Singular points of real quartic and quintic curves

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Abstract

There are thirteen types of singular points for irreducible real quartic curves and seventeen types of singular points for reducible real quartic curves. This classification is originally due to D. A. Gudkov. There are nine types of singular points for irreducible complex quartic curves and ten types of singular points for reducible complex quartic curves. There are 42 types of real singular points for irreducible real quintic curves and 49 types of real singular points for reducible real quintic curves. The classification of real singular points for irreducible real quintic curves is originally due to Golubina and Tai. There are 28 types of singular points for irreducible complex quintic curves and 33 types of singular points for reducible complex quintic curves. We derive the complete classification with proof by using the computer algebra system Maple. We clarify that the classification is based on computing just enough of the Puiseux expansion to separate the branches. Thus, the proof consists of a sequence of large symbolic computations that can be done nicely using Maple.

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1 Introduction

Leading mathematicians in the field of algebraic geometry, as well as other fields, have stated many times in books and talks that classification is one of the most fundamental problems of mathematics. Classifying singular points of algebraic curves of degree n is one of those problems. The first nontrivial degrees to consider are four and five. History and experience have

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shown that complete classifications for curves of specific degrees provide the data that allow conjectures to be formulated, which may become theorems about curves of all degrees. Such theorems allow further classifications, which in turn lead to new theorems, and so the process continues in a kind of synergy. The classification of singular points of curves has also been used in a technique for constructing curves with controlled topology, known as dissipation of singular points.

The classification of singular points of real quartic curves is originally due to D. A. Gudkov [8, 5, 6, 7, 9]. He determined the individual types of singular points, as well as all possible sets of singular points that real quartic curves can have. The classification of singular points of real quintic curves is originally due to Golubina and Tai [4]. They found 41 individual types of real singular points for real irreducible quintic curves. In this paper, we will derive the thirteen individual types of singular points for irreducible real quartic curves and the seventeen individual types of singular points for reducible real quartic curves. We will then derive, with proof, the classification of individual types of singular points for both irreducible and reducible real quintic curves. We found 42 individual types of singular points for irreducible quintic curves. We think that class 39 of Golubina and Tai should split into two distinct classes based on whether the two tangent lines are real and distinct or complex conjugate (these classes are represented by diagrams 3 and 4 below). We exhibit the 49 individual types of singular points for reducible real quintic curves. There are 28 individual types of singular points for irreducible complex quintic curves and 33 individual types of singular points for reducible complex quintic curves. Our description of the equivalence relation is new and our proof is new and gives a very nice illustration of the role that computer algebra can play in doing proofs. Furthermore, our proof is self-contained and is the most elementary proof possible, which makes the material accessible to the widest possible audience.

The classification of singular points of complex projective quintic curves appears in a paper by A. Degtyarev [3]. He not only exhibits the 28 individual types of singular points for complex irreducible quintics, but he exhibits all 221 sets of singular points, and furthermore proves that the rigid isotopy type of an irreducible complex quintic is defined uniquely by its set of singular points. Previous authors of papers on classifications of singular points of algebraic curves applied theorems that gave enough invariants to separate the singular points into distinct classes for the particular degree being considered. However, no general equivalence relation, applying to curves of arbitrary degree, was described.

The general question is how shall we classify singular points of real curves of a fixed degree. For each fixed degree n, we want a finite classification of singular points for all algebraic curves of degree n. Thus, in general, the local diffeomorphism type is not the desired criterion of classification for singular points. For irreducible quartic curves, there are only finitely many diffeomorphism types, but for reducible quartic curves, there are infinitely many equivalence classes with respect to local diffeomorphism. For example, in the Arnol'd notation [1], four lines intersecting at the origin represents an X_9 singular point which is really an infinite family of smoothly inequivalent singularities. Notice here that an irreducible real quintic curve can have an X_9 singular point. The tradition is to treat these as one class by fiat. In our scheme the X_9 family will appear naturally as a single class.

Now let us describe how we will classify the individual types of singular points that a real curve of a fixed degree can have. Given any polynomial equation $F(x, y) = 0$, it is possible to solve for y in terms of x in the form of fractional power series, called Puiseux expansions. There is an algorithm for doing this, and the software Maple computes such Puiseux expansions, even for curves with literal coefficients. Our classification is based on taking just enough of the Puiseux expansions to separate the "branches," and noting the exponents at which the "branches" separate. In other words, compute the Puiseux expansions to a power of x such that all expansions are unique. Then we will associate a tree-type graph, to which we will refer as a "tree diagram" or "diagram." These diagrams will be described in detail below and will codify how the "branches" separate and will serve to classify the type of the singular point. It follows from Section 10 of Milnor's book [10] that such a classification gives a finite number of types for each fixed degree. At this point let us remark that the term "branch" already has a traditional meaning in this context. We are really interested in the distinct Puiseux expansions. In [12], C. T. C. Wall has coined the term "pro-branch" for the distinct Puiseux expansions. In the same book, C. T. C. Wall defines the *exponent of contact* between two pro-branches to be the smallest exponent such that the corresponding terms in the two Puiseux expansions have unequal coefficients. Thus, the geometric meaning of our classification is that two singular points are equivalent if and only if their pro-branches have the same exponents of contact.

In studying a singular point of an algebraic curve, the first thing to look at is the Newton polygon. (Our Newton polygons will follow the style of Walker [11].) Corresponding to each segment of the Newton polygon, there is a quasihomogeneous polynomial [2, p. 195]. If all such quasihomogeneous polynomials have no multiple factors, then the Newton polygon already tells us the type of the singularity. (Note that in this case, we know right away the exponents at which all of the Puiseux expansions separate.) But if there is a multiple factor, then it is necessary to examine the situation more closely. For this, we turn to the Puiseux expansions. As indicated above, the relevant definition on which our classification is based is new and appeals to the Puiseux expansions in an invariant way.

Let us note that we will classify the *real* singular points. (It is possible for a real curve to have a complex conjugate pair of singular points. We will avoid this case.) By a simple translation of axes, we may assume that the singular point is at the origin. We will treat both irreducible and reducible curves, but note that the notions of irreducible and reducible are with respect to the complex numbers. Note also that we will not study reducible curves with multiple components.

The objects being classified are pairs whose first coordinate is a real quartic curve, specified by a polynomial with real coefficients, considered up to a real nonzero multiplicative constant, and the second coordinate is a singular point of the curve in the first coordinate. For example, let the quartic curve be given by $f(x, y) = 0$, where

$$\begin{aligned} f(x, y) = & a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy \\ & + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{40}x^4 + a_{31}x^3y \\ & + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4. \end{aligned}$$

Since we may assume that our singular point is at the origin, we have $a_{00} = 0$. Since the point is singular, $a_{10} = a_{01} = 0$. In this paper we will use the term "tangent cone" to refer to the terms of lowest degree in $f(x, y)$. The degree of these terms is called the multiplicity of the point. Let us remark here that our use of this term is slightly unconventional. The traditional use of the term tangent cone refers to the zero set of the terms of lowest degree, but for our work it is of the utmost importance to keep track of the multiplicities as well. With respect to quartic curves, if the point is of multiplicity four, then the curve must be reducible since any homogeneous polynomial of degree 4 must factor. Similarly with respect to quintic curves, if the point is of multiplicity five the curve must be reducible. Thus, for irreducible curves, we only need to study points of multiplicity four, three, or two in the quintic case and only points of multiplicity three and two in the quartic case.

Let us now explain how all of the cases are enumerated. First we choose the tangent cone by choosing the tangent lines together with their multiplicities. The choice of tangent lines can be fixed by a linear change of coordinates. Moreover, by rotation of axes, we may assume no tangent line is vertical. For each tangent cone, we consider all possible Newton polygons. For each Newton polygon, we first consider the case where none of the quasihomogeneous polynomials corresponding to the segments of the Newton polygon have a multiple factor. Then we consider the cases where there is a multiple factor. When there is a multiple factor, the choice of this factor can be fixed by a linear change of coordinates. For irreducible quartic curves, the only case of this kind is the one with quasihomogeneous terms $(y+x^2)^2$. In this case, Maple is used to compute the Puiseux expansions for the corresponding family of curves. The different types of singular points are then determined by the vanishing or nonvanishing of certain polynomials in the coefficients of this family; these polynomials come from executing the Newton-Puiseux algorithm and are given to us by the Maple computation of the Puiseux expansion. These polynomials are usually discriminants and they are associated with the first non-unique coefficient of the Puiseux expansions. The vanishing or nonvanishing of these polynomials give what we call the "conditions" and "cases" in the following sections.

Let us now discuss the issue of verifying the existence of irreducible curves that have a given type of singular point. Observe that for a given degree, the irreducible curves form a dense open subset in the Zariski topology on the space of all curves of that degree. When a segment of the Newton polygon contains a multiple quasihomogeneous factor, we use Maple to determine the different types of singular points corresponding to that family, and in this process, a sequence of polynomial conditions on the coefficients (which turn out to be discriminants) is obtained. With respect to the Zariski topology, if an irreducible curve is found at any stage of the sequence, then all prior stages contain irreducible curves. For all quartic and quintic curves it is clear that each family at the end of a sequence of computations contains an irreducible representative.

To be more specific, the details of the following outline will be carried out in the next section. For irreducible quartic curves, by a linear change of coordinates, as described above, it suffices to consider the following families:

Multiplicity 3.

$$\begin{aligned} y^3 + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 &= 0, &\qquad a &\neq 0\\ y^2(y-x) + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 &= 0, &\qquad a &\neq 0\\ y(y-x)(y-2x) + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 &= 0, &\qquad a &\neq 0\\ y(y^2+x^2) + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 &= 0, &\qquad a &\neq 0 \end{aligned}$$

Multiplicity 2.

$$\begin{aligned} y^2 - x^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^4&\\ +fx^3y + gx^2y^2 + hxy^3 + jy^4 &= 0 \end{aligned}$$

$$\begin{aligned} y^2 + x^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^4&\\ +fx^3y + gx^2y^2 + hxy^3 + jy^4 &= 0 \end{aligned}$$

$$\begin{aligned} y^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^4&\\ +fx^3y + gx^2y^2 + hxy^3 + jy^4 &= 0 \end{aligned}$$

$$(y+x^2)(y-x^2)+ax^3y+bxy^2+cx^2y^2+dy^3+exy^3+fy^4=0$$
$$y^2+x^4+ax^3y+bxy^2+cx^2y^2+dy^3+exy^3+fy^4=0$$
$$(y+x^2)^2+ax^3y+bxy^2+cx^2y^2+dy^3+exy^3+fy^4=0$$

Maple computation is needed only for the last family above. An interesting feature of the proof occurs at the end of this computation, where we show that every curve in the family

$$(y+x^2)^2+bx^3y+bxy^2+(1/4b^2+d)x^2y^2+dy^3+1/2bdxy^3+fy^4=0$$

is reducible. This is the key to establishing that the list of double points is complete.

