A new characterization of discrete valuation rings

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Abstract

The aim of this paper is to give a new characterization of discrete valuation rings, namely, it is shown that if $(D, \mathfrak{m})$ is a local (Noetherian) ring such that $\operatorname{depth} D > 0$ and $\mathfrak{m}^k$ is principal ideal for some integer $k \geq 1$, then D is a discrete valuation ring.

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1 Introduction

From Hensel's theory of p-adic numbers onwards, valuation theory has been a fundamental tool of number theory and the theory of function field in one variable. In contrast, Krull defined and studied valuation rings from a more ring-theoretic point of view (see [3]), which was used by Zariski in algebraic geometry.

The purpose of this paper is to give a new characterization of Noetherian valuation domains, which are often called *discrete valuation rings* (abbreviated to DVR). In other words, a discrete valuation ring is a valuation ring whose value group is isomorphic to $\mathbb{Z}$. We refer the reader to [2] for more details about valuation rings. Our theorem is the following.

Theorem 1.1. Let $(D, \mathfrak{m})$ be a local (Noetherian) ring such that $\operatorname{depth} D > 0$ and $\mathfrak{m}^k$ is principal for some integer $k \geq 1$. Then the following statements hold:
(i) D is a Gorenstein ring of dimensional one.
(ii) D is a discrete valuation ring.

One of our tools for proving Theorem 1.1 is the following:

Lemma 1.2. Let $(D, \mathfrak{m})$ be a local (Noetherian) ring such that $\operatorname{depth} D > 0$ and the ideal $\mathfrak{m}^k$ of D is principal, for some integer $k \geq 1$. Then the following statements hold:
(i) There exists an element $y \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $\mathfrak{m}^k = Dy^k$.
(ii) For all integers $n \geq k$, $\mathfrak{m}^n = Dy^n$.

Throughout this paper, $(D, \mathfrak{m})$ will always be a commutative Noetherian local ring with non-zero identity. For any D-module L, we use $\mu_D(L)$ to denote the minimal number of generators of L, which is equal to the dimension of the vector space $L/\mathfrak{m}L$ over field $D/\mathfrak{m}$ (see [4, Theorem 2.3]). The set of zero-divisors in D will be denoted by $Z_D(D)$. For a D-module M, the *socle* of M is denoted by $\operatorname{Soc}_D(M)$ and defined as $\operatorname{Hom}_D(D/\mathfrak{m}, M)$. Also, if N is a finitely generated non-zero D-module of depth n, then the number $r_D(N) = \dim_{D/\mathfrak{m}} \operatorname{Ext}^n_D(D/\mathfrak{m}, N)$ is called the *type of* N. For any unexplained notation and terminology we refer the reader to [2, 4] or [5].

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2 The results

The main goal of this section is to establish a new characterization of discrete valuation rings. The following lemma plays a key role in the proof of the main theorem.

Lemma 2.1. Let $(D, \mathfrak{m})$ be a local (Noetherian) ring such that $\operatorname{depth} D > 0$. Suppose that there exists an integer $k \geq 1$ such that the ideal $\mathfrak{m}^k$ of D is principal. Then the following statements hold:

(i) There exists an element $y \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $\mathfrak{m}^k = Dy^k$.

(ii) For all integers $n \geq k$, $\mathfrak{m}^n = Dy^n$.

Proof. In order to show (i), suppose the contrary is true. Then, for any $y \in \mathfrak{m} \setminus \mathfrak{m}^2$ we have $\mathfrak{m}^k \neq Dy^k$. Now, if $y^k \notin \mathfrak{m}^{k+1}$, then $\mu_D(Dy^k + \mathfrak{m}^{k+1}/\mathfrak{m}^{k+1}) = 1$. On the other hand, since by hypothesis $\mathfrak{m}^k$ is principle, it follows that $\mu_D(\mathfrak{m}^k/\mathfrak{m}^{k+1}) = 1$. Hence we obtain that

$$(Dy^k + \mathfrak{m}^{k+1})/\mathfrak{m}^{k+1} \cong \mathfrak{m}^k/\mathfrak{m}^{k+1} \cong R/\mathfrak{m},$$

and therefore $Dy^k + \mathfrak{m}^{k+1} = \mathfrak{m}^k$. Whence, by using Nakayama's Lemma we see that $\mathfrak{m}^k = Dy^k$, which is a contradiction. Consequently, for any $y \in \mathfrak{m} \setminus \mathfrak{m}^2$ we have $y^k \in \mathfrak{m}^{k+1}$.

