Third Hankel determinant for a subclass of close-to-star functions associated with exponential function

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Abstract

In the present paper, we consider a subclass of close-to-star functions associated to exponential function, in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Here the main purpose is to establish the upper bound for the third Hankel determinant for this class. Also, we investigate the same bounds for two-fold and three-fold symmetric functions.

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1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions defined in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ and having the Taylor-Maclaurin series of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$, consisting of functions which are univalent in E. For two analytic functions f and g in E, f is said to be subordinate to g (symbolically $f \prec g$) if there exists a function w with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$ such that $f(z) = g(w(z))$. Furthermore, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

In the theory of univalent functions, a noted result was Bieberbach's conjecture established by Bieberbach [4], which states that, for $f \in \mathcal{S}$, $|a_n| \leq n$, $n = 2, 3, ...$ This conjecture remained as a challenge for the mathematicians for a long time. Finally, L. De-Branges [6] proved this conjecture in 1985. During the course of proving this conjecture, various results related to the coefficients were established and it gives rise to some new subclasses of $\mathcal{S}$.

The well known classes of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{K}$ respectively and defined as

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}$$

and

$$\mathcal{K} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}.$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{CS}^*$ of close-to-star functions if there exists a starlike function g such that $Re\left(\frac{f(z)}{g(z)}\right) > 0$. The concept of close-to-star functions was established by Reade [22].

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For $g(z) = z$, MacGregor [15] introduced the class $\mathcal{R}_1$ consisting of analytic functions $f \in \mathcal{A}$ and satisfying the condition

$$Re\left(\frac{f(z)}{z}\right) > 0.$$

An exponential function is a mathematical function, which is used in many real world situations. Mainly, it is used to find the exponential decay or exponential growth. The exponential function $\varphi(z) = e^z$ has positive real part in E and $\varphi(E) = \{z \in \mathbb{C} : |\log z| < 1\}$ is symmetric with respect to the real axis and starlike with respect to 1.

From time to time, various subclasses of analytic functions were studied by associating to different kind of functions. Mendiratta et al.[17] investigated the class $S^*(e^z)$, the class of starlike functions associated with exponential function. Further Hai-Yan Zhang et al.[26] established the third Hankel determinant for the class $S^*(e^z)$. Recently Ganesh et al.[9] studied the classes $S_s^*(e^z)$ and $C_s(e^z)$, the classes of starlike functions with respect to symmetric points and convex functions with respect to symmetric points associated with exponential function.

Getting inspired from these works, now we define the following subclass of analytic functions by subordinating to e^z.

Let $\mathcal{R}_1(e^z)$ denote the class which consists of analytic functions $f \in \mathcal{A}$ and satisfying the condition

$$\frac{f(z)}{z} \prec e^z \iff \left|\log\frac{f(z)}{z}\right| < 1.$$

Pommerenke[19, 20] stated the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For $q = 2, n = 1$ and $a_1 = 1$, the Hankel determinant reduces to $H_2(1) = a_3 - a_2^2$, which is the well known Fekete-Szegö functional. Fekete and Szegö [8] then further generalised the estimate $|a_3 - \mu a_2^2|$ where μ is real and $f \in \mathcal{S}$.

Also for $q = 2, n = 2$, the Hankel determinant takes the form of $H_2(2) = a_2a_4 - a_3^2$, which is known as second Hankel determinant.

There is another very useful functional $J_{n,m}(f) = a_na_m - a_{m+n-1}$, $n, m \in \mathbb{N} - \{1\}$, which was investigated by Ma [14] and it is known as generalized Zalcman functional. The functional $J_{2,3}(f) = a_2a_3 - a_4$ is a specific case of the generalized Zalcman functional. Various authors computed the upper bound for the functional $J_{2,3}(f)$ over different subclasses of analytic functions as it plays very important role in finding the bounds for the third Hankel determinant.

