Low dimensional cohomologies of biparabolic subalgebras

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Abstract

The dimensions of zero and first regular cohomologies of a biparabolic subalgebra B of some simple Lie algebra are calculated. Namely, it is proved that if S and T are subsets of simple roots such as $B = H \oplus L^{R_+^S} \oplus L^{R_-^T}$, where H is a splitting Cartan subalgebra and R_+^S and R_-^T are the positive (negative) roots generated by S (by T respectively) then the dimension d_0 of the center of B is equal to the number of simple roots which is not contained in $S \cup T$. If $n = a_0 + a_1 + ... + a_r = b_0 + b_1 + ... + b_s$ where $a_i, b_i \in N$ are ordered partititions of n and B is the corresponding biparabolic subalgebra of $sl(n)$, then the dimension of outer derivations of B is equal to $(r + s - d_0)d_0$.

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1 Introduction.

It is well known (Whitehead) that the cohomologies of the finite-dimensional semisimple Lie algebra L over the field of characteristic zero with coefficients in finite-dimensional irreducible module M such that $LM \neq 0$ is always zero (see [5]):

$$H^i(L, M) = 0, i \geq 0.$$

If $L = M$ then cohomologies $H^i(L, L), i \geq 0$ are called regular. Consequently for finite-dimensional semisimple Lie algebra L over the field of characteristic zero the regular cohomologies are trivial:

$$H^i(L, L) = 0, i \geq 0.$$

For the generalization of semisimple Lie algebra, such as parabolic subalgebra G. Tolpygo [10] in 1972 proved that the triviality of regular cohomologies in all dimensions is still true. Borel subalgebras are partial cases of parabolic subalgebras, so in this way Tolpygo generalized the Leger and Luks [6] theorem on regular cohomologies of Borel subalgebras which asserts that regular cohomologies of Borel subalgebras are trivial.

For the further generalization of a semisimple algebra, of a biparabolic subalgebra, this issue turned out to be more difficult, the regular cohomologies of biparabolic subalgebras are usually nonzero.

At first, the cohomologies of biparabolic (seaweed) algebras was investigated in the article [4] of A. Elashvili and G. Rakviashvili. They proved that regular cohomologies of biparabolic subalgebra B of $sl(n)$ are trivial if the number of equal proper sums (see definition below) of suitable partitions of n is zero. Further G. Rakviashvili and E. Kuljanishvili [8] announced that the dimension of zero cohomologie groups of a biparabolic subalgebra B of $sl(n)$ is equal to the number of equal proper sums of suitable partitions of n of the biparabolic subalgebra B.

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In this paper we continue this research and generalize the results from [8] for zero cohomology groups of biparabolic sublagebras of any semisimple algebras (Theorem 1). Also, we provide in Theorem 2 the formula for the dimension of first regular cohomologie group of biparabolic subalgebra of $sl(n)$.

2 Zero regular cohomologies.

We need some notions and concepts such as roots, root spaces, Cartan, Borel, parabolic and biparabolic subalgebras.

Cartan subalgebra H is a nilpotent subalgebra that coincides to his normalizer. Cartan subalgebra of semisimple Lie algebra is commutative.

Let L be a semisimple Lie algebra over a field k of characteristic zero with a splitting Cartan subalgebra H [5]; therefore all characteristic roots of all adjoint representations $ad_L h, h \in H$, are in k.

For any semisimple Lie algebra L, there are roots [5] $R \subseteq H^*$ such that $0 \in R$ and

$$L = \oplus_{\alpha \in R} L_\alpha,$$

where

$$L_\alpha = \{x \in L | (\forall h \in H)[hx] = \alpha(h)(x)\}$$

is root space and $L_0 = H$.

If $(\alpha_1, \alpha_2, ..., \alpha_{n_0})$, $\alpha_i \in H^*$ is a simple system of roots of L then we may choose $e_i \in L_{\alpha_i}, f_i \in L_{-\alpha_i}, h_i \in H$ so that

$$[e_i h_i] = 2e_i, [f_i h_i] = -2f_i, [e_i f_i] = h_i.$$

If we abbreviate $[...[x_1 x_2]...x_r]$ as $[x_1 x_2 ... x_r]$, $x_i \in L$, then the set of elements

$$h_i, [e_{i_1} e_{i_2} ... e_{i_r}], [f_{i_1} f_{i_2} ... f_{i_r}], \tag{1}$$

is the basis of L [5].

The set of roots π is said to be parabolic [9] if i) $\pi + (-\pi) = R$, ii) if $\alpha, \beta \in \pi$ and $\alpha + \beta$ is verifying, then $\alpha + \beta \in \pi$. The subalgebra P of a semisimple Lie algebra L is said to be parabolic [9] if $P = H + L^\pi$, where $L^\pi = \oplus_{\alpha \in \pi} L_\alpha$ and roots set π is parabolic or equivalently if P contains the maximal solvable subalgebra - Borel subalgebra. As we mentioned before, a Borel subalgebra is parabolic.

