Framed Matrices and A_∞-Bialgebras *

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Abstract

We complete the construction of the biassociahedra KK, construct the free matrad $\mathcal{H}_\infty$, realize $\mathcal{H}_\infty$ as the cellular chains of KK, and define an A_∞-bialgebra as an algebra over $\mathcal{H}_\infty$. We construct the bimultiplihedra JJ, construct the relative free matrad $r\mathcal{H}_\infty$ as a $\mathcal{H}_\infty$-bimodule, realize $r\mathcal{H}_\infty$ as the cellular chains of JJ, and define a morphism of A_∞-bialgebras as a bimodule over $\mathcal{H}_\infty$. We prove that the homology of every A_∞-bialgebra over a commutative ring with unity admits an induced A_∞-bialgebra structure. We extend the Bott-Samelson isomorphism to an isomorphism of A_∞-bialgebras and determine the A_∞-bialgebra structure of $H_*(\Omega\Sigma X;\mathbb{Q})$. For each $n \geq 2$, we construct a space X_n and identify an induced nontrivial A_∞-bialgebra operation $\omega_2^n : H^*(\Omega X_n;\mathbb{Z}_2)^{\otimes 2} \to H^*(\Omega X_n;\mathbb{Z}_2)^{\otimes n}$.

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1 Introduction

Let $m, n \in \mathbb{N}, mn \geq 2$. The biassociahedron $KK_{n,m}$ is a contractible $(m+n-3)$-dimensional polytope with a single top dimensional cell e^{m+n-1}; in particular, $KK_{1,n} \cong KK_{n,1}$ is Stasheff's associahedron K_n [33]. Our construction of $KK_{n,m}$ in [31], which is valid for all m when $n \leq 3$ and for all n when $m \leq 3$, extends M. Markl's construction for $m+n \leq 6$ in [22] and [23].

In this paper we introduce and apply the theory of framed matrices to construct $KK_{m,n}$ for all m and n. A *framed matrix* is an equivalence class of paths of *generalized bipartition matrices* whose entries involve *augmented bipartitions*, which are pairs of partitions $(A_1|\cdots|A_r, B_1|\cdots|B_r)$ of finite sets of positive integers in which A_i and B_j are possibly null (cf. Sections 3 and 4).

Let $\Theta = \{\theta_m^n : \theta_1^1 = \mathbf{1}\}_{m,n\in\mathbb{N}}$ be a bigraded set with at most one element of bidegree (m,n) and let $F^{pre}(\Theta)$ denote the *free prematrad generated by* Θ introduced in [31]. Then $F^{pre}(\Theta)$ is the $\mathcal{A}_\infty$-operad when $\Theta = \{\theta_m^1\}_{m\geq 1}$ or $\Theta = \{\theta_1^n\}_{n\geq 1}$, and $F^{pre}(\Theta)/\sim$, where $A \sim B$ if $bideg\,(A) = bideg\,(B)$, is the *bialgebra prematrad* $\mathcal{H}^{pre} = \langle c_{n,m}\rangle_{m,n\in\mathbb{N}}$ when $\Theta = \{\mathbf{1}, \theta_2^1, \theta_1^2\}$ (see Example 6.10). The *free matrad* $\mathcal{H}_\infty$ is a proper submodule of $F^{pre}(\Theta)$ when Θ contains exactly one element of each bidegree.

Consider the canonical prematrad projection $\rho^{pre} : F^{pre}(\Theta) \to \mathcal{H}^{pre}$ under which

$$\rho^{pre}(\theta_m^n) = \begin{cases} c_{n,m}, & m+n \leq 3 \\ 0, & \text{otherwise.} \end{cases}$$

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Suppose we wish to define a differential ∂^{pre} such that $\rho^{pre} : (F^{pre}(\Theta), \partial^{pre}) \to (\mathcal{H}^{pre}, 0)$ is a free resolution in the category of prematrads. Then $H_*\left(F^{pre}(\Theta), \partial^{pre}\right) \approx \mathcal{H}^{pre}$ implies

$$\begin{aligned}
\partial^{pre}(\theta_2^1) &= \partial^{pre}(\theta_1^2) = 0 \\
\partial^{pre}(\theta_3^1) &= \gamma(\theta_2^1; \mathbf{1}, \theta_2^1) - \gamma(\theta_2^1; \theta_2^1, \mathbf{1}) \\
\partial^{pre}(\theta_2^2) &= \gamma(\theta_1^2; \theta_2^1) - \gamma(\theta_2^1\theta_2^1; \theta_1^2\theta_1^2) \\
\partial^{pre}(\theta_1^3) &= \gamma(\mathbf{1}, \theta_1^2; \theta_1^2) - \gamma(\theta_1^2, \mathbf{1}; \theta_1^2),
\end{aligned}$$

where $\gamma(\theta_2^1; \mathbf{1}, \theta_2^1) := \theta_2^1(\mathbf{1} \otimes \theta_2^1)$ and $\gamma(\theta_2^1\theta_2^1; \theta_1^2\theta_1^2) := \left(\theta_2^1 \otimes \theta_2^1\right)(2,3)\left(\theta_1^2 \otimes \theta_1^2\right)$. Although it is difficult to extend ∂^{pre} in such a way that acyclicity is easy to verify, there is a canonical differential ∂ on $\mathcal{H}_\infty$ such that the canonical projection $\varrho : (\mathcal{H}_\infty, \partial) \to (\mathcal{H}^{pre}, 0)$ is a free resolution in the category of prematrads.