For reducible quartic curves, Maple computation is used to examine the cases where an irreducible cubic is tangent to the line component and where two conics share a common tangent. (The other cases are enumerated by mathematical common sense, involving simple manipulations of the Newton polygons.)

For irreducible quintic curves, by a linear change of coordinates, as described above, it suffices to consider the following cases, indicated, for each multiplicity, by choice of tangent cone, number of Newton polygons, if greater than one, and choice of multiple quasihomogeneous factors, if applicable.

Multiplicity 4. y^4, $y^3(y-x)$, $y^2(y-x)^2$, $(x^2+y^2)^2$, $y^2(y-x)(y-2x)$, $y^2(x^2+y^2)$, $y(y^2-x^2)(y-gx)$, $y(y-x)(x^2+y^2)$, and $(x^2+y^2)(x^2+4y^2)$.

Short Maple computations are required only for the 3rd and 4th cases above because of multiple quasihomogeneous factors in the Newton polygon.

Multiplicity 3. y^3 (3 Newton Polygons), $y^2(y-x)$ (4 Newton Polygons), $(y-x)(y-2x)(y-3x)$, and $(y-x)(x^2+y^2)$.

Substantial Maple computation is required only for the family $y^2(y-x)-x^5+2x^3y+ax^4y+bx^2y^2+cx^3y^2+dxy^3+ex^2y^3+fy^4+gxy^4+hy^5$.

Multiplicity 2. y^2 (5 Newton Polygons), y^2-x^2, and y^2+x^2.

Substantial Maple computation is required for the family $(y+x^2)^2+ax^5+bx^3y+cxy^2+dx^4y+ex^2y^2+fy^3+gx^3y^2+hxy^3+jx^2y^3+ky^4+lxy^4+my^5$. Some spectacular factorizations are performed during the course of that computation and are indicated in the next section.

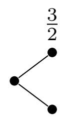
For reducible quintic curves, Maple computation is used to examine the cases where an irreducible conic is tangent to an irreducible cubic. (The other cases are enumerated by mathematical common sense, involving simple manipulations of the Newton polygons.)

Given an algebraic curve with a singular point at the origin, let us now describe how to associate a tree diagram to this singular point once we have the Puiseux expansions. Each time at least one "branch" separates, record the exponent where that happens. Place all such exponents in a row at the top. For each exponent in the top row, there corresponds a column of vertices. Each Puiseux expansion corresponds to exactly one vertex in that column, and those expansions with the same coefficients up to that exponent correspond to the same vertex. Braces will join those pairs of vertices, within a given column, that correspond to complex conjugate coefficients. In such a case, the only real solution of the original equation, satisfying the pair of expansions indicated by the braces, in a small enough neighborhood of the origin is $(0, 0)$.

In [12], C.T.C. Wall uses the term "pro-branches" to refer to the distinct Puiseux expansions belonging to a given singular point, and then defines a notion of *exponent of contact* between two pro-branches. It follows from [12, Lemma 4.1.1], that the diagram we assign to a singular point is invariant under a linear change of coordinates.

Example 1.1. $y^2 = -x^3$. Notice that $y = \pm\ i\ x^{3/2}$, which can also be written as $y = \pm(-x)^{3/2}$.

For each $x < 0$, there are two distinct real solutions for y. Hence, the diagram is as shown below (without braces!).



We start with one vertex on the left corresponding to the power zero. Line segments are drawn connecting the vertices from left to right, where each polygonal path from left to right corresponds to Puiseux expansions having the same set of coefficients up to a given exponent. The diagram stops at the first exponent where each vertex in that column corresponds to exactly one Puiseux expansion. The key point is that this tree diagram uniquely specifies the singularity type (up to permutations of vertices within columns) provided that no tangent line at the origin is vertical.

Example 1.2. $x^2y + x^4 + 2xy^2 + y^3 = 0$.

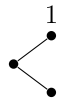
If $B := x^2y + x^4 + 2xy^2 + y^3$, the Maple command `puiseux(B,x=0,y,3)` tells us that the Puiseux expansions begin as follows:

$$y = -x + x^{3/2} \qquad \text{(branch\#1)}$$

$$y = -x - x^{3/2} \qquad \text{(branch\#2)}$$

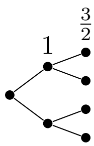
$$y = -x^2 \qquad \text{(branch\#3)}$$

In the next section, we will refer to the relevant truncated portion of the Puiseux expansion as the *Puiseux jet*. Notice that the coefficient of x in branch #1 and branch #2 is -1, while the coefficient of x in branch #3 is 0. So there is a splitting at the first power of x, which is indicated as



Next we must show the splitting of #1 from #2. Notice that the power of x at which #1 and #2 split is 3/2.

Now our diagram looks like the following:



The diagram is now complete; notice that there are three distinct vertices in the column labeled 3/2.

Let us clarify the nature of singular points corresponding to diagrams with braces. The real zero sets of x^2+y^2 and $(x^2+y^2)(x^2+4y^2)$ each consist of the origin alone, but these are to be regarded as distinct types of singular points. D. A. Gudkov established the notation A_1^* for the former and X_9^{**} for the latter. From our point of view the distinction between the two points is revealed algebraically by the Puiseux expansions. The geometric distinction between such special isolated real points is only revealed in the complex plane where the notion of exponent of contact has the meaning described earlier in this introduction.

Let us mention here some aspects of the computer algebra that is used. During the course of computing the Puiseux expansions, it is often necessary to use Maple to compute discriminants. Setting the discriminant equal to zero, computer algebra is used to solve this equation and then to substitute the result into the equation for the family of curves being treated at that stage. Sometimes these computations are too large to include in the paper (it would fill several pages), and in some cases Maple factors the discriminant so that it reduces to one or two lines of text, giving some sort of remarkable and curious algebraic identity. The end of a sequence of Puiseux expansion computations occurs when Maple factors the family of curves at that stage, again giving some sort of remarkable and curious algebraic identity. Many of these calculations can be seen in the Maple worksheets posted on the website of David Weinberg. Another feature of computer algebra that is used occurs in the section on reducible quintic curves. There, Gröbner bases are used to verify the existence or nonexistence of some singular points having diagrams with braces. More explanation of this will occur in that section.

Summary description of the equivalence relation. The geometric meaning of the equivalence relation is that two singular points are equivalent if and only if their pro-branches have the same exponents of contact. We have described a precise procedure for assigning a diagram to a singular point of an algebraic curve and this assignment is invariant under a linear change of coordinates. The diagram just codifies in the most convenient way all the information about the exponents of contact.

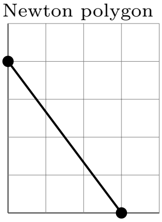
2 Classification of Singular Points for Quartic Curves.

2.1 Irreducible curves

2.1.1 Multiplicity 3

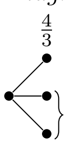
2.1.1.1 Tangent cone y^3.

Newton polygon



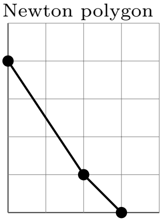
$y^3+ax^4+bx^3y+cx^2y^2+dxy^3+ey^4=0, a\neq 0$. Puiseux jets: $y=-a^{1/3}x^{4/3}$ (three expansions here).

Diagram type 1.



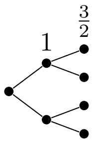
2.1.1.2 Tangent cone: $y^2(y-x)$.

Newton polygon



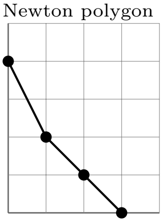
$y^2(y-x)+ax^4+bx^3y+cx^2y^2+dxy^3+ey^4$, $a\neq 0$. Puiseux jets (from Newton polygon; Maple not needed): $y=x$; $y=\pm\sqrt{a}x^{3/2}$.

Diagram type 2.



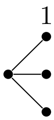
2.1.1.3 Tangent cone: $y(y-x)(y-2x)$.

Newton polygon



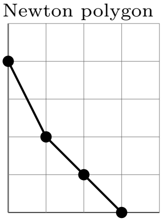
$y(y-x)(y-2x)+ax^4+bx^3y+cx^2y^2+dxy^3+ey^4=0$, $a\neq 0$. Puiseux jets: $y=0$; $y=x$; $y=2x$.

Diagram type 3.



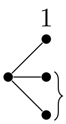
2.1.1.4 Tangent cone: $y(y^2+x^2)$.

Newton polygon



$y(y^2+x^2)+ax^4+bx^3y+cx^2y^2+dxy^3+ey^4=0$, $a\neq 0$. Puiseux jets: $y=0$; $y=\pm ix$.

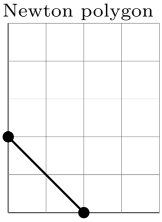
Diagram type 4.



2.1.2 Multiplicity 2.

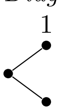
2.1.2.1 Tangent cone: $(y-x)(y+x)=y^2-x^2$.

Newton polygon



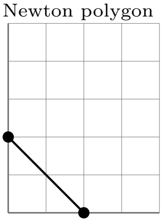
$y^2-x^2+ax^3+bx^2y+cxy^2+dy^3+ex^4+fx^3y+gx^2y^2+hxy^3+jy^4=0$. Puiseux jets: $y=x$; $y=-x$.

Diagram type 5.



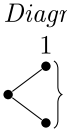
2.1.2.2 Tangent cone: y^2+x^2.

