a consequence of Proposition 2.2 and Corollary 2.6, $\text{KSB}(E, F)$ is a linear subspace of $\text{SB}(E, F)$.

Theorem 2.7. Let X be a dense linear subspace of E and $T : X \rightarrow F$ a spectrally compact operator. Then T has a unique spectrally compact extension $\tilde{T} : E \rightarrow F$.

Proof. Since T is bounded we can extend T to a bounded operator $\tilde{T} : E \rightarrow F$. Suppose (x_n) is a spectrally bounded sequence in E and let $M \geq 0$ be such that $\text{r}(x_n) \leq M$ for all $n \in \mathbb{N}$. Suppose $V = \{\lambda \in \mathbb{C} : |\lambda| < M\}$, then $\text{sp}(x_n) \subseteq V$ for all $n \in \mathbb{N}$. Since $x_1 \in \overline{X}$, there is a sequence (x_{1n}) in X such that $x_{1n} \rightarrow x_1$. By [1, Theorem 3.4.2], there exists $0 < \delta_1 < 1$ such that $\|x_1 - y\| < \delta_1$ implies that $\text{sp}(y) \subseteq V$, that is $\text{r}(y) \leq M$. Take $\varepsilon_1 < \delta_1$, then there exists $n_1 \in \mathbb{N}$ such that

$$\|x_{1n} - x_1\| < \varepsilon_1 < 1 \text{ for every } n \geq n_1.$$

Thus $\text{sp}(x_{1n}) \subseteq V$ for all $n \geq n_1$. Similarly for x_2 there exists a sequence (x_{2n}) in X such that $x_{2n} \rightarrow x_2$. Again by [1, Theorem 3.4.2], there exists $0 < \delta_2 < \frac{1}{2}$ such that $\|x_2 - y\| < \delta_2$ implies that $\text{sp}(y) \subseteq V$. Take $0 < \varepsilon_2 < \delta_2$, there exists $n_2 \in \mathbb{N}$ such that

$$\|x_{2n} - x_2\| < \varepsilon_2 < \frac{1}{2} \text{ for every } n \geq n_2.$$

Therefore $\text{sp}(x_{2n}) \subseteq V$ for all $n \geq n_2$. An inductive argument gives us a sequence $(y_k) \subseteq X$ ($y_k = x_{kn_k}$ ($k \in \mathbb{N}$)) such that $x_k - y_k \rightarrow 0$ and $\text{sp}(y_k) \subseteq V$. Since T is spectrally compact, there is a subsequence (y_{k_j}) such that $Ty_{k_j} \rightarrow y_0$ for some $y_0 \in F$, as $j \rightarrow \infty$. Since \tilde{T} is continuous

$$\tilde{T}x_{k_j} - \tilde{T}y_{k_j} = \tilde{T}(x_{k_j} - y_{k_j}) \rightarrow 0.$$

So $\tilde{T}x_{k_j} \rightarrow y_0$ and hence \tilde{T} is spectrally compact. Q.E.D.

Proposition 2.8. If $T \in \text{SB}(E, F)$ and $S \in \text{KSB}(F, G)$ for some spectrally normed spaces E, F and G , then $ST : E \rightarrow G$ is spectrally compact.

Proof. Suppose that (x_n) is a spectrally bounded sequence in E , that is there exists $M \geq 0$ such that $\text{r}(x_n) \leq M$ for all $n \in \mathbb{N}$. Since $T \in \text{SB}(E, F)$, by Proposition 1.1

$$\text{r}(Tx_n) \leq \|T\|_\sigma \text{r}(x_n) \leq M\|T\|_\sigma \text{ for every } n \in \mathbb{N}.$$

So (Tx_n) is a spectrally bounded sequence in F and hence it has a subsequence (Tx_{n_j}) such that (STx_{n_j}) converges, because S is spectrally compact. Therefore $ST \in \text{KSB}(E, G)$. Q.E.D.

Corollary 2.9. $\text{KSB}(E)$ is a right ideal of $\text{SB}(E)$, and if $E \subseteq C(Y)$ for some compact Hausdorff space Y , then $\text{KSB}(E)$ is a two sided ideal of $\text{SB}(E)$.

Proof. By Proposition 2.8, $\text{KSB}(E)$ is a right ideal of $\text{SB}(E)$. Suppose that $E \subseteq C(Y)$, $T \in \text{SB}(E)$ and $S \in \text{KSB}(E)$. Let (x_n) be a spectrally bounded sequence in E . By Theorem 2.5, it has a subsequence (x_{n_j}) such that $Sx_{n_j} \rightarrow y$ for some $y \in E$, as $j \rightarrow \infty$. Since $E \subseteq C(Y)$, by [3, Proposition 2.9], T is bounded and hence $T(Sx_{n_j}) \rightarrow Ty$. Thus TS is spectrally compact by Theorem 2.5. Q.E.D.