The precise definition of $\mathcal{H}_\infty$ requires the notion of *coherent* framed matrices defined in terms of the S-U diagonal Δ_P on the permutahedra $P = \{P_n\}_{n\geq 1}$ [29], and once defined, $\mathcal{H}_\infty$ is the *minimal* submodule of $F^{pre}(\Theta)$. But unlike the resolution differential defined in [31], which fails to preserve the underlying coherence in bidegree $(4,4)$ (cf. Example 5.3), the more delicate resolution differential defined here preserves the underlying coherence in every bidegree.

Let $\mathfrak{n} = \{1, 2, \ldots, n\}$ and let P_n denote the $(n-1)$-dimensional permutahedron. Guided by the combinatorial join of permutahedra $P_m *_c P_n = P_{m+n}$ defined in [31], we introduce the *balanced framed join* $\mathfrak{m} \circledast_{pp} \mathfrak{n}$ and the *reduced balanced framed join* $\mathfrak{m} \circledast_{kk} \mathfrak{n} = \mathfrak{m} \circledast_{pp} \mathfrak{n} / \sim$. The bipermutahedron $PP_{n,m}$ is the geometric realization of $\mathfrak{m} \circledast_{pp} \mathfrak{n}$ and the biassociahedron $KK_{n+1,m+1} = PP_{n,m}/\sim$ is the geometric realization of $\mathfrak{m} \circledast_{kk} \mathfrak{n}$, where $\sim$ extends the standard relation $K_{n+1} = P_n/\sim$. Thus $KK_{m,n} = K_m *_{kk} K_n$ is the reduced balanced framed join of the associahedra K_m and K_n for all m and n.

Let $KK = \{KK_{n,m}\}_{mn\geq 2}$. We realize $\mathcal{H}_\infty$ as the cellular chains $C_*(KK)$ and define an *A_∞-bialgebra* as an algebra over $\mathcal{H}_\infty$. We define a *relative prematrad*, construct the *relative free matrad* $r\mathcal{H}_\infty$ as a $\mathcal{H}_\infty$-bimodule, realize $r\mathcal{H}_\infty$ as the cellular chains of bimultiplihedra $JJ = \{JJ_{n,m}\}_{mn\geq 1}$, and define a *morphism* $G : A \Rightarrow B$ *of A_∞-bialgebras* as a bimodule over $\mathcal{H}_\infty$.

Let R be a commutative ring with unity. An A_∞-bialgebra is a graded R-module H together with a family of multilinear operations $\{\omega_m^n \in Hom_{m+n-3}(H^{\otimes m}, H^{\otimes n})\}_{m,n\geq 1}$ and a chain map $\alpha : C_*(KK) \to \mathcal{E}nd_{TH}$ such that $\alpha\left(e^{m+n-3}\right) = \omega_m^n$. Given an A_∞-bialgebra B whose homology $H_*(B)$ is a free R-module, and a homology isomorphism $g : H_*(B) \to B$, we prove that the A_∞-bialgebra structure on B pulls back along g to an A_∞-bialgebra structure on $H_*(B)$, and any two such structures so obtained are isomorphic. This is a special case of our main result:

Theorem 11.5. *Let A be a free R-module, let B an A_∞-bialgebra, and let $g : A \to B$ be a homology isomorphism. Then*

(i) (Existence) g *induces an A_∞-bialgebra structure ω_A on A and extends to a map $G : A \Rightarrow B$ of A_∞-bialgebras and*

(ii) (Uniqueness) (ω_A, G) *is unique up to isomorphism.*

Our proof of Theorem 11.5 follows from a new transfer algorithm based on the interpretation of $JJ_{n,m}$ as a subdivision of $KK_{n,m} \times I$. In the special case of A_∞-(co)algebras, in which J_n is interpreted as a subdivision of $K_n \times I$, our approach differs significantly from the standard perturbation method for transferring A_∞-structures. Indeed, our definition of a morphism between two A_∞-(bi)structures in terms $C_*(JJ_{n,m})$ allows us to extend a given morphism $A \to B$ of DG modules to a morphism $A \Rightarrow B$ of A_∞-(bi)structures inductively. For some remarks on the history of perturbation theory see [14] and [15].

We conclude with several applications. Given a topological space X and a field F, the bialgebra structure of simplicial singular chains $S_*(\Omega X; F)$ of Moore base pointed loops induces an A_∞-bialgebra structure on $H_*(\Omega X; F)$ whose A_∞-algebra substructure was observed by Kadeishvili [17] and whose A_∞-coalgebra substructure was observed by Gugenheim [11]. Furthermore, the A_∞-coalgebra structure on $H_*(X;F)$ extends to an A_∞-bialgebra structure on the tensor algebra $T^a\tilde{H}_*(X;F)$, which is trivial if and only if the A_∞-coalgebra structure on $H_*(X;F)$ is trivial, and the Bott-Samelson isomorphism $t_* : T^a\tilde{H}_*(X;F) \overset{\approx}{\to} H_*(\Omega\Sigma X;F)$ extends to an isomorphism of A_∞-bialgebras (Theorem 12.2). The A_∞-bialgebra structure of $H_*(\Omega\Sigma X;\mathbb{Q})$ provides the first nontrivial rational homology invariant for $\Omega\Sigma X$ (Corollary 12.3). Finally, for each $n \geq 2$, we construct a space X_n and identify a nontrivial A_∞-bialgebra operation $\omega_2^n : H^*(\Omega X_n;\mathbb{Z}_2)^{\otimes 2} \to H^*(\Omega X_n;\mathbb{Z}_2)^{\otimes n}$ defined in terms of the action of the Steenrod algebra $\mathcal{A}_2$ on $H^*(X_n;\mathbb{Z}_2)$.

2 Combinatorial and Topological Tools

We begin with a review of the combinatorial and topological tools we need for our constructions.