Newton polygon



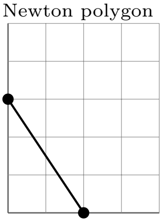
$y^2+x^2+ax^3+bx^2y+cxy^2+dy^3+ex^4+fx^3y+gx^2y^2+hxy^3+jy^4=0$. Puiseux jets: $y=\pm ix$.

Diagram type 6.

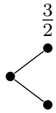


2.1.2.3 Tangent cone: y^2.

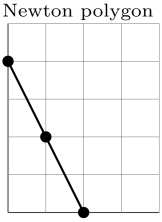
Newton polygon



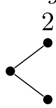
$y^2 + ax^3 + bx^2y + cxy^2 + dy^3 + ex^4 + fx^3y + gx^2y^2 + hxy^3 + jy^4 = 0$, $a \neq 0$. Puiseux jets: $y = \pm\sqrt{a}\, x^{3/2}$.
Diagram type 7.



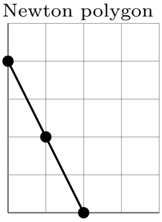
Newton polygon



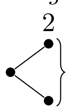
Quasihomogeneous factors: $(y+x^2)(y-x^2)$. $(y+x^2)(y-x^2)+ax^3y+bxy^2+cx^2y^2+dy^3+exy^3+fy^4 = 0$. Puiseux jets: $y = x^2$; $y = -x^2$.
Diagram type 8.



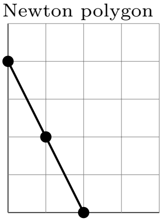
Newton polygon



Quasihomogeneous factors: $y^2 + x^4$. $y^2 + x^4 + ax^3y + bxy^2 + cx^2y^2 + dy^3 + exy^3 + fy^4 = 0$. Puiseux jets: $y = \pm ix^2$.
Diagram type 9.

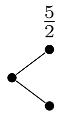


Newton polygon



Quasihomogeneous factors: $(y + x^2)^2$. $A := (y + x^2)^2 + ax^3y + bxy^2 + cx^2y^2 + dy^3 + exy^3 + fy^4 = 0$.

Notice that the quasihomogeneous polynomial $(y + x^2)^2$ has a double root. Thus, the family above contains several different types of singular points. We will determine polynomial conditions on the coefficients that will give all the different types of singular points by using Maple to compute a succession of Puiseux expansions. We begin by computing the Puiseux expansion of A using the Maple command `puiseux(A, x = 0, y, 0)`. Notice that the zero in the last argument instructs Maple to exhibit just enough of the Puiseux expansion to separate the "branches"! So, we obtain the Puiseux jets: $y = -x^2 + (a - b)^{1/2}x^{5/2}$ under the condition $a \neq b$. This gives *Diagram type 10*:



In the case $a = b$, we get the Puiseaux jets $y = -x^2 + x^3\mathrm{RootOf}(-bZ + Z^2 + c - d)$ under the condition $c \neq \frac{1}{4}b^2 + d$, and hence the two diagram types as follows:

Diagram type 11. if $b^2 - 4(c-d) > 0$

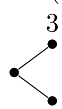
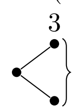
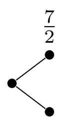


Diagram type 12. if $b^2 - 4(c-d) < 0$



In the case $c = \frac{1}{4}b^2 + d$, the Puiseux jets are $y = -x^2 + (e - \frac{1}{2}bd)^{1/2}x^{7/2}$ under the condition $e \neq \frac{1}{2}bd$, thus give rise to *Diagram type 13*:



Finally, in the case that $e = \frac{1}{2}bd$, the family has become $H := (y + x^2)^2 + bx^3y + bxy^2 + (\frac{1}{4}b^2 + d)x^2y^2 + dy^3 + \frac{1}{2}bdxy^3 + fy^4$. We cannot show that H is reducible by using the Maple command `factor(H)`. However, the Maple command `factor(H - fy`^{4}`)` shows that $H - fy^4 = \frac{1}{4}(2x^2 + 2y + bxy)(bxy + 2x^2 + 2dy^2 + 2y)$. Therefore, $H = \frac{1}{4}(2x^2 + bxy + 2y)^2 + \frac{d}{2}(2x^2 + bxy + 2y)y^2 + fy^4$, which is homogeneous in $(2x^2 + bxy + 2y)$ and y^2, and thus factors into $H = (2x^2 + y^2(d - (d^2 - 4f)^{1/2}) + bxy + 2y)(2x^2 + y^2(d + (d^2 - 4f)^{1/2}) + bxy + 2y)$.

This completes the classification of singular point types for irreducible real quartic curves.

2.2 Reducible Curves

2.2.1 Degrees of factors: $3, 1$.

If the straight line does not pass through $(0, 0)$, then there are three cases:

Diagram type 1.

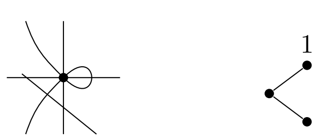


Diagram type 2.

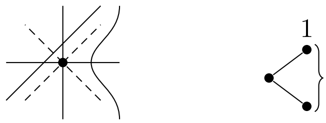
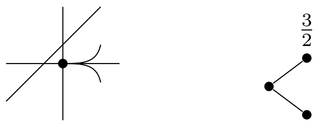


Diagram type 3.



If the straight line does pass through $(0, 0)$, then there are five cases:

Diagram type 4.

Diagram type 5.

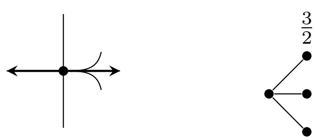
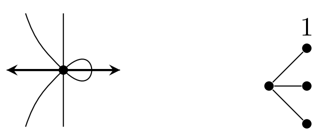


Diagram type 6.

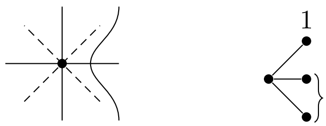


Consider the family $(y^2-x^2+ax^3+bx^2y+cxy^2+dy^3)(y-x) = 0$. By using Maple, we obtain the Puiseux jets $y = x+0x^2$, $y = x+\frac{1}{2}(-a-b-c-d)x^2$, $y = -x$ under the condition $a+b+c+d \neq 0$. This yields

Diagram type 7

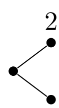
In the case $a + b + c + d = 0$, the cubic is reducible, so we are done.

Diagram type 8



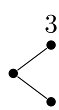
Consider the family $(y - x + ax^2 + bxy + cy^2 + dx^3 + ex^2y + fxy^2 + gy^3)(y - x) = 0$. By using Maple, we obtain the Puiseux jets: $y = x + 0x^2$, and $y = x + (-a - b - c)x^2$ under the condition $a + b + c \neq 0$.

Diagram type 9.



In the case $a = -b - c$, we get the Puiseux jets $y = x + 0x^3$ and $y = x + (-f - g - e - d)x^3$ under the condiion $f + g + e + d \neq 0$.

Diagram type 10.

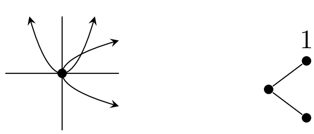


In the case $f + g + e + d = 0$, the cubic is reducible, so we are done.

2.2.2 Degrees of factors: $2, 2$.

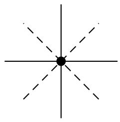
The first case is that we have two distinct tangents at $(0, 0)$.

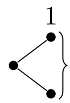
Diagram type 1.



This corresponds to $(y+ix+ix^2)(y-ix-ix^2) = y^2+x^2+2x^3+x^4 = y^2+x^2(1+x)^2$.

Diagram type 2.





The second case is that two tangents coincide at $(0,0)$, i.e., $(y+ax^2+bxy+cy^2)(y+dx^2+exy+fy^2)=0$. Notice that $a \neq 0$ and $d \neq 0$. Calculation using Maple yields the Puiseux jets $y=-ax^2$ and $y=-dx^2$ under the condition $a \neq d$ with the following diagrams:

Diagram type 9.

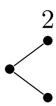
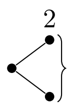


Diagram type 11.



In the case $a=d$, we get the Puiseux jets $y=-dx^2+edx^3$ and $y=-dx^2+bdx^3$ under the condition $b \neq e$:

Diagram type 10.

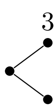
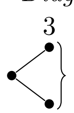


Diagram type 12.



In the case $b=e$, we get the Puiseux expansions $y=-dx^2+edx^3+(-e^2d-cd^2)x^4$ and $y=-dx^2+edx^3+(-e^2d-fd^2)x^4$, under the condition: $c \neq f$:

Diagram type 13.

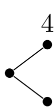
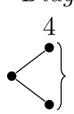


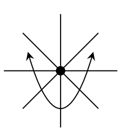
Diagram type 14.



If $c=f$, then we just have a curve of degree 2 with multiplicity 2.

2.2.3 Degrees of factors: $2,1,1$.

In the following list, the geometric situation on the left hand side yields the diagram on the right hand side.



or

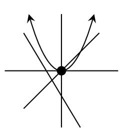
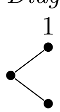


Diagram type 1



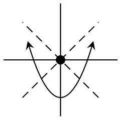
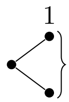


Diagram type 2



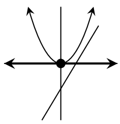
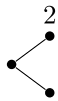


Diagram type 9



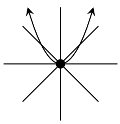
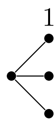


Diagram type 6



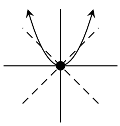
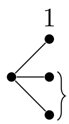


Diagram type 8



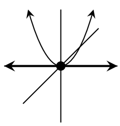
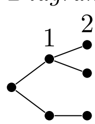


Diagram type 7



2.2.4 Degrees of factors: $1, 1, 1, 1$.

Again, the geometric situation on the left hand side yields the diagram on the right hand side.

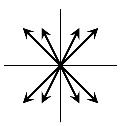
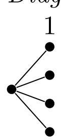


Diagram type 15



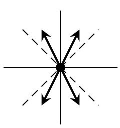
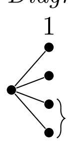


Diagram type 16



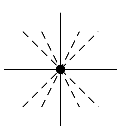
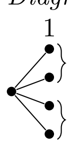


Diagram type 17



This completes the classification of singular point types for reducible real quartic curves.