Furthermore, in the case $q = 3, n = 1$, the Hankel determinant yields

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix},$$

which is known as the third Hankel determinant.
For $f \in \mathcal{S}$ and $a_1 = 1$, we have

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

On applying the triangle inequality, the above equation yields

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \quad (1)$$

Numerous work has been done on the estimation of second Hankel determinant by various authors including Noor [18], Ehrenborg [7], Layman [11], Singh [24], Mehrok and Singh [16] and Janteng et al. [10]. The estimation of third Hankel determinant is little bit complicated. Babalola [3] was the first researcher who successfully obtained the upper bound of third Hankel determinant for the classes of starlike functions, convex functions and the class of functions with bounded boundary rotation. Further a few researchers including Shanmugam et al. [23], Bucur et al. [5], Altinkaya and Yalcin [1], Singh and Singh [25] have been actively engaged in the study of third Hankel determinant for various subclasses of analytic functions

In the present investigation, we study the bounds for the Fekete-Szegö inequality, second Hankel determinant, Zalcman functional and third Hankel determinant for the class $\mathcal{R}_1(e^z)$. Also, we establish the upper bound of third Hankel determinant for two-fold and three-fold symmetric functions.

By $\mathcal{P}$, we denote the class of analytic functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

whose real parts are positive in E.

To derive our main results, we shall make use of the following lemmas:
Lemma 1 [2] If $p \in \mathcal{P}$, then

$$|p_k| \leq 2, k \in \mathbb{N}, \quad (2)$$

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}, \quad (3)$$

$$|p_{i+j} - \mu p_i p_j| \leq 2, 0 \leq \mu \leq 1, \quad (4)$$

and for complex number ρ, we have

$$|p_2 - \rho p_1^2| \leq 2\max\{1, |2\rho - 1|\}. \quad (5)$$

Lemma 2 [12, 13] If $p \in \mathcal{P}$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some $|x| \leq 1$ and $|z| \leq 1$.

Lemma 3 [2] Let $p \in \mathcal{P}$, then

$$|Jp_1^3 - Kp_1p_2 + Lp_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|.$$

In particular, it is proved in [21] that

$$|p_1^3 - 2p_1p_2 + p_3| \leq 2.$$

2 Bounds of $|H_3(1)|$ for the class $\mathcal{R}_1(e^z)$

Theorem 2.1 If $f \in \mathcal{R}_1(e^z)$, then

$$|a_2| \leq 1, \tag{6}$$

$$|a_3| \leq 1, \tag{7}$$

$$|a_4| \leq 1, \tag{8}$$

and

$$|a_5| \leq \frac{37}{24}. \tag{9}$$

Proof. Since $f \in \mathcal{R}_1(e^z)$, by definition of subordination, we have

$$\frac{f(z)}{z} = e^{w(z)}. \tag{10}$$

Define $p(z) = \dfrac{1 + w(z)}{1 - w(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + ...$, which implies $w(z) = \dfrac{p(z) - 1}{p(z) + 1}$.

Also $\dfrac{f(z)}{z} = 1 + a_2z + a_3z^2 + a_4z^3 + a_5z^4 + ...$

Again $e^{w(z)} = 1 + \dfrac{1}{2}p_1z + \left(\dfrac{p_2}{2} - \dfrac{p_1^2}{8}\right)z^2$
$+ \left(\dfrac{p_1^3}{48} - \dfrac{p_1p_2}{4} + \dfrac{p_3}{2}\right)z^3 + \left(\dfrac{p_1^4}{384} + \dfrac{p_1^2p_2}{16} - \dfrac{p_3p_1}{4} - \dfrac{p_2^2}{8} + \dfrac{p_4}{2}\right)z^4 + ...$

Therefore, (10) yields

$$1 + a_2z + a_3z^2 + a_4z^3 + a_5z^4 + ... = 1 + \frac{1}{2}p_1z + \left(\frac{p_2}{2} - \frac{p_1^2}{8}\right)z^2$$

$$+ \left(\frac{p_1^3}{48} - \frac{p_1p_2}{4} + \frac{p_3}{2}\right)z^3 + \left(\frac{p_1^4}{384} + \frac{p_1^2p_2}{16} - \frac{p_3p_1}{4} - \frac{p_2^2}{8} + \frac{p_4}{2}\right)z^4 + ... \tag{11}$$