2.1 Partitions and Permutahedra

By an *ordered set* we mean the empty set or a finite strictly increasing set of positive integers. We denote the special ordered sets $\mathfrak{o} := \varnothing$ and $\mathfrak{n} := \{1,2,\ldots,n\}$, $n \geq 1$, and define $\min\varnothing = \max\varnothing := 0$.

Let $\mathbf{a}$ be an ordered set and let $\#\mathbf{a}$ denote its cardinality. When $A_1|\cdots|A_n$ is an ordered partition of A in which $A_i = \varnothing$, we write $A_1|\cdots|A_{i-1}|0|\cdots|A_n$. The set of *(ordered) partitions of* $\mathbf{a}$ is defined and denoted by

$$P(\mathbf{a}) := \begin{cases} \{0\}, & \mathbf{a} = \varnothing \\ \{\text{standard ordered partitions of } \mathbf{a}\}, & \text{otherwise,} \end{cases}$$

and decomposes as the disjoint union of subsets $P_n(\mathbf{a}) := \{A_1|\cdots|A_n \in P(\mathbf{a})\}$ of *length n partitions of* $\mathbf{a}$, i.e., $P(\mathbf{a}) = \bigcup_{1\leq n\leq \#\mathbf{a}} P_n(\mathbf{a})$.

Let $\alpha \in P_n(\mathbf{a})$ and let $|\alpha|$ denote its *dimension*. When $\mathbf{a} = \varnothing$, define $|\mathbf{a}| := 0$. When $\mathbf{a} \neq \varnothing$, define $|\alpha| := \#\mathbf{a} - n$ via the correspondence $P_n(\mathbf{a}) \leftrightarrow \{\text{codimension } n-1 \text{ cells of } P_{\#\mathbf{a}}\}$, where $P_{\#\mathbf{a}}$ denotes the permutahedron whose cells (or faces) are indexed by the partitions in $P(\mathbf{a})$. Thus, when $\mathbf{a} \neq \varnothing$ we identify $P(\mathbf{a})$ with the permutahedron $P_{\#\mathbf{a}}$.

The set of *augmented (ordered) partitions of* $\mathbf{a}$ is defined and denoted by

$$P'(\mathbf{a}) := \begin{cases} \{\underbrace{0|\cdots|0}_{k} : k \in \mathbb{N}\}, & \mathbf{a} = \varnothing \\ \{A_1|\cdots|A_k : k \in \mathbb{N}, \text{ where } A_{i_1}|\cdots|A_{i_r} \in P(\mathbf{a}) \\ \quad \text{for some } i_1 < \cdots < i_r\}, & \mathbf{a} \neq \varnothing. \end{cases}$$

Let $\alpha = A_1|\cdots|A_n \in P'(\mathbf{a})$; its *length* $l(\alpha) := n$. Then $P'_n(\mathbf{a}) := \{\alpha \in P'(\mathbf{a}) : l(\alpha) = n\}$ is the subset of *length n augmented partitions in* $P'(\mathbf{a})$. Note that $P'_1(\mathbf{a}) = P_1(\mathbf{a}) = \{\mathbf{a}\}$.

Let $\alpha := A_1|\cdots|A_{n+1} \in P'_{n+1}(\mathbf{a})$. Given a subset $M^k \subseteq A_k$, the *(partitioning) action of M^k on α* is the partition

$$\partial_{M^k}\alpha := A_1|\cdots|A_{k-1}|M^k|A_k \smallsetminus M^k|A_{k+1}|\cdots|A_{n+1} \tag{2.1}$$

(cf. [29]). The set M^k is *extreme* if $M^k = \varnothing$ or $M^k = A_k$. Note that when $M^k \neq \varnothing$, the index k can be omitted.

Given an ordered subset $\lambda \subseteq \mathfrak{n}$, let $k = \#\lambda$. Define $\lambda^0 := 0$ and $\lambda^{k+1} := n+1$; when $k > 0$, write $\lambda = \left\{\lambda^1 < \cdots < \lambda^k\right\}$. The λ-*projection*

$$\mu_\lambda : P'_{n+1}(\mathbf{a}) \to P'_{k+1}(\mathbf{a}) \tag{2.2}$$

is defined by $\mu_\lambda(\alpha) := \bar{A}_1|\cdots|\bar{A}_{k+1}$, where $\bar{A}_i = A_{\lambda^{i-1}+1} \cup \cdots \cup A_{\lambda^i}$. In the extreme cases $\lambda = \varnothing$ and $\lambda = \mathfrak{n}$ we have $\mu_\varnothing(\alpha) = \mathbf{a}$ and $\mu_\mathfrak{n}(\alpha) = \alpha$. When $\lambda_i = \mathfrak{n} \smallsetminus \{i\}$, $1 \leq i \leq n$, we write

$$\alpha[i] := \mu_{\lambda_i}(\alpha) = A_1|\cdots|A_i \cup A_{i+1}|\cdots|A_{n+1}. \tag{2.3}$$

Then $\partial_{A_k}\alpha[k] = \alpha$ for all k. The *canonical projection*

$$\pi : P'(\mathbf{a}) \to P(\mathbf{a})$$

is given by discarding empty blocks when $\mathbf{a} \neq \varnothing$ and by $\pi(0|\cdots|0) := \varnothing$ otherwise.

Define the *dimension* and *vacuosity* of α by

$$|\alpha| := |\pi(\alpha)| \quad \text{and} \quad v(\alpha) := l(\alpha) - l(\pi(\alpha)), \tag{2.4}$$

respectively; then $v(\alpha)$ counts the number of empty blocks in α.

Let $\mathbf{a}$ and $\mathbf{b}$ be ordered sets. When $\mathbf{a} \subseteq \mathbf{b}$, the precise way in which $\mathbf{a}$ embeds in $\mathbf{b}$ is encoded by a special element of $P'(\mathbf{a})$, called the *embedding partition of* $\mathbf{a}$ *in* $\mathbf{b}$. When $\mathbf{a} \neq \varnothing$, this particular partition is constructed block-by-block as the elements of $\mathbf{a}$ are analyzed sequentially.