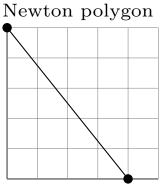
3 Classification of Singular Points of Quintic Curves

3.1 Irreducible curves

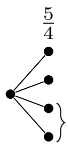
3.1.1 Multiplicity 4.

3.1.1.1 Tangent cone: y^4.

Newton polygon

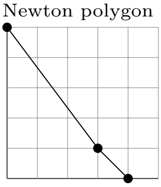


$y^4+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5=0$, $a\neq 0$. Puiseux jets: $y=(-a)^{1/4}x^{5/4}$.
Diagram type 1.

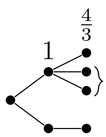


3.1.1.2 Tangent cone: $y^3(y-x)$.

Newton polygon

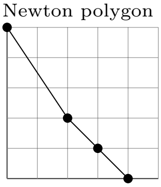


$y^3(y-x)+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5=0, a\neq 0$. Puiseux jets: $y=x$ and $y=(-a)^{1/3}x^{4/3}$.
Diagram type 2.

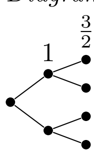


3.1.1.3 Tangent cone: $y^2(y-x)^2$

Newton polygon



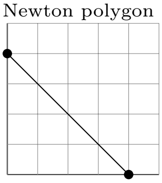
$A=y^2(y-x)^2+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5=0, a\neq 0$. The Puiseux expansion is computed by using the Maple command `puiseux(A,x=0,y,0)`. Puiseux jets: $y=(-a)^{1/2}x^{3/2}$ and $y=x-(a+b+c+d+e+f)^{1/2}x^{3/2}$, under the condition $a+b+c+d+e+f\neq 0$.
Diagram type 3.



If $a+b+c+d+e+f=0$, then each curve of the form A is reducible.

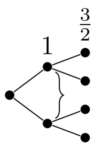
3.1.1.4 Tangent cone: $(x^2 + y^2)^2$.

Newton polygon



$B = (x^2+y^2)^2+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5 = 0$ under the conditions $a-c+e \neq 0$ or $b+f-d \neq 0$. Using Maple, we obtain the Puiseux jets $y = \mathrm{RootOf}(Z^2+1)x + (a + b\mathrm{RootOf}(Z^2 + 1) + f\mathrm{RootOf}(Z^2 + 1) - c + e - d\mathrm{RootOf}(Z^2 + 1))^{1/2}x^{3/2}$. (Note: these are four expansions.)

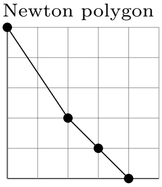
Diagram type 4.



If $a - c + e = 0$ and $b + f - d = 0$, then, using Maple, each curve in B is reducible.

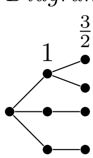
3.1.1.5 Tangent cone: $y^2(y - x)(y - 2x)$.

Newton polygon



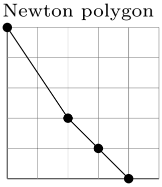
$y^2(y-x)(y-2x)+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5 = 0, a \neq 0$. Puiseux jets: $y = x$, $y = 2x$ and $y = (-\frac{1}{2}a)^{1/2}x^{3/2}$.

Diagram type 5.



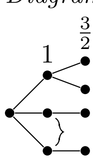
3.1.1.6 Tangent cone: $y^2(x^2 + y^2)$.

Newton polygon



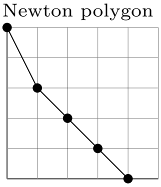
$y^2(x^2+y^2)+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5 = 0$, $a \neq 0$. Puiseux jets: $y = \pm ix$ and $y = (-a)^{1/2}x^{3/2}$.

Diagram type 6.



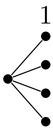
3.1.1.7 Tangent cone: $y(y^2-x^2)(y-gx)$.

Newton polygon



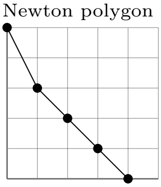
$y(y^2-x^2)(y-gx)+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5=0$, $a\neq 0$, $g\neq 0,\pm 1$. Puiseux jets: $y=0x$, $y=x$, $y=-x$, and $y=gx$.

Diagram type 7.



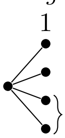
3.1.1.8 Tangent cone: $y(y-x)(x^2+y^2)$.

Newton polygon



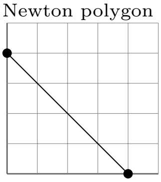
$y(y-x)(x^2+y^2)+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5=0$, $a\neq 0$. Puiseux jets: $y=0x$, $y=x$, and $y=\pm ix$.

Diagram type 8.



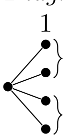
3.1.1.9 Tangent cone: $(x^2+y^2)(x^2+4y^2)$.

Newton polygon



$(x^2+y^2)(x^2+4y^2)+ax^5+bx^4y+cx^3y^2+dx^2y^3+exy^4+fy^5=0$. Puiseux jets: $y=\pm ix$ and $y=\pm 2ix$.

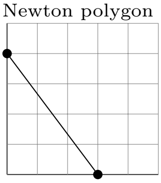
Diagram type 9.



3.1.2 Multiplicity 3.

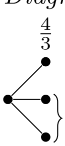
3.1.2.1 Tangent cone: y^3.

Newton polygon

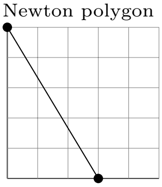


$y^3+ax^4+bx^3y+cx^2y^2+dxy^3+ey^4+fx^5+gx^4y+hx^3y^2+jx^2y^3+kxy^4+ly^5=0$, $a\neq 0$. Puiseux jet: $y=(-a)^{1/3}x^{4/3}$.

Diagram type 10.

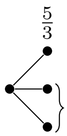


Newton polygon

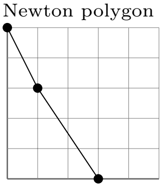


$y^3 + ax^5 + bx^2y^2 + cxy^3 + dy^4 + ex^4y + fx^3y^2 + gx^2y^3 + hxy^4 + jy^5 = 0$, $a \neq 0$. Puiseux jet: $y(-a)^{1/3}x^{5/3}$.

Diagram type 11.

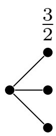


Newton polygon



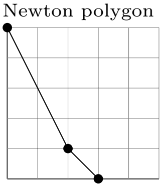
$x^5 - x^3y + y^3 + ax^4y + bx^2y^2 + cx^3y^2 + dxy^3 + ex^2y^3 + fy^4 + gxy^4 + hy^5 = 0$. Puiseux jets: $y = 0x^{3/2}$ and $y = x^{3/2}$.

Diagram type 12.



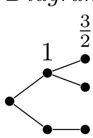
3.1.2.2 Tangent cone: $y^2(y - x)$.

Newton polygon

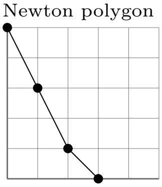


$y^2(y - x) + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + fx^5 + gx^4y + hx^3y^2 + jx^2y^3 + kxy^4 + ly^5 = 0$, $a \neq 0$. Puiseux jets: $y = x$ and $y = a^{1/2}x^{3/2}$.

Diagram type 13.

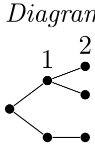


Newton polygon

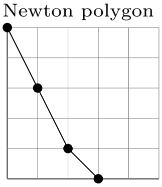


Quasihomogeneous factors: $x(x^2 - y)(x^2 + y)$. $y^2(y - x) + x^5 + ax^4y + bx^2y^2 + cx^3y^2 + dxy^3 + ex^2y^3 + fy^4 + gxy^4 + hy^5 = 0$. Puiseux jets: $y = x$, $y = x^2$, and $y = -x^2$.

Diagram type 14.

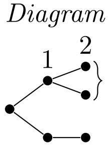


Newton polygon

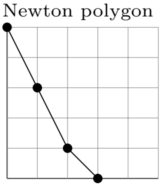


Quasihomogeneous factors: $x(-y^2 - x^4)$. $y^2(y - x) - x^5 + ax^4y + bx^2y^2 + cx^3y^2 + dxy^3 + ex^2y^3 + fy^4 + gxy^4 + hy^5 = 0$. Puiseux jets: $y = x$ and $y = \pm ix^2$.

Diagram type 15.

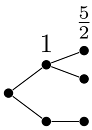


Newton polygon

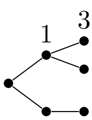
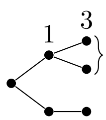


Quasihomogeneous factors: $-x(y-x^2)^2$. $y^2(y-x) - x^5 + 2x^3y + ax^4y + bx^2y^2 + cx^3y^2 + dxy^3 + ex^2y^3 + fy^4 + gxy^4 + hy^5 = 0$.