Two parabolic subalgebras P and $P^{'}$ are said to be weakly opposite [9] if $P + P^{'} = L$. By the definition [9] a biparabolic subalgebra (also named earlier as seaweed subalgebra) B of the semisimple Lie algebra L with the splitting Cartan subalgebra H is the intersection of two weakly opposite subalgebras P and $P^{'}$, i.e. $B = P \cap P^{'}$. If S, T are subsets of $(\alpha_1, \alpha_2, ..., \alpha_{n_0})$, then lets denote R_+^S and R_-^T positive (negative) roots generated by S (by T respectively). Then the direct sum

$$P = H \oplus L^{R_+^S} \oplus L^{R_-^T},$$

where

$$L^{R_+^S} = \oplus_{\alpha \in R_+^S} L_\alpha, \quad L^{R_-^T} = \oplus_{\alpha \in R_-^T} L_\alpha$$

is the biparabolic subalgebra of L and it is said to be standard biparabolic subalgebra. Each biparabolic subalgebra is isomorphic to some standard biparabolic subalgebra, therefore in following, we shall consider only such biparabolic subalgebras.

Because zero regular cohomology group $H^0(B,B)$ is isomorphic to the center $Z(B)$, we formulate the following theorem:

Theorem 1. The dimension of the center $Z(P) \simeq H^0(P,P)$ of the biparabolic subalgebra B is equal to the number of simple roots α_i, $\alpha_i \notin S \cup T$.

Proof. Let $e_i \in L_{\alpha_i}$ corresponds to $\alpha_i \in S$ and $f_i \in L_{-\alpha_i}$ corresponds to $\alpha_i \in T$. If $z \in Z(P)$ then

$$z = \sum_{i=1}^{n_0} a_i h_i + \sum_I b_I [e_{i_1} e_{i_2} ... e_{i_r}] + \sum_I c_I [f_{i_1} f_{i_2} ... f_{i_r}].$$

Let us multiply z by h_j. Then

$$0 = [zh_j] = \sum_I b_I [[e_{i_1} e_{i_2} ... e_{i_r}] h_j] + \sum_I c_I [[f_{i_1} f_{i_2} ... f_{i_r}] h_j] =$$

$$= \sum_I \sum_s b_I [e_{i_1} e_{i_2} ... [e_{i_s} h_j] ... e_{i_r}] + \sum_I \sum_s c_I [f_{i_1} f_{i_2} ... [f_{i_s} h_j] ... f_{i_r}] =$$

$$= \sum_I \sum_s b_I \alpha_{i_s}(h_j) [e_{i_1} e_{i_2} ... e_{i_r}] + \sum_I \sum_s c_I \alpha_{i_s}(h_j) [f_{i_1} f_{i_2} ... f_{i_r}]. \quad (2)$$

Since (1) are the basis, then the coefficients in (2) are zeros:

$$b_I \sum_s \alpha_{i_s}(h_j) = 0, c_I \sum_s \alpha_{i_s}(h_j) = 0.$$

Since

$$[e_{i_1} e_{i_2} ... e_{i_r}] \in L_{\alpha_{i_1} + \alpha_{i_2} + ... + \alpha_{i_r}},$$

then $\sum_s \alpha_{i_s} \neq 0$ and there is h_j such that

$$\sum_s \alpha_{i_s}(h_j) \neq 0$$

and consequently, $b_I = 0, c_I = 0$ for all I.

So the elements of $Z(P)$ have the form

$$z = \sum_{i=1}^{n_0} a_i h_i.$$

Let multiply z on e_j:

$$[e_j z] = \sum_{i=1}^{n_0} a_i [e_j h_i] = (\sum_{i=1}^{n_0} a_i \alpha_j(h_i)) e_j = 0.$$

Therefore, we get the system of linear equations with n_0 variables

$$\sum_{i=1}^{n_0} a_i \alpha_j(h_i) = 0, j = 1, 2, ..., n_0. \quad (3)$$

The multiplication of z by f_j gives the same system of linear equations:

$$[f_j z] = \sum_{i=1}^{n_0} a_i [f_j h_i] = -(\sum_{i=1}^{n_0} a_i \alpha_j(h_i)) f_j = 0,$$

$$\sum_{i=1}^{n_0} a_i \alpha_j(h_i) = 0, j = 1, 2, ..., n_0.$$

If B coincides with the semisimple Lie algebra L then determinant of the $n_0 \times n_0$ matrix $(\alpha_j(h_i))$ is nonzero, because the center of semisimple Lie algebra is equal to zero. Also, all equations of the system are linearly independent. If at least one from e_i and f_j belongs to B for $\alpha_i \in (\alpha_1, \alpha_2, ..., \alpha_{n_0})$ then the number of equations does not decrease. But if e_j and f_j simultaneously do not belong to B, then in (3) the number of equations decreases by one, and the dimension of solutions of (3) decreases also by one. The theorem is proved.