Theorem 2.10. Let E and F be spectrally normed spaces with $F \subseteq C(Y)$, where Y is a compact Hausdorff space. The following statements hold:

- (i) The inclusion mapping $\iota : \text{SB}(E, F) \rightarrow \text{B}(E, F)$ is contractive,
- (ii) if there exists $M \geq 0$ satisfying $\|x\| \leq M$, for every x in $U_E = \{x \in E : r(x) \leq 1\}$, the inequality $\|\iota(T)\| \leq \|T\|_\sigma \leq M\|\iota(T)\|$ holds for every T in $\text{SB}(E, F)$,
- (iii) $\text{SB}(E, F)$ is a Banach space whenever F is a closed subspace of $C(Y)$,
- (iv) under the hypothesis in (ii), ι has closed range whenever F is a closed subspace of $C(Y)$,
- (v) $\iota(\text{KSB}(E, F)) \subseteq \text{K}(E, F)$ and under the assumptions in (ii), $\text{KSB}(E, F)$ is a $\|\cdot\|_\sigma$ -closed subspace of $\text{K}(E, F)$.

Proof. (i) Since $F \subseteq C(Y)$, by [3, Proposition 2.9], we have

$$\text{SB}(E, F) \subseteq \text{B}(E, F),$$

and $\|T\| \leq \|T\|_\sigma$ for every $T \in \text{SB}(E, F)$. Thus the inclusion mapping $\iota : \text{SB}(E, F) \rightarrow \text{B}(E, F)$ is contractive.

(ii) By hypothesis, $\|x\| \leq M$ for every $x \in U_E$. Therefore, given $x \in U_E$ and T in $\text{SB}(E, F)$, we have

$$r(T(x)) \leq \|T(x)\| \leq \|\iota(T)\|M.$$

Since x was arbitrarily chosen in U_E , we deduce, via Proposition 1.2 or [3, Proposition 2.4], that $\|T\|_\sigma \leq M\|\iota(T)\|$.

(iii) Let (T_n) be a $\|\cdot\|_\sigma$ -Cauchy sequence in $\text{SB}(E, F)$. In this case, there exists $M \geq 0$ satisfying $\|T_n\|_\sigma \leq M$, for every $n \in \mathbb{N}$. Since $F \subseteq C(Y)$, by [3, Proposition 2.9], we have $\text{SB}(E, F) \subseteq \text{B}(E, F)$ and $\|\cdot\| \leq \|\cdot\|_\sigma$ on $\text{SB}(E, F)$. We deduce that every $\|\cdot\|_\sigma$ -Cauchy sequence in $\text{SB}(E, F)$ is a $\|\cdot\|$ -Cauchy sequence in $\text{B}(E, F)$. It follows that there exists $T \in \text{B}(E, F)$ such that $\|T_n - T\| \rightarrow 0$ (F is a Banach space).

Let us fix an arbitrary $x \in E$. Since F is commutative the spectral radius is subadditive. Thus, the inequality

$$\begin{aligned} r(T(x)) &\leq r((T - T_n)(x)) + r(T_n(x)) \\ &\leq \|T_n - T\| \|x\| + \|T_n\|_\sigma r(x) \\ &\leq \|T_n - T\| \|x\| + Mr(x) \end{aligned}$$

holds for every $n \in \mathbb{N}$. Taking limit in $n \rightarrow \infty$ we have $r(T(x)) \leq Mr(x)$. The required statement follows because x is arbitrary.

(iv) Follows from (i), (ii), and (iii).

(v) Proposition 2.5 above implies that $\iota(\text{KSB}(E, F)) \subseteq \text{K}(E, F)$. Now suppose that $\|T_n - T\|_\sigma \rightarrow 0$ where $(T_n) \subseteq \text{KSB}(E, F)$ and $T \in \text{SB}(E, F)$. It follows from (i) that $\|\iota(T_n) - \iota(T)\| \rightarrow 0$ in $\text{B}(E, F)$. The sequence $(\iota(T_n))$ lies in $\text{K}(E, F)$. Therefore, $\iota(T)$ is a compact operator and, since U_E is bounded, there exists $M \geq 0$ such that $T(U_E) = \iota(T)(U_E) \subseteq \overline{\iota(T)(M(E_1))}$ is compact. This shows that $T \in \text{KSB}(E, F)$. Q.E.D.

Corollary 2.11. If $E = F \subseteq C(Y)$ for a compact Hausdorff space Y , then $\text{KSB}(E)$ is a closed two sided ideal of $\text{SB}(E)$.

3 Spectral adjoint

We recall the following definition from [3]:

Definition 3.1. For a spectrally normed space E , $(\text{SB}(E, \mathbb{C}), \|\cdot\|_\sigma)$ is called the *spectral dual* of E and is denoted by E^σ .

Remark 3.2. By Proposition 1.2, for every spectrally normed space E , the space E^σ is normed. In fact it is a Banach space, see [3, Proposition 3.2]. If E^* denote the dual of E then by Theorem 2.10 (i) or [3, Proposition 2.9], we have a contractive embedding from E^σ into E^* . In other words, $E^\sigma \subseteq E^*$ and $\|f\| \leq \|f\|_\sigma$ for every $f \in E^\sigma$.