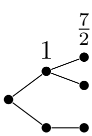
Now we use Maple to compute Puiseux expansions. Under the condition $a+b+1 \neq 0$, we obtain the Puiseux jets $y = x$ and $y = x^2 + (a+b+1)^{1/2}x^{5/2}$, giving *Diagram type 16*:



In the case $b = -a-1$ under the condition $d \neq -\frac{1}{4}(1-2a+a^2+4c)$, we get the Puiseaux jets $y = x$ and $y = x^2 + x^3\mathrm{RootOf}(Z^2 + (a-1)Z - d - c)$, and hence the two diagram types as follows:

Diagram type 17.	*Diagram type 18.*
if $1-2a+a^2+4d+4c > 0$	if $1-2a+a^2+4d+4c < 0$
	

In the case $d = -\frac{1}{4}(1-2a+a^2+4c)$ under the condition $\frac{1}{8} - \frac{1}{8}a - \frac{1}{8}a^2 - \frac{1}{2}c + \frac{1}{2}ac + \frac{1}{8}a^3 + e + f \neq 0$, we get the Puiseux jets $y = x$ and $y = x^2 + x^3(\frac{1}{2} - \frac{1}{2}a) + (\frac{1}{8} - \frac{1}{8}a - \frac{1}{8}a^2 - \frac{1}{2}c + \frac{1}{2}ac + \frac{1}{8}a^3 + e + f)^{1/2}x^{7/2}$, giving *Diagram type 19*:



In the case $f = -(\frac{1}{8} - \frac{1}{8}a - \frac{1}{8}a^2 - \frac{1}{2}c + \frac{1}{2}ac + \frac{1}{8}a^3 + e)$, we define

$$\begin{aligned} D_1 =& 256c^2 - 128c + 384ca^2 - 256ca + 80 - 32a^2 \\ & - 64a + 80a^4 - 64a^3 - 512e + 1024g + 512ae. \end{aligned}$$

Under the condition that $D_1 \neq 0$, we get the Puiseux jets $y = x$ and $y = x^2 + x^3(\frac{1}{2} - \frac{1}{2}a) + x^4\mathrm{RootOf}(16Z^2 + (16c - 20 - 4a^2 + 24a)Z + 8e - 14a - 16g - 8ca^2 - a^4 - 8c + 5 + 16ac - 2a^3 + 12a^2 - 8ea)$, giving the following two diagram types:

Diagram type 20. if $D_1 > 0$

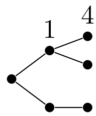
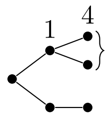


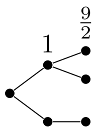
Diagram type 21. if $D_1 < 0$



If now $D_1 = 0$, we define

$$D_2 = \frac{3}{32} - \frac{7a}{64} - \frac{3e}{8} - \frac{c}{8} + h - \frac{a^3}{32} - \frac{ca}{8} + \frac{ca^2}{8} + \frac{ea}{4} + \frac{a^4}{32} + \frac{a^3c}{8} + \frac{a^2e}{8} + \frac{c^2a}{4} + \frac{ce}{2} + \frac{a^5}{64}.$$

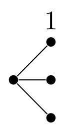
If $D_2 \neq 0$, we get the Puiseux jets $y = x$ and $y = x^2 + (\frac{1}{2} - \frac{1}{2}a)x^3 + (\frac{a^2}{3} + \frac{5}{8} - \frac{c}{2} - \frac{3}{4}a)x^4 + {D_2}^{1/2}x^{9/2}$, and *Diagram type 22*:



Finally, if $D_2 = 0$, the Maple command `factor` applied to the resulting family shows that each curve in the family is reducible. Thus, we are done analyzing the tangent cone $y^2(y - x)$.

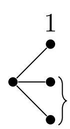
3.1.2.3 Tangent cone: $(y - x)(y - 2x)(y - 3x)$.

Thus, $(y-x)(y-2x)(y-3x) + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + fx^5 + gx^4y + hx^3y^2 + jx^2y^3 + kxy^4 + ly^5 = 0$, and we get the Puiseux jets $y = x$, $y = 2x$, and $y = 3x$, and hence *Diagram type 23*:



3.1.2.4 Tangent cone: $(y - x)(x^2 + y^2)$.

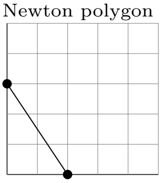
Thus $(y-x)(x^2+y^2) + ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 + fx^5 + gx^4y + hx^3y^2 + jx^2y^3 + kxy^4 + ly^5 = 0$. We get the Puiseux jets $y = x$ and $y = \pm ix$, and consequently *Diagram type 24*:



3.1.3 Multiplicity 2

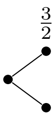
3.1.3.1 Tangent cone: y^2.

Newton polygon

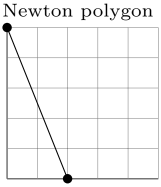


$y^2+ax^3+bx^2y+cxy^2+dy^3+ex^4+fx^3y+gx^2y^2+hxy^3+jy^4+kx^5+lx^4y+mx^3y^2+nx^2y^3+pxy^4+qy^5=0$, $a \neq 0$. Puiseux jets: $y=(-a)^{1/2}x^{3/2}$.

Diagram type 25.

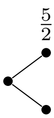


Newton polygon

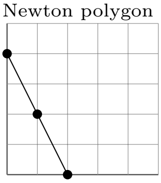


$y^2-x^5+axy^2+by^3+cx^3y+dx^2y^2+exy^3+fy^4+gx^4y+hx^3y^2+jx^2y^3+kxy^4+ly^5=0$. Puiseux jets: $y=\pm x^{5/2}$.

Diagram type 26.

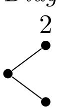


Newton polygon

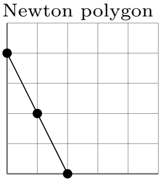


Quasihomogeneous factors: $(y-x^2)(y+x^2)$. $(y-x^2)(y+x^2)+ax^5+bx^3y+cxy^2+dx^4y+ex^2y^2+fy^3+gx^3y^2+hxy^3+jx^2y^3+ky^4+lxy^4+my^5=0$. Puiseux jets: $y=\pm x^2$.

Diagram type 27.

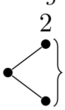


Newton polygon

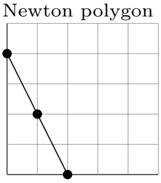


Quasihomogeneous factors: y^2+x^4. $y^2+x^4+ax^5+bx^3y+cxy^2+dx^4y+ex^2y^2+fy^3+gx^3y^2+hxy^3+jx^2y^3+ky^4+lxy^4+my^5=0$. Puiseux jets: $y=\pm ix^2$.

Diagram type 28.

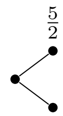


Newton polygon



Quasihomogeneous factors: $(y+x^2)^2$. In order to find all remaining singular points corresponding to the tangent cone y^2, we should consider the family $(y+x^2)^2+ax^5+bx^3y+cxy^2+dx^4y+ex^2y^2+fy^3+gx^3y^2+hxy^3+jx^2y^3+ky^4+lxy^4+my^5=0$.

We now perform a sequence of Maple calculations of Puiseux expansions for this family. Under the condition $b-a-c \neq 0$, we get the Puiseux jets $y=-x^2+(b-a-c)^{1/2}x^{5/2}$, giving *Diagram type 26*:



In the case $b=a+c$ under the condition $a^2-2ac+c^2-4e+4f+4d \neq 0$, we get the Puiseux jets $y=-x^2+x^3\mathrm{RootOf}(Z^2+(a-c)Z+e-f-d)$, corresponding to the following two diagram types:

Diagram type 29.
if $a^2-2ac+c^2-4e+4f+4d>0$

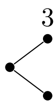
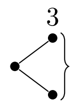
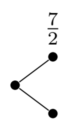


Diagram type 30.
if $a^2-2ac+c^2-4e+4f+4d<0$



In the case $e=1/4(a^2-2ac+c^2+4f+4d)$ under condition $g \neq \frac{a^2c}{2}-\frac{ac^2}{4}-\frac{a^3}{4}+\frac{fa}{2}-\frac{fc}{2}-\frac{da}{2}+\frac{dc}{2}+h$, we getthe Puiseux jets $y=-x^2+x^3(-\frac{a}{2}+\frac{c}{2})+(\frac{a^2c}{2}-\frac{ac^2}{4}-\frac{a^3}{4}+\frac{fa}{2}-\frac{da}{2}+\frac{dc}{2}-g+h)^{1/2}x^{7/2}$, giving *Diagram type 31*:



In the case $g=\frac{a^2c}{2}-\frac{ac^2}{4}-\frac{a^3}{4}+\frac{fa}{2}-\frac{fc}{2}-\frac{da}{2}+\frac{dc}{2}+h$, we define

$$\begin{aligned} D_1 = & 256f^2+256fc^2-256fa^2-512df+256c^2a^2+256a^4+512da^2 \\ & +256d^2-512dca+1024j-512hc+512ha-512a^3c-1024k. \end{aligned}$$

Under the condition $D_1 \neq 0$, we get the Puiseux jets $y=-x^2+x^3(-\frac{a}{2}+\frac{c}{2})+x^4\mathrm{RootOf}(16Z^2+(16f+8c^2-8a^2-16d)Z+8dca-16j-4dc^2+8hc-6c^2a^2+16k-8ha-3a^4+c^4-4da^2+8a^3c)$, corresponding to the following two diagram types:

Diagram type 32.
if $D_1>0$

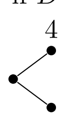
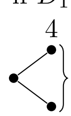


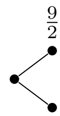
Diagram type 33.
if $D_1<0$



If now $D_1 = 0$ (solved for k), we define

$$\begin{aligned} D_2 =& -4adf + 8l + 4dc^2a + 4fca^2 - 4dca^2 - 4jc - 4hca - fc^3 + 2cf^2 \\ & - 2c^3a^2 - 2cd^2 + 2af^2 + 2c^2a^3 + 2ad^2 + 2hc^2 - 4hf + 4hd. \end{aligned}$$

Under the condition $D_2 \neq 0$, we get the Puiseux jets (in parametric form) $x = -\frac{1}{8}D_2T^2$ and $y = -\frac{1}{64}{D_2}^2T^4 - \frac{1}{1024}c{D_2}^3T^6 - \frac{1}{16384}(c^2 + 2f + 2ca - 2d){D_2}^4T^8 - \frac{1}{32768}{D_2}^5T^9$ giving *Diagram type 34*:



In the case $D_2 = 0$ (solved for l), we continue and define

$$\begin{aligned} D_3 =& 20480c^2a^2d + 16384m - 4096fc^3a - 8192fc^2a^2 + 8192hc^2a \\ & + 4096fc^2d - 8192hcd - 16384d^2ca - 8192jca + 4096a^2d^2 \\ & + 4096c^2a^4 + 4096h^2 + 4096a^2f^2 + 8192adh - 8192ca^2h \\ & - 8192haf + 4096d^3 - 1024f^2c^2 + 4096f^2d + 1024d^2c^2 - 8192d^2f \\ & - 8192jf + 8192jd + 1024c^4a^2 - 8192c^3a^3 + 16384fcad \\ & - 2048dc^3a - 8192a^3dc - 8192a^2df + 8192ca^3f. \end{aligned}$$