Comparing the coefficients of z, z^2, z^3 and z^4 in (11), we obtain

$$a_2 = \frac{p_1}{2}, \tag{12}$$

$$a_3 = \frac{p_2}{2} - \frac{p_1^2}{8}, \tag{13}$$

$$a_4 = \frac{p_1^3}{48} - \frac{p_1 p_2}{4} + \frac{p_3}{2}, \tag{14}$$

and

$$a_5 = \frac{p_1^4}{384} - \frac{p_2^2}{8} - \frac{p_3 p_1}{4} + \frac{p_1^2 p_2}{16} + \frac{p_4}{2}. \tag{15}$$

Using (2) in (12), we can easily get (6).
(13) can be expressed as

$$|a_3| = \frac{1}{2}\left|p_2 - \frac{1}{4}p_1^2\right|.$$

Using (5), it yields

$$|a_3| \leq \frac{1}{2}\left[2\max\left\{1, \frac{1}{2}\right\}\right],$$

which proves (7).
On applying Lemma 3 in (14), it gives (8).
Now (15) can be expressed as

$$|a_5| = \frac{1}{2}\left|\frac{1}{2}(p_4 - \frac{1}{2}p_2^2) + \frac{1}{2}(p_4 - p_1 p_3) + \frac{1}{8}p_1^2 p_2 + \frac{1}{192}p_1^4\right|.$$

Applying triangle inequality and using (2) and (4) in the above equation, the result (9) is obvious.

Theorem 2.2 If $f \in \mathcal{R}_1(e^z)$, then

$$|a_3 - a_2^2| \leq 1. \tag{16}$$

Proof. From (12) and (13), we have

$$|a_3 - a_2^2| = \frac{1}{2}\left|p_2 - \frac{3}{4}p_1^2\right|. \tag{17}$$

Application of (5) in (17) lead us to (16).

Theorem 2.3 If $f \in \mathcal{R}_1(e^z)$, then

$$|a_2 a_3 - a_4| \leq 1. \tag{18}$$

The result is sharp.
Proof. From (12), (13) and (14), we have

$$|a_2 a_3 - a_4| = \frac{1}{12}\left|-p_1^3 + 6p_1 p_2 - 6p_3\right|. \tag{19}$$

Using Lemma 2 and applying triangle inequality, (19) takes the form of

$$|a_2a_3 - a_4| \leq \frac{1}{12}\left[\frac{1}{2}p_1^3 + \frac{3}{2}p_1(4-p_1^2)|x|^2 + 3(4-p_1^2) - 3(4-p_1^2)|x|^2\right].$$

Putting $p_1 = p \in [0,2]$ and $|x| = t \in [0,1]$, it yields

$$|a_2a_3 - a_4| \leq \frac{1}{12}\left[\frac{1}{2}p^3 + \frac{3}{2}p(4-p^2)t^2 + 3(4-p^2) - 3(4-p^2)t^2\right] = F(p,t).$$

Now

$$\frac{\partial F}{\partial t} = \frac{1}{4}(p-2)(4-p^2)t,$$

which is a decreasing function of t.
So $\max\{F(p,t)\} = F(p,0) = \frac{1}{12}\left[\frac{1}{2}p^3 - 3p^2 + 12\right] = G(p)$.
Now $G'(p) = 0$ gives $p = 0$. Also $G''(p) = \frac{1}{4}(p-2)$, which is negative for each $p \in [0,2]$.
This implies $\max\{G(p)\} = G(0) = 1$, which proves (18).
The result is sharp for the function $f_(z) = ze^{z^3}$.

Theorem 2.4 If $f \in \mathcal{R}_1(e^z)$, then

$$|a_2a_4 - a_3^2| \leq 1. \tag{20}$$

The bound is sharp.
Proof. Using (12), (13) and (14), we have

$$|a_2a_4 - a_3^2| = \frac{1}{192}\left|48p_1p_3 - p_1^4 - 48p_2^2\right|.$$

Substituting for p_2 and p_3 from Lemma 2 and letting $p_1 = p$, we get

$$|a_2a_4 - a_3^2| = \frac{1}{192}\left| - p^4 - 48x^2(4-p^2) + 24p(4-p^2)(1-|x|^2)z\right|.$$