We consider E^σ as a spectrally normed space via the spectral structure inherited from the embedding into $C((E^\sigma)_1^*)$. Moreover, in this spectral structure $\|f\|_\sigma = r(f)$ for all $f \in E^\sigma$.

If $T : E \rightarrow F$ is a spectrally bounded operator, the linear operator $T^\sigma : F^\sigma \rightarrow E^\sigma$ defined by $T^\sigma g = g \circ T$ (for $g \in F^\sigma$), is said to be the *spectral adjoint* of T . The following proposition is [3, Corollary 3.7].

Proposition 3.3. The spectral adjoint T^σ of a spectrally bounded operator T is a spectrally bounded operator with $\|T^\sigma\|_\sigma \leq \|T\|_\sigma$.

We show that a similar result holds for spectrally compact operators.

Theorem 3.4. Suppose that $T : E \rightarrow F$ is a spectrally compact operator then $T^\sigma : F^\sigma \rightarrow E^\sigma$ is spectrally compact.

Proof. We consider a spectral structure on F^σ as in Remark 3.2. Let $B \subseteq F^\sigma$ be a spectrally bounded set, so there exists $C > 0$ such that

$$\|g\|_\sigma = r(g) \leq C \text{ for every } g \in B.$$

We shall show that $T^\sigma(B)$ is totally bounded. Let $\varepsilon > 0$ be given. Since T is spectrally compact, $T(U_E) = \{Tx : r(x) \leq 1\}$ is relatively compact and hence is totally bounded in F . So there are $x_1, \dots, x_n \in U_E$ such that for every $x \in U_E$ there exists $j \in \{1, \dots, n\}$ for which

$$\|Tx - Tx_j\| < \frac{\varepsilon}{3C}.$$

We define a linear operator $S : F^\sigma \rightarrow \mathbb{C}^n$ by

$$S(g) = (g(Tx_1), \dots, g(Tx_n)) \text{ for every } g \in F^\sigma.$$

Since each $g \in B$ is a bounded linear functional and T is compact, S is a compact operator and hence $\overline{S(B)}$ is a compact set. Therefore $S(B)$ is totally bounded, that is there exist g_1, \dots, g_m in B such that for each $g \in B$ there exists $k \in \{1, \dots, m\}$ with

$$\|Sg - Sg_k\|_e < \frac{\varepsilon}{3},$$

where $\|\cdot\|_e$ denotes the Euclidean norm on \mathbb{C}^n . Thus for every $g \in B$ there is $k \in \{1, \dots, m\}$ such that

$$|g(Tx_j) - g_k(Tx_j)|^2 \leq \sum_{\ell=1}^n |g(Tx_\ell) - g_k(Tx_\ell)|^2 = \|S(g - g_k)\|_e^2 < \frac{\varepsilon^2}{3^2},$$

for every $j \in \{1, \dots, n\}$. Fix arbitrary $x \in U_E$ and $g \in B$. There is $j \in \{1, \dots, n\}$ such that

$$\|Tx - Tx_j\| < \frac{\varepsilon}{3C}.$$

There is $k \in \{1, \dots, m\}$ such that

$$\|Sg - Sg_k\|_e < \frac{\varepsilon}{3}.$$

For every $g \in B$ we have $\|g\| \leq \|g\|_\sigma \leq C$, thus

$$\begin{aligned} |g(Tx) - g_k(Tx)| &\leq |g(Tx) - g(Tx_j)| + |g(Tx_j) - g_k(Tx_j)| \\ &\quad + |g_k(Tx_j) - g_k(Tx)| \\ &\leq \|g\| \cdot \|Tx - Tx_j\| + \frac{\varepsilon}{3} + \|g\| \cdot \|Tx_j - Tx\| \\ &< C \cdot \frac{\varepsilon}{3C} + \frac{\varepsilon}{3} + C \cdot \frac{\varepsilon}{3C} = \varepsilon. \end{aligned}$$

Therefore

$$\begin{aligned} \|T^\sigma g - T^\sigma g_k\|_\sigma &= \sup\{\mathfrak{r}((T^\sigma g - T^\sigma g_k)(x)) : \mathfrak{r}(x) \leq 1\} \\ &= \sup\{|(T^\sigma g - T^\sigma g_k)(x)| : \mathfrak{r}(x) \leq 1\} \\ &= \sup\{|g(Tx) - g_k(Tx)| : \mathfrak{r}(x) \leq 1\} \leq \varepsilon. \end{aligned}$$

So for each $g \in B$ there exists $k \in \{1, \dots, n\}$ such that $\|T^\sigma g - T^\sigma g_k\|_\sigma \leq \varepsilon$.
It follows that $T^\sigma(B)$ is totally bounded. Q.E.D.

We close with an open question:

Question 3.5. Let T^σ , the spectral adjoint of a linear mapping T between spectrally normed spaces, be spectrally compact. Is it true that T is also spectrally compact?

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