Definition 2.1. Let $\mathbf{a} \subseteq \mathbf{b}$. The **embedding partition of a in b**, denoted by $EP_\mathbf{b}\mathbf{a}$, is the following element of $P'(\mathbf{a})$:
If $\mathbf{a} = \varnothing$, define $EP_\mathbf{b}\mathbf{a} := 0|\cdots|0 \in P'_{\#\mathbf{b}+1}(\mathbf{a})$.
If $\mathbf{a} \neq \varnothing$, write $\mathbf{a} = \{a_1 < \cdots < a_m\}$ and define $a_0 := 0$ and $a_{m+1} := \infty$.
For $i = 0, 1, \ldots, m$: define $j_i := \#\{b \in \mathbf{b} : a_i < b < a_{i+1}\}$.
Then $EP_\mathbf{b}\mathbf{a} := \mathbf{b}_1|\cdots|\mathbf{b}_q$, where

1. $\mathbf{b}_k = \varnothing$ for all $k \leq j_0$ and $k > q - j_m$,
2. $a_1 \in \mathbf{b}_{j_0+1}$, and
3. for $i = 1, 2, \ldots, m-1$: if $a_i \in \mathbf{b}_k$,
   then $a_{i+1} \in \mathbf{b}_{k+j_i}$ and $\mathbf{b}_{k+1} = \cdots = \mathbf{b}_{k+j_i-1} = \varnothing$ when $j_i > 1$.

Heuristically speaking, the empty blocks at the extremes are place holders for the extreme elements of $\mathbf{b} \smallsetminus \mathbf{a}$. The elements of a non-empty block $\mathbf{b}_k$ are consecutive in $\mathbf{b}$, and $\mathbf{b}_k$ is maximal in the sense that $\max \mathbf{b}_k$ and $\min \mathbf{b}_{k+1}$ are *not* consecutive in $\mathbf{b}$ whenever $k < q$ and $\mathbf{b}_{k+1} \neq \varnothing$. For example, $EP_\mathbf{b}\mathbf{b} = \mathbf{b}$ and $EP_\mathbf{9}\{2,3,7,9\} = 0|23|0|0|7|9$. Note that

$$\#(\mathbf{b} \smallsetminus \mathbf{a}) = \sum_{i=0}^{m} j_i = q - 1. \tag{2.5}$$

The algorithmic formulation in Definition 2.1 is due to D. Freeman and the second author in [8].

2.2 Partitions and Planar Leveled Trees

Denote the sets of down-rooted and up-rooted Planar Leveled Trees (PLTs, aka *planar rooted trees with levels*) with n leaves by $\vee(n)$ and $\wedge(n)$, respectively, and consider a PLT $T \in \vee(n)$ with k levels. Set $T = T_1$ and let $(\Upsilon^{n_{11}}, \ldots, \Upsilon^{n_{1s_1}})$ be the sequence of top level corollas in T_1. Then $\mathbf{n}_1 = (n_{11}, \ldots, n_{1s_1})$ is the *first leaf sequence of* T. Form the subtree T_2 by removing the top level of T_1 and let $(\Upsilon^{n_{21}}, \ldots, \Upsilon^{n_{2s_2}})$ be the sequence of top level corollas in T_2. Then $\mathbf{n}_2 = (n_{21}, \ldots, n_{2s_2})$ is the *second leaf sequence of* T. Continue in this manner until the process terminates and the k^{th} *leaf sequence* $\mathbf{n}_k = (n_{k1})$ is obtained ($s_k = 1$). Then $(\mathbf{n}_1, \ldots, \mathbf{n}_k)$ is the *leaf decomposition of* T.

To recover the down-rooted tree T from its leaf decomposition $(\mathbf{n}_1, \ldots, \mathbf{n}_k)$, set $T^{\mathbf{n}_k} = \Upsilon^{n_{k1}}$ and construct the subtree $T^{\mathbf{n}_{k-1},\mathbf{n}_k}$ by attaching corolla $\Upsilon^{n_{k-1,j}}$ to leaf j of $T^{(\mathbf{n}_k)}$. Construct the subtree $T^{\mathbf{n}_{k-2},\mathbf{n}_{k-1},\mathbf{n}_k}$ by attaching corolla $\Upsilon^{n_{k-2,j}}$ to leaf j of $T^{\mathbf{n}_{k-1},\mathbf{n}_k}$. Continue in this manner until the process terminates and $T = T^{\mathbf{n}_1,\ldots,\mathbf{n}_k}$ is obtained. Dually, a tree $T \in \wedge(n)$ has the leaf decomposition $(\mathbf{n}_k, \ldots, \mathbf{n}_1)$ and $T = T_{\mathbf{n}_k,\ldots,\mathbf{n}_1}$, where $\mathbf{n}_1$ is the bottom level leaf sequence of T.