Now, we may assume that

$$\mathfrak{m} = Dy_1 + \cdots + Dy_s,$$

where $y_i \in \mathfrak{m} \setminus \mathfrak{m}^2$, for all $i = 1, \ldots, s$. Therefore, we have $y_i^k \in \mathfrak{m}^{k+1}$, for all $i = 1, \ldots, s$.

Next, we claim that $\mathfrak{m}^{sk} = \mathfrak{m}^{sk+1}$. To achieve this, let w be an arbitrary element of $\mathfrak{m}^{sk}$. Then there exist elements $c_{i_1 \ldots i_s} \in R$ such that

$$w = \sum_{i_1 + \cdots + i_s = sk} c_{i_1 \ldots i_s} y_1^{i_1} \ldots y_s^{i_s}.$$

It is easy to see that there exists $j \in \{1, \ldots, s\}$ such that $i_j \geq k$. Then, we have

$$y_j^{i_j} = y_j^{i_j - k} y_j^k \in \mathfrak{m}^{i_j - k}\mathfrak{m}^{k+1}.$$

Therefore

$$y_1^{i_1} \ldots y_j^{i_j} \ldots y_s^{i_s} \in \mathfrak{m}^{sk+1},$$

and so $w \in \mathfrak{m}^{sk+1}$. This shows that $\mathfrak{m}^{sk} = \mathfrak{m}^{sk+1}$. Accordingly, Nakayama's Lemma implies that $\mathfrak{m}^{sk} = 0$. Hence, $\mathfrak{m} \in \operatorname{Ass}_R(R)$, which is a contradiction; so that (i) is true.

To prove (ii), it is enough for us to show that $\mathfrak{m}^n = \mathfrak{m}^i(Dy^{n-i})$, for all $n \geq k$ and for all $0 \leq i \leq k$, where $\mathfrak{m}^k = Dy^k$. We prove this by induction on n. When $n = k$, for all $0 \leq i \leq k$, by (i) we have

$$\mathfrak{m}^k = Dy^k = (Dy^i)(Dy^{k-i}) \subseteq \mathfrak{m}^i(Dy^{k-i}) \subseteq \mathfrak{m}^k,$$

and therefore $\mathfrak{m}^k = \mathfrak{m}^i(Dy^{k-i})$; so that there is nothing to prove in the case $n = k$.

Now, suppose, inductively, that $n > k$ and we have shown that $\mathfrak{m}^n = \mathfrak{m}^i(Dy^{n-i})$, for all $0 \leq i \leq k$. We shall show that $\mathfrak{m}^{n+1} = \mathfrak{m}^i(Dy^{n+1-i})$, for all $0 \leq i \leq k$. To do this end, we can write $n = k + l$, for some integer $l \geq 1$; and then we have

$$\mathfrak{m}^{n+1} = \mathfrak{m}^k\mathfrak{m}^{l+1} = (Dy^k)\mathfrak{m}^{l+1} \subseteq (Dy)\mathfrak{m}^{k+l} = (Dy)\mathfrak{m}^i(Dy^{k+l-i}) \subseteq \mathfrak{m}^i(Dy^{k+l+1-i}) \subseteq \mathfrak{m}^{n+1},$$

and so $\mathfrak{m}^{n+1} = \mathfrak{m}^i(Dy^{n+1-i})$. The inductive step is complete. Q.E.D.

We are now ready to state and prove the main theorem of this paper which provides a new characterization discrete valuation rings.