Since $|p| = |p_1| \leq 2$, by using (2), we may assume that $p \in [0,2]$. Then by using triangle inequality and $|z| \leq 1$ with $|x| = t \in [0,1]$, we obtain

$$|a_2a_4 - a_3^2| \leq \frac{1}{192}\left[p^4 + 24p(4-p^2) + 48(4-p^2)t^2 - 24p(4-p^2)t^2\right] = F(p,t).$$

Then

$$\frac{\partial F}{\partial t} = \frac{1}{4}(2-p)(4-p^2)t \geq 0.$$

Therefore $F(p,t)$ is an increasing function of t.
So $\max\{F(p,t)\} = F(p,1) = \frac{1}{192}\left[p^4 - 48p^2 + 192\right] = H(p)$.

Now $H'(p) = 0$ gives $p = 0$. Also $H''(p) = \frac{1}{192}\left[12p^2 - 96\right]$, which is negative for each $p \in [0, 2]$.
This implies $\max\{H(p)\} = H(0) = 1$, which proves (20).
The result is sharp for the function $f_(z) = ze^{z^2}$.

Theorem 2.5 If $f \in \mathcal{R}_1(e^z)$, then

$$|H_3(1)| \leq \frac{85}{24}. \tag{21}$$

Proof. By using (7), (8), (9), (16), (18) and (20) in (1), the result (21) can be easily obtained.

3 Bounds of $|H_3(1)|$ for two-fold and three-fold symmetric functions

A function f is said to be n-fold symmetric if it satisfies the following condition:

$$f(\xi z) = \xi f(z)$$

where $\xi = e^{\frac{2\pi i}{n}}$ and $z \in E$.
By $S^{(n)}$, we denote the set of all n-fold symmetric functions which belong to the class S.
The n-fold symmetric function have the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1}. \tag{22}$$

An analytic function f of the form (22) belongs to the family $\mathcal{R}_1^{(n)}(e^z)$ if and only if

$$\frac{f(z)}{z} = e^{\frac{p(z)-1}{p(z)+1}}, p \in \mathcal{P}^{(n)},$$

where

$$\mathcal{P}^n = \left\{p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} p_{nk} z^{nk}, z \in E\right\}. \tag{23}$$

Theorem 3.1 If $f \in \mathcal{R}_1^{(2)}(e^z)$, then

$$|H_3(1)| \leq 1. \tag{24}$$

Proof. If $f \in \mathcal{R}_1^{(2)}(e^z)$, so by definition of subordination, we have

$$\frac{f(z)}{z} = e^{w(z)}.$$

Define $p(z) = \frac{1 + w(z)}{1 - w(z)}$, which implies $w(z) = \frac{p(z) - 1}{p(z) + 1}$. Therefore, we have

$$\frac{f(z)}{z} = e^{\frac{p(z)-1}{p(z)+1}}. \tag{25}$$

Using (12)-(15),(22), and (23) for $n = 2$, (25) yields

$$a_3 = \frac{1}{2}p_2, \tag{26}$$

$$a_5 = \frac{1}{2}p_4 - \frac{1}{8}p_2^2. \tag{27}$$

Also

$$H_3(1) = a_3a_5 - a_3^3. \tag{28}$$

Using (26) and (27) in (28), it yields

$$H_3(1) = \frac{1}{4}p_2\left(p_4 - \frac{3}{4}p_2^2\right). \tag{29}$$

On applying triangle inequality and using (2) and (4), we can easily get the result (24).

Theorem 3.2 If $f \in \mathcal{R}_1^{(3)}(e^z)$, then

$$|H_3(1)| \leq 1. \tag{30}$$

Proof. If $f \in \mathcal{R}_1^{(3)}(e^z)$, so there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$\frac{f(z)}{z} = e^{\frac{p(z)-1}{p(z)+1}}. \tag{31}$$

Using (22) and (23) for $n = 3$, (31) gives

$$a_4 = \frac{1}{2}p_3. \tag{32}$$

Also

$$H_3(1) = -a_4^2. \tag{33}$$

Using (32) in (33), it yields

$$H_3(1) = -\frac{1}{4}p_3^2. \tag{34}$$

Using (2) in (34), (30) can be easily obtained.

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