Let $\mathbf{a}$ be a non-empty ordered set. Given $B_1|\cdots|B_k \in P(\mathbf{a})$, identify $B_1|\cdots|B_k$ with the down-rooted PLT $T^{\mathbf{n}_1,\ldots,\mathbf{n}_k}$, where $(\mathbf{n}_1, \ldots, \mathbf{n}_k)$ is constructed as follows: Set $A_1 = \mathbf{a}$ and let $\beta_{11}|\cdots|\beta_{1s_1} := EP_{A_1}B_1$; then $\mathbf{n}_1 = (\#\beta_{11} + 1, \ldots, \#\beta_{1s_1} + 1)$. Set $A_2 = A_1 \smallsetminus B_1$ and consider $B_2|\cdots|B_k \in P(A_2)$. Let $\beta_{21}|\cdots|\beta_{2s_2} := EP_{A_2}B_2$; then $\mathbf{n}_2 = (\#\beta_{21} + 1, \ldots, \#\beta_{2s_2} + 1)$. Continue in this manner until the process terminates and $\mathbf{n}_k$ is obtained. Denote the componentwise correspondence between partitions and leaf sequences established above by $\varepsilon(B_i) = \mathbf{n}_i$. Then the map

$$\overset{\vee}{\varepsilon} : P(\mathbf{a}) \to \vee(\#\mathbf{a} + 1) \tag{2.6}$$

given by $\overset{\vee}{\varepsilon}(B_1|\cdots|B_k) = T^{\varepsilon(B_1),\ldots,\varepsilon(B_k)}$ is a bijection, and $\overset{\vee}{\varepsilon}{}^{-1}$ is given by the following standard construction when $\mathbf{a} = \mathfrak{n}$: Given $T \in \vee(n+1)$, number the leaves of T from left-to-right and assign the label ℓ to the vertex of T at which the branch containing leaf ℓ meets the branch containing leaf $\ell + 1$. Let B_i denote the set of vertex labels in level i; then $\overset{\vee}{\varepsilon}(B_1|\cdots|B_k) = T$. For example, $\overset{\vee}{\varepsilon}(5|13|24) = T^{(11112),(221),(3)} \in \vee(6)$ is pictured in Figure 1.

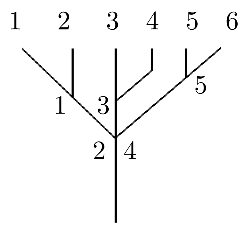


Figure 1. The down-rooted PLT corresponding to 5|13|24.

Dually, let $\tau : P(\mathbf{a}) \to P(\mathbf{a})$ denote the *reversing map*

$$\tau(B_1|\cdots|B_k) = B_k|\cdots|B_1, \tag{2.7}$$

let σ be a reflection of the plane in some horizontal axis, and define

$$\overset{\wedge}{\varepsilon} := \sigma\overset{\vee}{\varepsilon}\tau : P(\mathbf{a}) \to \wedge(\#\mathbf{a} + 1). \tag{2.8}$$

Let $(\mathbf{n}_1,\ldots,\mathbf{n}_k)$ and $(\mathbf{n}'_k,\ldots,\mathbf{n}'_1)$ be the leaf decompositions of $\overset{\vee}{\varepsilon}(B_1|\cdots|B_k)$ and $\overset{\wedge}{\varepsilon}(B_1|\cdots|B_k)$, respectively. Then the (typically non-isomorphic) PLTs $T^{\mathbf{n}_1,\ldots,\mathbf{n}_k}$ and $T_{\mathbf{n}'_k,\ldots,\mathbf{n}'_1}$ index the same cell of P_n (cf. [20], [29]).

2.3 The Combinatorial Join of Permutahedra

In this subsection we present a slight reformulation of the combinatorial join constructed in [31].

The n *right-shift* of an ordered set $\mathbf{a}$ is the ordered set

$$\mathbf{a}+n:=\begin{cases}\varnothing, & \mathbf{a}=\varnothing\\ \{a+n:a\in\mathbf{a}\}, & \mathbf{a}\neq\varnothing.\end{cases}$$

The *partitioned union* of $A_1|\cdots|A_n\in P'_n(\mathbf{a})$ *with* $B_1|\cdots|B_n\in P'_n(\mathbf{b})$ is the partition $A_1|\cdots|A_n\Cup B_1|\cdots|B_n:=$

$$\begin{cases}A_1\cup(B_1+\max\mathbf{a})|\cdots|A_n\cup(B_n+\max\mathbf{a}), & \min\mathbf{b}\leq\max\mathbf{a}\\ A_1\cup B_1|\cdots|A_n\cup B_n, & \text{otherwise.}\end{cases} \tag{2.9}$$

When $n=1$, formula (2.9) defines the *ordered set union of* $\mathbf{a}$ *with* $\mathbf{b}$. Note that ordered set union is associative but non-commutative in general; however, $\mathfrak{m}\Cup\mathfrak{n}=\mathfrak{n}\Cup\mathfrak{m}$.

Define $P'_n(\mathbf{a})\Cup P'_n(\mathbf{b}):=\{\alpha\Cup\beta:(\alpha,\beta)\in P'_n(\mathbf{a})\times P'_n(\mathbf{b})\}$; then clearly, $P'_n(\mathbf{a})\Cup P'_n(\mathbf{b})\subseteq P'_n(\mathbf{a}\Cup\mathbf{b})$. Conversely, given $C_1|\cdots|C_n\in P'_n(\mathbf{a}\Cup\mathbf{b})$, for each $i\in\mathfrak{n}$ let

$$A_i=C_i\cap\mathbf{a}\text{ and }B_i=\begin{cases}(C_i\smallsetminus\mathbf{a})-\max\mathbf{a}, & \text{if }\min\mathbf{b}\leq\max\mathbf{a}\\ C_i\smallsetminus\mathbf{a}, & \text{otherwise.}\end{cases}$$

Then $C_1|\cdots|C_n=A_1|\cdots|A_n\Cup B_1|\cdots|B_n\in P'_n(\mathbf{a})\Cup P'_n(\mathbf{b})$ so that $P'_n(\mathbf{a})\Cup P'_n(\mathbf{b})=P'_n(\mathbf{a}\Cup\mathbf{b})$.