Under the condition $D_3 \neq 0$, we get the Puiseux jets $y = -x^2 + x^3(\frac{1}{2}c) + x^4(-\frac{1}{4}c^2 - \frac{1}{2}f - \frac{1}{2}ca + \frac{1}{2}d) + x^5\mathrm{RootOf}(64Z^2 + (64dc + 64ad - 64ca^2 - 64ac^2 + 64h - 64af - 128fc - 16c^3)Z + (-144c^2a^2d - 64m + 88fc^3a + 128fc^2a^2 - 64hc^2a - 80fc^2d + 64hcd - 64hcf + 96d^2ca + 32jca + 64f^2ca - 16d^3 + c^6 + 68f^2c^2 - 16f^2d + 16fc^4 + 12d^2c^2 + 32d^2f - 8hc^3 + 32jf - 32jd + 20c^4a^2 + 64c^3a^3 + 8ac^5 - 8dc^4 - 160fcad - 32dc^3a))$, corresponding to the following two diagram types:

Diagram type 35. if $D_3 > 0$

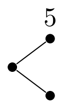
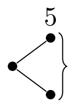


Diagram type 36. if $D_3 < 0$

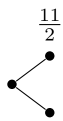


Now let $D_3 = 0$ (solved for m) and define

$$\begin{aligned} D_4 =& 72c^2a^3d + 8afc^2d - 56ahcd + 24ahcf + 8a^3d^2 + 8c^2a^5 + 8ah^2 \\ & + 8a^3f^2 + 14c^4a^3 - 28c^3a^4 - a^2c^5 - 12ch^2 - 3f^2c^3 - d^2c^3 \\ & + 72fca^2d - 8fc^3a^2 - 40fc^2a^3 + 40hc^2a^2 - 60d^2ca^2 - 12f^2ca^2 \\ & + 16a^2dh - 16ca^3h - 16ha^2f - 36dc^3a^2 - 16a^4dc - 16a^3df \end{aligned}$$

$$
\begin{aligned}
&+ 16ca^4f - 4fc^3d + 8hc^2d + 12hc^2f - 14af^2c^2 + 4afc^4 + 30ad^2c^2 \\
&- 8ahc^3 + 2adc^4 - 32ad^2f + 8dcf^2 + 16daf^2 - 16dhf - 8cd^3 \\
&+ 16ad^3 + 16hd^3 + 16jh - 16jaf + 16jad - 16jca^2 - 8jfc - 8jdc \\
&+ 8jac^2.
\end{aligned}
$$

Under the condition $D_4 \neq 0$, we get the Puiseux jets (in parametric form) $x = -\frac{1}{32}D_4T^2$ and $y = -\frac{1}{1024}D - 2^2T^4 - \frac{1}{65536}cD_4{}^3T^6 - \frac{1}{4194304}(c^2 + 2f + 2ca - 2d)D_4{}^4T^8 - \frac{1}{268435456}(4af - 4ad + 4ca^2 + 8fc - 4dc + ac^2 + c^3 - 4h)D_4{}^5T^{10} + \frac{1}{1073741824}D_4{}^6T^{11}$, giving *Diagram type 37*:



In the case $D_4 = 0$ (solved for j), we define

$$D_5 = 1024(4a^2 - 28ca + c^2 + 24d - 8f)(-2h + 2af - 2ad + 2ca^2 + fc + dc - ac^2)^2.$$

(This is the discriminant of the quadratic polynomial exhibited in the coefficient of x^6 in the Puiseux jet immediately below. It is interesting that Maple factored this discriminant.) Unter the condition $D_5 \neq 0$, we get the Puiseux jets $y = -x^2 + \frac{1}{2}cx^3 + (-\frac{1}{4}c^2 - \frac{1}{2}f - \frac{1}{2}ca + \frac{1}{2}d)x^4 + (\frac{1}{2}af - \frac{1}{2}ad + \frac{1}{2}ca^2 + fc - \frac{1}{2}dc + \frac{1}{2}ac^2 + \frac{1}{8}c^3 - \frac{1}{2}h)x^5 + x^6\mathrm{RootOf}(256Z^2 + (128a^3c - 160dc^2 + 1024fca - 896dca - 128a^2d - 128ah + 640c^2a^2 + 160ac^3 + 608fc^2 - 448hc - 512df + 128a^2f + 32c^4 256d^2 + 256f^2)Z + (64d^4 + c^8 + 10ac^7 - 160a62d^3 + 256fac^5 - 224fdc^4 + 224h^2ca - 480hc^3a^2 - 128hcd^2 - 512hc^2a^3 - 256had^2 - 152hac^4 + 144hdc^3 - 576dc^4a^2 + 440d^2c^3a - 104dc^5a - 736a^4c^2d + 608a^3cd^2 - 864a^3c^3d + 832d^2c^2a^2 - 352d^3ca - 28hc^5 + 192h^2c^2 - 96h^2d - 104d^3c^2 - 10dc^6 + 40d^2c^4 + 288a^5c^3 + 240a^3c^5 + 320a^4c^4 + 64f^4 + 32fh^2 + 96a^2f^3 - 256fd^3 + 38fc^6 + 312f^3c^2 - 256f^3d + 376f^2c^4 + 384d^2f^2 + 1304f^2c^3a + 1600f^2c^2a^2 - 696f^2c^2d - 256hcf^2 + 544f^3ca - 128haf^2 - 352a^2df^2 + 480ca^3f^2 + 416fa^2d^2 + 672fc^2a^4 + 424fd^2c^2 - 528fhc^3 + 1040fc^4a^2 + 1440fc^3a^3 - 1568f^2cad - 2560fc^2a^2d - 1120fhc^2a + 512fhcd + 1376fd^2ca + 384fadh - 640fca^2h - 1456fdc^3a - 1088fa^3dc + 608hdc^2a + 768hdca^2 + 64c^6a^2))$, giving the following two types of diagrams:

Diagram type 38. if $D_5 > 0$

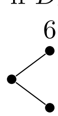
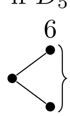


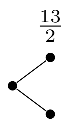
Diagram type 39. if $D_5 < 0$



In the case $h = \frac{1}{2}(2af - 2ad + 2ca^2 + fc + dc - ac^2)$, the `factor` command in Maple tells us that each curve in the resulting family is reducible; in fact, the family becomes $\frac{1}{4}(dy + 1 - acy + ax)(acy^2 + 2x^2 + cxy + 2y + fy^2 - dy^2)^2$. In the case $f = \frac{1}{8}(4a^2 - 28ac + c^2 + 24d)$, we define

$$
\begin{aligned}
D_6 = {} & 512a^9 + 12288c^3a^6 - 106176c^4a^5 - 122592c^5a^4 - 31936c^6a^3 \\
& + 6144h^2a^3 + 768h^2c^3 + 32768d^3a^3 + 32768d^3c^3 - 48c^6h \\
& + 3360c^7a^2 + 96c^7d - 102c^8a + 24576a^5d^2 + 24576c^2a^7 \\
& + 3072d^2c^5 + 340608c^4a^3d + 16896c^2a^5d - 301056c^2a^3d^2 \\
& - 316416c^3a^2d + 97536c^5a^2d + 24576h^2ad - 27648h^2ca^2 \\
& + 24576h^2dc - 26112h^2ac^2 + 98304d^3ca^2 + 98304d^3ac^2 \\
& - 36096hc^2a^4 - 52034hc^4a^2 + 3264hc^5a - 49152hc^2d^2 \\
& - 3072hc^4d - 61440d^2ca^4 + 27648ca^5h - 24576a^4dh \\
& - 49152a^2d^2h - 6432c^6ad - 49152a^6dc + 308736c^3a^4d \\
& - 4096h^3 + c^9 + 215040c^2a^2dh - 98304d^2c^4a + 101376hc^3ad \\
& + 86016hcda^3 - 98304hcd^2a - 118272c^3a^3h + 6144da^7 - 3072a^6h \\
& - 6912a^8c.
\end{aligned}
$$

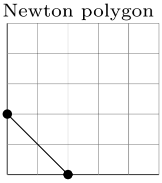
Under the condition $D_6 \neq 0$, we get the Puiseux jets $x = \frac{1}{32768}D_6T^2$ and $y = -\frac{1}{1073741824}{D_6}^2T^4 + \frac{1}{70368744177664}c{D_6}^3T^6 - \frac{1}{18446744073709551616}(4a^2 + 16d - 20ca + 5c^2){D_6}^4T^8 + \frac{1}{604462909807314587353088}(-8h + 4a^3 + 16ad - 12ca^2 + 40dc - 47ac^2 + 4c^3)D - 6^5T^{10} - \frac{1}{39614081257132168796771975168}(8a^4 + 48a^2 + 64d^2 - 44a^3c - 88dca + 34c^2a^2 + 112dc^2 - 129ac^3 + 7c^4 - 8ah - 28hc){D_6}^6T^{12} + \frac{1}{40564819207303340847894502572032}{D_6}^7T^{13}$, giving *Diagram type 40*:



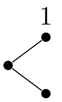
In the case: $D_6 = 0$, the `factor` command in Maple tells us that each curve in the resulting family is reducible; in fact, the family becomes $\frac{1}{256}(dy + 1 - acy + ax)(16x^2 + 16y + 4a^2y^2 + 16dy^2 + 8cxy - 20acy^2 + c^2y^2)^2$. This completes the case of the quasihomogeneous factors $(y + x^2)^2$.

3.1.3.2 Tangent cone: $y^2 - x^2$.

Newton polygon

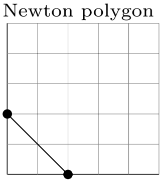


$y^2 - x^2 +$ higher terms $= 0$. Puiseux jets: $y = \pm x$.
Diagram type 41.

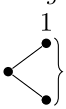


3.1.3.3 Tangent cone: $y^2 + x^2$.

Newton polygon



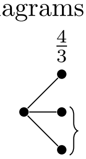
$y^2 + x^2 +$ higher terms $= 0$. Puiseux jets: $y = \pm ix$.
Diagram type 42.

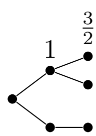


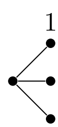
3.2 Reducible Curves

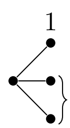
3.2.1 Factor of degree one

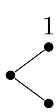
If the reducible curve has a factor of degree one, either the line component passes through the origin or it doesn't. If it does not pass through the origin, then it is only necessary to list the diagrams from the degree four case. If the line component does pass through the origin, then, by careful scrutiny of the Newton polygon, all the corresponding diagrams can easily be obtained by modifying the diagrams in the preceding case. We now list the diagrams in this case of a factor of degree one.