Theorem 2.2. Assume that $(D, \mathfrak{m})$ is a local (Noetherian) ring such that $\operatorname{depth} D > 0$ and the ideal $\mathfrak{m}^k$ of D is principal for some integer $k \geq 1$. Then the following statements hold:

(i) There exists $y \in \mathfrak{m} \backslash \mathfrak{m}^2$ such that $\operatorname{Soc}(D/Dy^{k+1}) \cong D/\mathfrak{m}$ and D/Dy^{k+1} is a zero-dimensional Gorenstein ring.

(ii) D is a Gorenstein ring of dimensional one.

(iii) $\min\{l \in \mathbb{N} \mid \mathfrak{m}^l \text{ is a principal ideal of } D\} = 1$.

(iv) D is a discrete valuation ring.

Proof. In order to show (i), in view of Lemma 2.1 here exists an element $y \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $\mathfrak{m}^k = Dy^k$. Suppose now that $z := y^{k+1}$, and we show that $\operatorname{Soc}(D/Dz) \cong D/\mathfrak{m}$. It is sufficient for us to show that the ideal $(Dz :_D \mathfrak{m})$ of D is principal. To do this, as $\operatorname{depth} D > 0$, it follows that $\mathfrak{m} \nsubseteq Z_D(D)$, and so $\mathfrak{m}^k \nsubseteq Z_D(D)$. Thus $y^k \notin Z_D(D)$, and hence y is a D-regular sequence. Whence, it is easy to see that

$$(Dy^{k+1} :_D Dy) = Dy^k.$$

Consequently, as $Dy^{k+1} = \mathfrak{m}^{k+1}$, we have

$$Dy^k = (Dz :_D Dy) \supseteq (Dz :_D \mathfrak{m}) = (\mathfrak{m}^{k+1} :_D \mathfrak{m}) \supseteq Dy^k,$$

so that $(Dz :_D \mathfrak{m}) = Dy^k$, as required.

For the second part, first note that in view of the proof of Lemma 2.1, $z := y^{k+1}$ is a D-regular sequence. On the other hand, since by hypothesis the ideal $\mathfrak{m}^k$ is principal, it follows from the Krull's principal ideal theorem $\operatorname{height} \mathfrak{m} \leq 1$. Now, since $\operatorname{depth} D > 0$, it yields that $\operatorname{depth} D = \dim D = 1$, so that D is a Cohen-Macaulay ring. Therefore the ring D/Dz is also Cohen-Macaulay and $\dim D/Dz = 0$. Consequently, as $\operatorname{Soc}(D/Dz) \cong D/\mathfrak{m}$, it follows that D/Dz is a Cohen-Macaulay ring of type 1, and so in view of [1, Theorem 3.2.10] D/Dz is Gorenstein.

For (ii) use (i) and [1, Proposition 3.1.19].

In order to prove (iii), suppose that

$$t := \min\{l \in \mathbb{N} \mid \mathfrak{m}^l \text{ is a principal ideal of } D\},$$

and assume the contrary is true, i.e., $t > 1$. Then $\mathfrak{m}^t = Du$ is principal, where u is a D-regular sequence. Hence, in view of (i) and [1, Proposition 3.1.19], D/Du is Gorenstein. Thus by [4, Theorem 18.1] we get that $\operatorname{Soc}(D/Du) \cong D/\mathfrak{m}$. Now, as $\mathfrak{m}^{t-1}/\mathfrak{m}^t \subseteq \operatorname{Soc}(D/Du)$ and by Nakayama's Lemma $\mathfrak{m}^{t-1} \neq \mathfrak{m}^t$, it follows that $\mathfrak{m}^{t-1}/\mathfrak{m}^t \cong D/\mathfrak{m}$. Therefore, according to [4, Theorem 2.3], $\mathfrak{m}^{t-1}$ is a principal ideal of D, which is a contradiction with the choice of t.

Finally, in order to show (iv), according to (iii) the maximal ideal $\mathfrak{m}$ of D is principal. Thus as $\dim D = 1$ it follows that D is a regular local ring. Therefore in view of [4, Ex. 3.3] every non-zero ideal of D is a power of $\mathfrak{m}$ and so D is a discrete valuation rings. Q.E.D.

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