Definition 2.2. Let $\mathbf{a}$ and $\mathbf{b}$ be ordered sets. The **combinatorial join of** $P_m(\mathbf{a})$ **with** $P_n(\mathbf{b})$ is the set

$$P_m(\mathbf{a})*_c P_n(\mathbf{b}):=\{\alpha\Cup\beta\in P_r(\mathbf{a}\Cup\mathbf{b}):(\alpha,\beta)\in P'_r(\mathbf{a})\times P'_r(\mathbf{b}),1\leq r\leq m+n\}.$$

Define

$$P(\mathbf{a})*_c P(\mathbf{b}):=\bigcup_{(1,1)\leq(m,n)\leq(\#\mathbf{a},\#\mathbf{b})}P_m(\mathbf{a})*_c P_n(\mathbf{b}).$$

Since $P_r(\mathbf{a}\Cup\mathbf{b})\subset P'_r(\mathbf{a}\Cup\mathbf{b})=P'_r(\mathbf{a})\Cup P'_r(\mathbf{b})$, an element $c\in P(\mathbf{a})*_c P(\mathbf{b})$ decomposes uniquely as $c=\alpha\Cup\beta$ for some $(\alpha,\beta)\in P'_r(\mathbf{a})\times P'_r(\mathbf{b})$. Hence

$$P(\mathbf{a})*_c P(\mathbf{b})=P(\mathbf{a}\Cup\mathbf{b}). \tag{2.10}$$

Thus in view of (2.10), the *combinatorial join of permutahedra* $P_m*_c P_n=P_{m+n}$.

Let $c\in P(\mathbf{a}\Cup\mathbf{b})$ and let e_c be the cell of P_{m+n} indexed by c. Then c decomposes uniquely as $c=\alpha\Cup\beta$ for some $(\alpha,\beta)\in P'_r(\mathbf{a})\times P'_r(\mathbf{b})$, and the projections $\pi(\alpha)\in P(\mathbf{a})$ and $\pi(\beta)\in P(\mathbf{b})$ index cells $e_\alpha\subset P_m$ and $e_\beta\subset P_n$. Thus, there is a decomposition map

$$W:\{\text{cells of }P_{m+n}\}\to\{\text{cells of }P_m\}\times\{\text{cells of }P_n\}$$

given by $W(e_c) = e_\alpha \times e_\beta$. Furthermore, if $e \times e' \subset P_m \times P_n$ is indexed by $A_1|\cdots|A_s \times B_1|\cdots|B_t \in P(\mathbf{a}) \times P(\mathbf{b})$, set

$$\alpha \times \beta = A_1|\cdots|A_s|0|\cdots|0 \ \times \ 0|\cdots|0|B_1|\cdots|B_t \in P'_{s+t}(\mathbf{a}) \times P'_{s+t}(\mathbf{b});$$

then $W(e_{\alpha \Cup \beta}) = e_\alpha \times e_\beta = e \times e'$ and W is a surjection. For example, when $m = n = 1$, $W(e_{12}) = e_1 \times e_1$, $W\left(e_{1|2}\right) = e_{1|0} \times e_{0|1}$, and $W\left(e_{2|1}\right) = e_{0|1} \times e_{1|0}$. Of course, W is not an injection since $e_1 \times e_1 = e_{1|0} \times e_{0|1} = e_{0|1} \times e_{1|0}$. Note that $|e_{\alpha \Cup \beta}| \geq |e_\alpha| + |e_\beta|$, where equality holds when $\alpha \Cup \beta$ is a shuffle permutation of $(\mathfrak{m}; \mathfrak{n} + m)$.

Example 2.3. Let us compute the combinatorial join $P_1 *_c P_2 = P_3$:

r	$\alpha \in P'_r(\mathbf{1})$	$\beta \in P'_r(\mathbf{2})$	$\alpha \Cup \beta \in P_r(\mathbf{1} \Cup \mathbf{2})$
1	1	12	$1 \Cup 12 = 123$
2	$1\|0$ $0\|1$	$1\|2$ $2\|1$ $0\|12$ $12\|0$	$1\|0 \Cup 1\|2 = 12\|3$ $1\|0 \Cup 2\|1 = 13\|2$ $1\|0 \Cup 0\|12 = 1\|23$ $0\|1 \Cup 1\|2 = 2\|13$ $0\|1 \Cup 2\|1 = 3\|12$ $0\|1 \Cup 12\|0 = 23\|1$
3	$1\|0\|0$ $0\|1\|0$ $0\|0\|1$	$0\|1\|2$ $0\|2\|1$ $1\|0\|2$ $2\|0\|1$ $1\|2\|0$ $2\|1\|0$ $12\|0\|0$ $0\|12\|0$ $0\|0\|12$	$1\|0\|0 \Cup 0\|1\|2 = 1\|2\|3$ $1\|0\|0 \Cup 0\|2\|1 = 1\|3\|2$ $0\|1\|0 \Cup 1\|0\|2 = 2\|1\|3$ $0\|1\|0 \Cup 2\|0\|1 = 3\|1\|2$ $0\|0\|1 \Cup 1\|2\|0 = 2\|3\|1$ $0\|0\|1 \Cup 2\|1\|0 = 3\|2\|1$

Remark 2.4. M. Markl's description of the permutahedron P_{m+n} in [23] is a translation of the description of the combinatorial join decomposition $P_{m+n} = P_m *_c P_n$ defined in terms of partitions as above into a description defined in terms of PLTs.

2.4 Diagonals on Permutahedra and Associahedra

Let X be an n-dimensional polytope that admits a cellular projection $p : P_{n+1} \to X$ and a realization as a subdivision of the n-cube I^n. In this subsection we review the S-U diagonal Δ_P (see [29]) and discuss the diagonal Δ_X induced by p. The diagonal $\Delta_{K_{n+2}}$ is obtained by setting $X = K_{n+2}$.