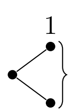
1. 

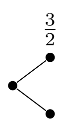
2. 

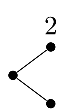
3. 

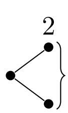
4. 

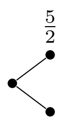
5. 

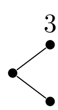
6. 

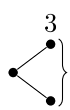
7. 

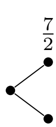
8. 

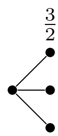
9. 

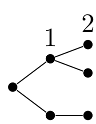
10. 

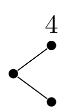
11. 

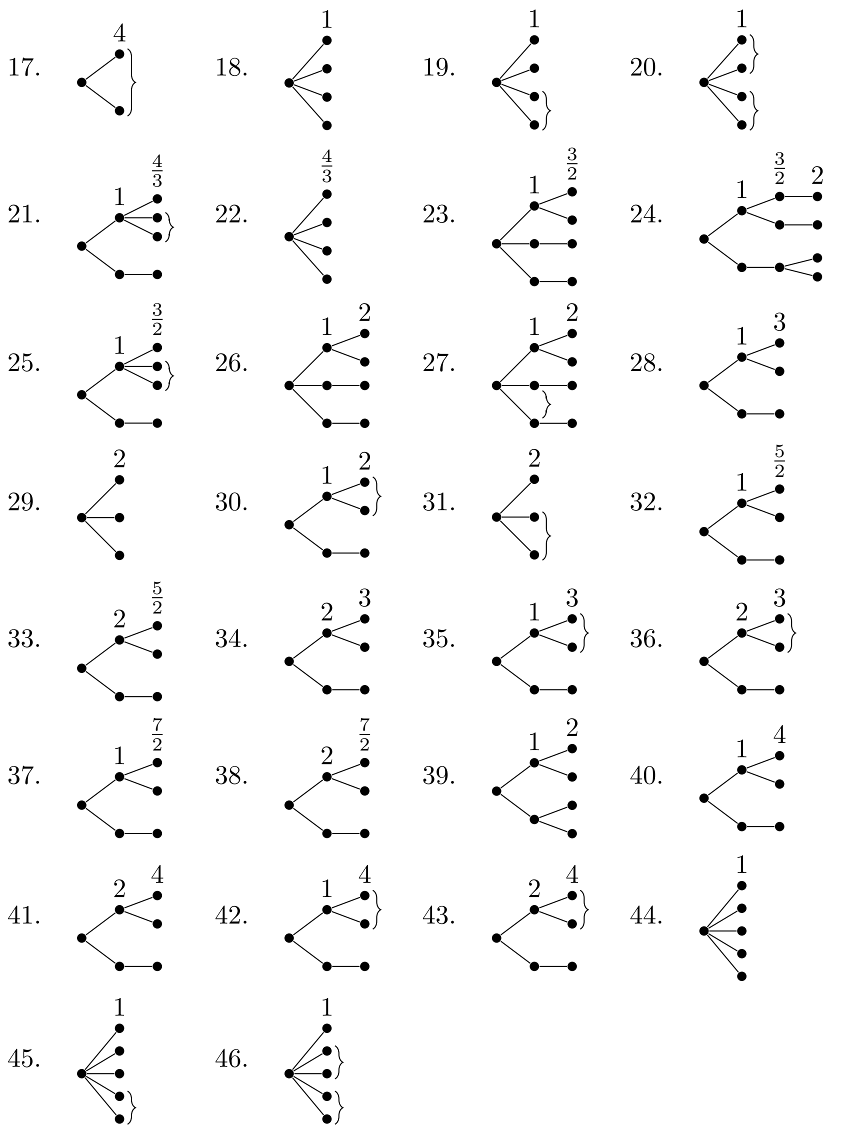
12. 

13. 

14. 

15. 

16. 



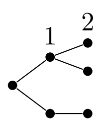
3.2.2 Factor of degree 2

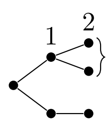
We may assume that the factors of degrees two and three are each irreducible; otherwise there is a factor of degree one, and we are in the previous case. If the factor of degree three has a cusp or acnode at the origin, we obtain nothing new. Furthermore, we only need consider the case where the tangent line to the conic factor agrees with a tangent line to the cubic factor.

Some further argument is needed to justify the existence of *complex-conjugate-type* singular points (singular points whose diagrams contain braces). The Maple computations below do not reveal whether these types of singular points exist. Real and imaginary parts with literal coefficients are inserted for each coefficient in each factor of the family of reducible curves where each family consists of two or more irreducible components. This is expanded and simplified by computer algebra. A system of equations is formed by setting equal to zero all coefficients in the imaginary part. Gröbner bases are used to find conditions under which this system has a solution. These conditions are substituted (by computer algebra) into the original family (having the real and imaginary parts). Puiseux expansions are recomputed. It is a very beautiful feature of the computation that each time the discriminant is negative (Maple returns a factorization of the discriminant into even powers, the whole expression having a minus sign in front.) The calculation ends when the Maple factorization of the equation of the family of curves has a multiple factor. Unfortunately, these calculations are too lengthy to include in this paper. There are interesting phenomena to observe here. Not every diagram without braces is accompanied by the corresponding diagram with braces.

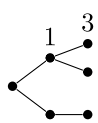
If the cubic has a crunode at the origin, consider the family $(y(y-x)+ax^3+bx^2y+cxy^2+dy^3)(y+ex^2+fxy+gy^2)=0$. We will now use Maple to calculate Puiseux expansions.

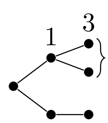
Under the condition $a \neq -e$, we get the Puiseux jets $y = x$, $y = ax^2$, and $y = -ex^2$ corresponding to diagrams 15. and 30. above:





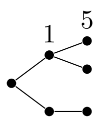
In the case $a = -e$ and under the condition $f \neq e-b$, we get the Puiseux jets $y = x$, $y = -ex^2 + efx^3$, and $y = -ex^2 + (e^2 - be)x^3$, corresponding to diagrams 28. and 35. in the list above:





In the case $f = e - b$ under the condition $b \neq e - c - g$, we get the Puiseux jets $y = x$, $y = -ex^2 + (e^2 - be)x^3 + (3be^2 - eb^2 + ce^2 - 2e^3)x^4$, and $y = -ex^2 + (e^2 - be)x^3 + (-e^3 - e^2g + 2be^2 - eb^2)x^4$, giving diagrams 40. and 42. in the above list:

In the case $b = e - c - g$ under the condition $g \neq d$, we get the Puiseux jets $y = x$, $y = -ex^2 + (ce + eg)x^3 + (-e^2g - eg^2 - 2ecg - ec^2)x^4 + (3e^2cg + 3eg^2c + 3egc^2 + ec^3 + 3e^2g^2 + eg^3)x^5$, and $y = -ex^2 + (ce + eg)x^3 + (-e^2g - eg^2 - 2ecg - ec^2)x^4 + (e^3g - e^3d + 3e^2g^2 + 3e^2cg + eg^3 + 3eg^2c + 3egc^2 + ec^3)x^5$, giving diagram 47.:



If $g = d$, then each curve in the resulting family has a linear factor.

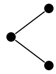
The final case is where the cubic has a simple point at the origin with tangent $y = 0$. $y = (ax^2 + bxy + cy^2)(y + dx^2 + exy + fy^2 + gx^3 + hx^2y + jxy^2 + ky^3) = 0$. Under the condition $a \neq d$, we get the Puiseux jets $y = -dx^2$ and $y = -ax^2$, giving diagrams 8. and 9.:

In the case $a = d$ under the condition $g \neq de - db$, we get the Puiseux jets $y = -dx^2 + (de - g)x^3$ and $y = -dx^2 + dbx^3$, corresponding to diagrams 11. and 12.:

In the case $g = de - db$ under the condition $h \neq be + fd - dc - b^2$, we get the Puiseux jets $y = -dx^2 + dbx^3 + (-d^2c - db^2)x^4$ and $y = -dx^2 + dbx^3 + (-bde - fd^2 + hd)x^4$, giving diagrams 16. and 17.:

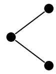
In the case $h = be + fd - dc - b^2$ under the condition $j \neq bf + ce - 2bc$, we obtain the Puiseux jets $y = -dx^2 + dbx^3 + (-d^2c - db^2)x^4 + (db^3 + bfd^2 + d^2bc + d^2ce - d^2j)x^5$ and $y = -dx^2 + dbx^3 + (-d^2c - db^2)x^4 + (3d^2bc + db^3)x^5$, and thus diagram 48.:

5



Finally, in the case $j = bf + ce - 2bc$ and under the condition $k \neq cf - c^2$, we obtain the Puiseux jets $y = -dx^2 + dbx^3 + (-d^2c - db^2)x^4 + (3d^2bc + db^3)x^5 + (-6d^2b^2c - db^4 - 2d^3c^2)x^6$ and $y = -dx^2 + dbx^3 + (-d^2c - db^2)x^4 + (3d^2bc + db^3)x^5 + (-db^4 - 6d^2b^2c - d^3cf - d^3c^2 + d^3k)x^6$, corresponding to diagram 49.:

6



If $k = cf - c^2$, then each curve in the resulting family has a linear factor and a multiple component; and this completes the classification.

References

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[6] D. A. Gudkov, N. A. Kirsanova, and G. F. Nebukina. Points of inflection and double tangents of fourth order curves II (Russian), 1983. VINITI 17–83.

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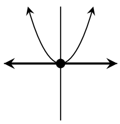
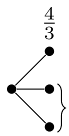
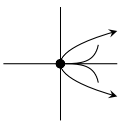
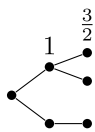
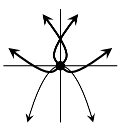
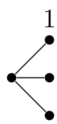
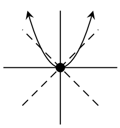
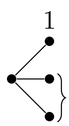
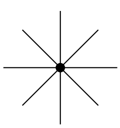
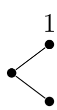
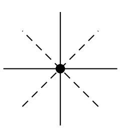
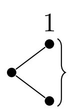
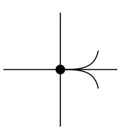
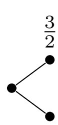
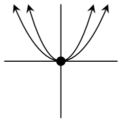
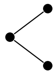
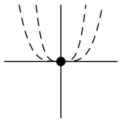
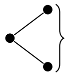
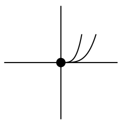
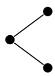
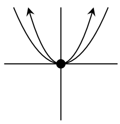
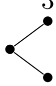
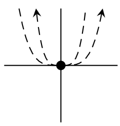
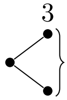
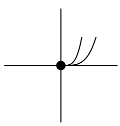
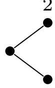
Appendix

A Summary of Classification for Quartic Curves.