Let $sl(n)$ be the special linear Lie algebra and fix two ordered partitions ([1], [3], [7]) which we call a double partition of n:

$$n = a_0 + a_1 + ... + a_r = b_0 + b_1 + ... + b_s; a_i, b_i \in N. \tag{4}$$

From (4) follows for example that $e_{a_i, a_i+1}, \ e_{b_j+1, b_j} \notin B$. Then the suitable biparabolic subalgebra B of $sl(n)$ consists from matrices that have the geometrical seaweed form [3] just like on the following picture:

$$\begin{pmatrix} a_{11} & a_{21} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} & 0 & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} & a_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{67} \end{pmatrix}$$

If $a_0 + a_1 + ... + a_i = b_0 + b_1 + ... + b_j$ where $i \neq n, \ j \neq n$, then we say that we have equal proper sum in double partitions. In the picture above, the double partition of biparabolic subalgebra of $sl(7)$ is $7 = 2 + 3 + 2 = 3 + 2 + 1 + 1$, and the number of equal proper sums of double partition is one.

Corollary. The dimension of the biparabolic subalgebra of $sl(n)$ is the number of equal proper sums of double partition (4).

Proof. Indeed the number of equal proper sums of double partition is equal to the number of upper and lower vertices which coincide with each other. These vertices correspond to the matrices $(e_{i,i+1}), \ \ (e_{i+1,i})$, which correspond to e_i and f_i.

3 The first regular cohomologies.

First regular cohomologie group $H^1(B, B)$ of B is isomorphic to outer derivations $Out(B)$, i.e. to the factor of derivations of B by internal derivations of B. Let denote

$$d_1 = dimOut(P) = dimH^1(P, P), d_0 = dimZ(P),$$

where $Out(P)$ is outer derivations of P and $Z(P)$ is the center of P.
Theorem 2. $d_1 = (r + s - d_0)d_0$.
Proof. In [4, Theorem 1] it is proved that

$$H^n(B, B) \simeq H^n(B, Z(B)).$$

So

$$H^1(B, B) \simeq H^1(B, Z(B)). \tag{5}$$

Further

$$H^1(B, Z(B)) \simeq Der(B, Z(B))/Int(B, Z(B)).$$

We have

$$Der(B, Z(B)) \simeq Hom(B/[B, B], Z(B))$$

and since $Z(B)$ is the center of B, then

$$Int(B, Z(B)) = 0.$$

So

$$H^1(B, B) \simeq Hom(B/[B, B], Z(B)). \tag{6}$$

Since

$$dimHom(B/[B, B], Z(B)) = (dimB - dim[B, B])d_0.$$

we must prove that

$$(r + s - d_0)d_0 = (dimB - dim[B, B])d_0,$$
$$dim[B, B] = dimB - r - s + d_0. \tag{7}$$

The Cartan subalgebra H of B is the set of traceless diagonal matrices of $sl(n)$ and its dimension is $n - 1$.

We denote by e_{ij} the matrix (e_{ij}). If e_{ij} does not belong to H, then there is $h \in H$ such that $\{e_{ij}, e_{ji}, h\} \simeq sl(2)$, so

$$[e_{ij}h] = 2e_{ij}, [e_{ji}h] = -2e_{ji}, [e_{ij}e_{ji}] = h.$$

This means that $h, e_{ij}, e_{ji} \in [B, B]$. If $e_{ij} \in B$ and $e_{ji} \notin B$ then $e_{ij} \in [B, B]$ and $h, e_{ji} \notin [B, B]$. So if an element from B is not diagonal matrix, then it belongs to $[B, B]$, therefore B and $[B, B]$, can only differ by diagonal matrices. To find out which diagonal matrices of B does not belong to $[B, B]$ it is sufficient to consider multiplication rule elements of the type $e_{i,i+1}$ and $e_{i+1,i}$.

If at least one from $e_{i,i+1}$ and $e_{i+1,i}$ does not belong to B, then

$$[e_{i,i+1}, e_{i+1,i}] = e_{ii} - e_{i+1,i+1} = e_i - e_{i+1}$$

does not belong to $[B, B]$. From the remark after (4) it follows that the number such i that $e_{i,i+1} \notin B$ is equal to r and similarly, the number such i that $e_{i+1,i} \notin B$ is equal to s. So the number of such i that $e_{i,i+1}$ or $e_{i+1,i}$ does not belong to B is equal to $r + s$ minus the number of such i, that $e_{i,i+1}$ and $e_{i+1,i}$ simultaneously does not belong to B which is $r + s - d_0$ by Corollary. This completes the proof of Theorem 2.

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