The permutahedron P_n can be realized as a subdivision of I^{n-1} in the following way: Identify the faces of P_n with the partitions of $\mathfrak{n}$; then P_1 is identified with the partition 1. If P_{n-1} has been constructed and $a = A_1|\cdots|A_p$ is a face of P_{n-1}, define $a_0 := 0$, $a_j := \#(A_{p-j+1} \cup \cdots \cup A_p)$ for $0 < j < p$, $a_p := \infty$, and $\frac{1}{2^\infty} := 0$. Define $I(a) := I_1 \cup I_2 \cup \cdots \cup I_p$, where $I_j := [1 - 2^{-a_{j-1}}, 1 - 2^{-a_j}]$; then $P_n = \bigcup_{a \in P_{n-1}} a \times I(a)$, where faces of $a \times I(a)$ are identified with partitions of $\mathfrak{n}$ as follows (see Figures 1 and 2):

Face of $a \times I\,(a)$	**Partition of** $\mathfrak{n}$	
$a \times 0$	$A_1 \vert \cdots \vert A_p \vert n$	
$a \times (I_j \cap I_{j+1})$	$A_1 \vert \cdots \vert A_{p-j} \vert n \vert A_{p-j+1} \vert \cdots \vert A_p,$	$1 \leq j \leq p-1$
$a \times 1$	$n \vert A_1 \vert \cdots \vert A_p,$	
$a \times I_j$	$A_1 \vert \cdots \vert A_{p-j+1} \cup n \vert \cdots \vert A_p,$	$1 \leq j \leq p.$

A vertex common to P_n and I^{n-1} is a *cubical vertex*. Thus a is a cubical vertex of P_n if and only if $a|n$ and $n|a$ are cubical vertices of P_{n+1}.

Let e be a cell of P_n and denote the set of vertices of e by $\mathcal{V}_e$. The subset $\mathcal{V}_e \subseteq S_n$ determines the components of $\Delta_P(e)$ in the following way: Let $\sigma = x_1|\cdots|x_n \in \mathcal{V}_e$. Reading σ from left-to-right and from right-to-left, construct the partitions $\overleftarrow{\sigma}_1|\cdots|\overleftarrow{\sigma}_p$ and $\overrightarrow{\sigma}_q|\cdots|\overrightarrow{\sigma}_1$ of maximal decreasing subsets and form the *Strong Complementary Pair* (SCP)

$$a_\sigma \times b_\sigma : \overleftarrow{\sigma}_1|\cdots|\overleftarrow{\sigma}_p \times \overrightarrow{\sigma}_q|\cdots|\overrightarrow{\sigma}_1 \in P(\mathfrak{n}) \times P(\mathfrak{n}).$$

Then $\sigma = \max a_\sigma = \min b_\sigma$, $\min \overleftarrow{\sigma}_j < \max \overleftarrow{\sigma}_{j+1}$ for all $j < p$, and $\min \overrightarrow{\sigma}_i < \max \overrightarrow{\sigma}_{i+1}$ for all $i < q$. Let $\{c_{ij}\} := \overrightarrow{\sigma}_i \cap \overleftarrow{\sigma}_j$; then $a_\sigma \times b_\sigma$ is represented by the $q \times p$ *step matrix* $C_\sigma = (c_{ij})$ *over* $\mathfrak{n} \cup \{0\}$, whose positive entries in each row and column are contiguous and increasing.

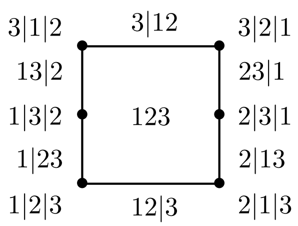


Figure 1: P_3 as a subdivision of $P_2 \times I$.

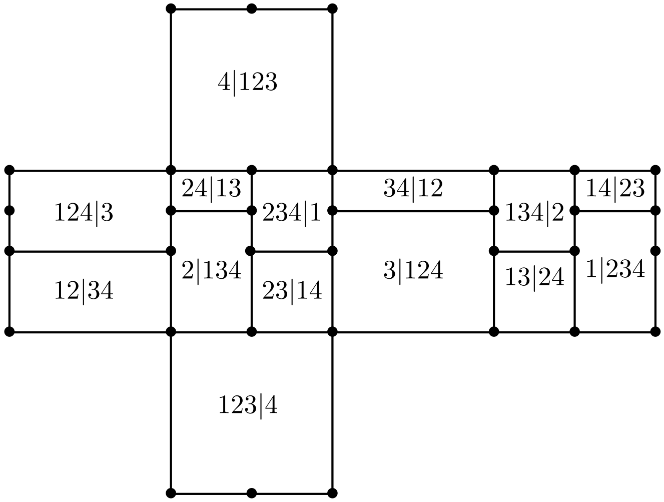


Figure 2: The facets of P_4 as a subdivision of I^3.

Example 2.5. Consider the vertex $\sigma = 2|1|3|5|4$ of P_5; then $\overleftarrow{\sigma}_1|\overleftarrow{\sigma}_2|\overleftarrow{\sigma}_3 = 21|3|54$ and $\overrightarrow{\sigma}_3|\overrightarrow{\sigma}_2|\overrightarrow{\sigma}_1 = 2|135|4$ so that $a_\sigma \times b_\sigma = 21|3|54 \times 2|135|4$ and

$$C_\sigma = \begin{pmatrix} 0 & 0 & 4 \\ 1 & 3 & 5 \\ 2 & 0 & 0 \end{pmatrix}.$$

Let $a = A_1|\cdots|A_p \in P(\mathfrak{n})$. For $1 \leq j < p$, choose a subset $M_j \subseteq (A_j \smallsetminus \{\min A_j\})$ such that $\min M_j > \max A_{j+1}$ when $M_j \neq \varnothing$. Define the *right-shift M_j action*

$$R_{M_j}(a) := \begin{cases} A_1|\cdots|A_j \smallsetminus M_j|A_{j+1} \cup M_j|\cdots|A_k, & M_j \neq \varnothing \\ a, & M_j = \varnothing. \end{cases}$$

Choose $M_1 \subset A_1$ so that $R_{M_1}(a)$ is defined, choose $M_2 \subset A_2 \cup M_1$ so that $R_{M_2}R_{M_1}(a)$ is defined, choose $M_3 \subset A_3 \cup M_2$ so that $R_{M_3}R_{M_2}R_{M_1}(a)$ is defined, and so on. Having chosen $M := (M_1, M_2, \ldots, M_{p-1})$, denote the composition $R_{M_{p-1}} \cdots R_{M_2}R_{M_1}(a)$ by $R_{\mathbf{M}}(a)$.