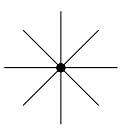
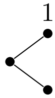
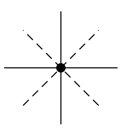
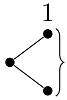
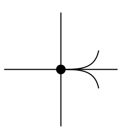
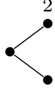
In this appendix, we summarize the classification by providing tables that show, for each singular point type, a simple example, together with a picture, the tree diagram, and the name of the singularity according to the Arnol'd notation. (Please note that in the table of reducible curves, the example will not always perfectly match the picture.)

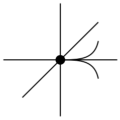
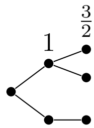
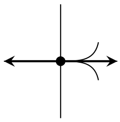
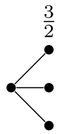
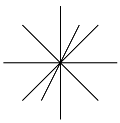
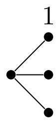
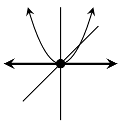
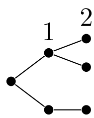
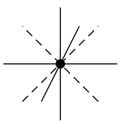
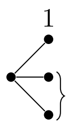
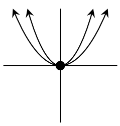
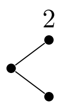
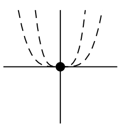
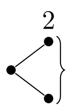
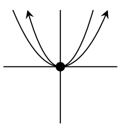
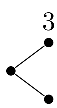
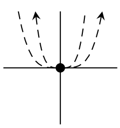
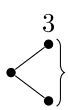
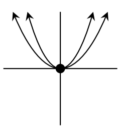
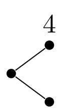
A.1 Irreducible curves

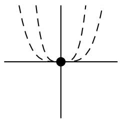
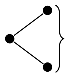
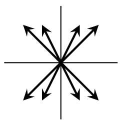
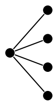
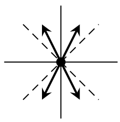
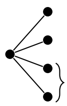
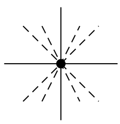
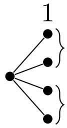
Name	*Picture*	*Diagram*	*Example*
1. E_6			$y^3 - x^4 = 0$
2. D_5			$x^4 + xy^2 + y^3 = 0$
3. D_4			$x^4 + x^2y + xy^2 + y^3 = 0$
4. D_4^*			$x^4 + x^2y + y^3 = 0$
5. A_1			$y^2 - x^2 + x^4 = 0$
6. A_1^*			$y^2 + x^2 + x^4 = 0$
7. A_2			$y^2 + x^3 + x^4 = 0$

Name	*Picture*	*Diagram*	*Example*
8. A_3		2 	$y^2 - x^4 + y^3 = 0$
9. A_3^*		2 	$y^2 + x^4 + y^3 = 0$
10. A_4		$\frac{5}{2}$ 	$y^2 + 2x^2y + x^4 + x^3y = 0$
11. A_5		3 	$y^2 + 2x^2y + x^4 + y^3 = 0$
12. A_5^*		3 	$y^2 + 2x^2y + x^4 - y^3 = 0$
13. A_6		$\frac{7}{2}$ 	$y^2+2x^2y+x^4+x^2y^2+\frac{1}{4}y^4+y^3 = 0$

A.2 Reducible curves

Name	*Picture*	*Diagram*	*Example*
1. A_1		1 	$(y-1)(y-2)(y-x)(y+x) = 0$
2. A_1^*		1 	$(y-1)(y-2)(x^2+y^2) = 0$
3. A_2		$\frac{3}{2}$ 	$(y-1)(y^2-x^3) = 0$

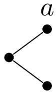
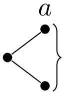
Name	*Picture*	*Diagram*	*Example*
4. D_5			$(y-x)(y^2-x^3)=0$
5. E_7			$y(y^2-x^3)=0$
6. D_4			$(y-x)(y+x)(y-x^2)=0$
7. D_6			$y(y-x)(y-x^2)=0$
8. D_4^*			$(y-x)(y^2+x^2-x^3)=0$
9. A_3			$(y-x^2)(y+x^2)=0$
10. A_3^*			$y^2+x^4=0$ (Reducible over $\mathbb{C}$)
11. A_5			$(y+x^2)(y+x^2+xy)=0$
12. A_5^*			$x^4+2x^2y+y^2x^2+y^2=0$ (Reducible over $\mathbb{C}$)
13. A_7			$(y+x^2+xy)(y+x^2+xy+y^2)=0$

Name	*Picture*	*Diagram*	*Example*
14. A_7^*		4 	$x^4+2x^3y+2x^2y+y^2x^2+2xy^2+y^4+y^2=0$ (Reducible over $\mathbb{C}$)
15. X_9		1 	$(y-x)(y+x)(y-2x)(y+2x)=0$
16. X_9^*		1 	$(y-x)(y+x)(x^2+y^2)=0$
17. X_9^{**}		1 	$(x^2+y^2)(x^2+4y^2)=0$

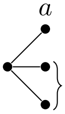
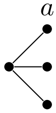
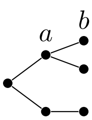
B Summary of Classification for Quintic Curves.

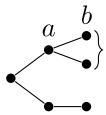
B.1 Irreducible curves

B.1.1 Multiplicity 2

a 	$\frac{2}{2} \le a \le \frac{13}{2}$	$A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ A_6,\ A_7,\ A_8,\ A_9,\ A_{10},\ A_{11},\ A_{12}$
a 	$1 \le a \le 6$	$A_1^*,\ A_3^*,\ A_5^*,\ A_7^*,\ A_9^*,\ A_{11}^*$

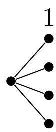
B.1.2 Multiplicity 3

a 	$a = \frac{4}{3}, \frac{5}{3}, 1$	$E_6,\ E_8,\ D_4^*$
a 	$a = \frac{3}{2}, 1$	$E_7,\ D_4$
	$a = 1$ $b = \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \frac{6}{2}, \frac{7}{2}, \frac{8}{2}, \frac{9}{2}$	$D_5,\ D_6\ ,D_7,\ D_8,\ D_9,\ D_{10},\ D_{11}$

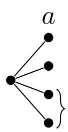


$a = 1; b = 2, 3, 4$ $D_6^*,\ D_8^*,\ D_{10}^*$

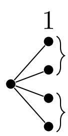
B.1.3 Multiplicity 4



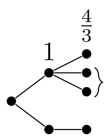
X_9



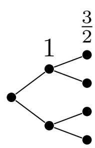
$a = 1, \frac{5}{4}$ $X_9^*,\ W_{12}$



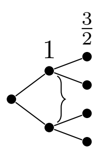
X_9^{**}



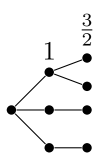
Z_{11}



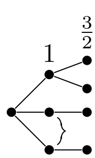
$Y_{1,1}^1$



$Y_{1,1}^{1*}$



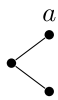
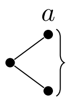
$X_{1,1}$



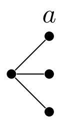
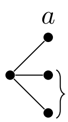
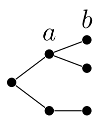
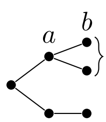
$X_{1,1}^*$

B.2 Reducible curves

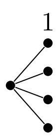
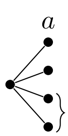
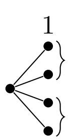
B.2.1 Multiplicity 2

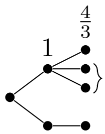
	$a = \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \frac{6}{2}, \frac{7}{2}, \frac{8}{2}, 5, 6$	$A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ A_6,\ A_7,\ A_9,\ A_{11}$
	$a = 1, 2, 3, 4$	$A_1^*,\ A_3^*,\ A_5^*,\ A_7^*$

B.2.2 Multiplicity 3

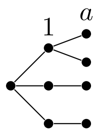
	$a = 1, \frac{3}{2}, 2$	$D_4\ E_7,\ J_{10}$
	$a = 1, \frac{4}{3}, 2$	$D_4^*,\ E_6,\ J_{10}^*$
	$a = 1$ and $b = \frac{3}{2}, \frac{4}{2}, \frac{5}{2}, \frac{6}{2}, \frac{7}{2}, \frac{8}{2}, 5$, or $a = 2$ and $b = \frac{5}{2}, \frac{6}{2}, \frac{7}{2}, \frac{8}{2}$	$D_5,\ D_6,\ D_7,\ D_8,\ D_9,\ D_{10}, D_{12}, J_{11}, J_{12}, J_{13}, J_{14}$
	$a = 1; b = 2, 3, 4$, or $a = 2; b = 3, 4$	$D_6^*,\ D_8^*,\ D_{10}^*,\ J_{12}^*,\ J_{14}^*$

B.2.3 Multiplicity 4

		X_9
	$a = 1, \frac{4}{3}$	$X_9^*,\ W_{13}$
		X_9^{**}

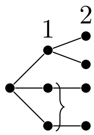


Z_{11}

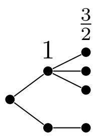


$a = \frac{3}{2}, 2$

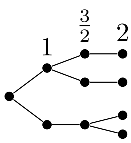
$X_{1,1},\ X_{1,2}$



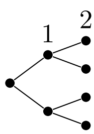
$X^{*1}_{1,2}$



Z_{12}

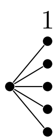


$Y^{1}_{1,2}$

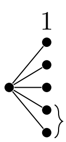


$Y^{1}_{2,2}$

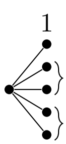
B.2.4 Multiplicity 5



N_{16}



N^{*}_{16}



N^{**}_{16}