Dually, let $b = B_q|\cdots|B_1 \in P(\mathfrak{n})$. For $1 \leq i < q$, choose a subset $N_i \subseteq (B_i \smallsetminus \{\min B_i\})$ such that $\min N_i > \max B_{i+1}$ when $N_i \neq \varnothing$. Define the *left-shift N_i action*

$$L_{N_i}(b) := \begin{cases} B_q|\cdots|B_{i+1} \cup N_i|B_i \smallsetminus N_i|\cdots|B_1, & N_i \neq \varnothing \\ b, & N_i = \varnothing. \end{cases}$$

Choose $N_1 \subset B_1$ so that $L_{N_1}(b)$ is defined, choose $N_2 \subset B_2 \cup N_1$ so that $L_{N_2}L_{N_1}(b)$ is defined, choose $N_3 \subset B_3 \cup N_2$ so that $L_{N_3}L_{N_2}L_{N_1}(b)$ is defined, and so on. Having chosen $N := (N_1, N_2, \ldots, N_{q-1})$, denote the composition $L_{N_{q-1}} \cdots L_{N_2}L_{N_1}(b)$ by $L_{\mathbf{N}}(b)$.

Now choose $\mathbf{M}$ and $\mathbf{N}$ so that $R_{\mathbf{M}}(a_\sigma)$ and $L_{\mathbf{N}}(b_\sigma)$ are defined, and form the *Complementary Pair* (CP) $R_{\mathbf{M}}(a_\sigma) \times L_{\mathbf{N}}(b_\sigma)$. Define

$$A_\sigma \times B_\sigma := \bigcup_{\mathbf{M},\mathbf{N}} \{R_{\mathbf{M}}(a_\sigma) \times L_{\mathbf{N}}(b_\sigma)\};$$

then

$$\Delta_P(e) = \bigcup_{\sigma \in \mathcal{V}_e} A_\sigma \times B_\sigma. \tag{2.11}$$

Note that $R_{M_j}(a_\sigma)$ and $L_{N_i}(b_\sigma)$ are realized by a right-shift M_j action and a down-shift N_i action on C_σ. Thus *the components of $\Delta_P(e^{n-1})$ are generated by all possible right-shift actions together with all possible down-shift actions on all possible step matrices over $\mathfrak{n} \cup \{0\}$.*

Example 2.6. On the top dimensional cell e^2 of P_3, $\Delta_P(e^2)$ is the union of

$$\begin{array}{ll} A_{1|2|3} \times B_{1|2|3} = \{1|2|3 \times 123\}, & A_{1|3|2} \times B_{1|3|2} = \{1|32 \times 13|2\}, \\ A_{2|1|3} \times B_{2|1|3} = \{21|3 \times 2|13,\ 21|3 \times 23|1\}, & A_{2|3|1} \times B_{2|3|1} = \{2|31 \times 23|1\}, \\ A_{3|1|2} \times B_{3|1|2} = \{31|2 \times 3|12,\ 1|32 \times 3|12\}, & A_{3|2|1} \times B_{3|2|1} = \{321 \times 3|2|1\}. \end{array}$$

Note that $R_{M_j}(a_\sigma)$ and $L_{N_i}(b_\sigma)$ are realized by a right-shift M_j action and a down-shift N_i action on C_σ. Thus *the components of $\Delta_P(e^{n-1})$ are generated by all possible right-shift actions together with all possible down-shift actions on all possible step matrices over $\mathfrak{n} \cup \{0\}$.*

Example 2.7. Continuing Example 2.5, the left-shift action $L_{\{5\}}(2|135|4) = 25|13|4$ on the right-hand factor of the SCP $a_\sigma \times b_\sigma = 12|3|45 \times 2|135|4$ produces the CP $12|3|45 \times 25|13|4$ and induces the corresponding down-shift action on the step matrix

$$C_\sigma = \left(\begin{array}{ccc} 0 & 0 & 4 \\ 1 & 3 & 5 \\ 2 & 0 & 0 \end{array} \right) \xrightarrow{L_{\{5\}}} \left(\begin{array}{ccc} 0 & 0 & 4 \\ 1 & 3 & 0 \\ 2 & 0 & 5 \end{array} \right).$$

Define Δ_P on cellular chains $C_*(P_n)$ by

$$\Delta_P(e^{n-1}) := \sum_{\substack{e_1 \times e_2 \in A_\sigma \times B_\sigma \\ \sigma \in S_n}} sgn(e_1, e_2)\ e_1 \otimes e_2, \tag{2.12}$$

where $sgn(e_1, e_2)$ denotes the sign specified in [29], and on proper cells by extending comultiplicatively. Define the boundary operator ∂ on $C_*(P_n)$ by

$$\partial e^{n-1} := \sum_{A|B \sqsubset P_n} (-1)^{\#A} sgn(A|B)\ A|B \tag{2.13}$$

and on proper cells by extending as a derivation, i.e.,

$$\partial(A_1|\cdots|A_k) = \sum (-1)^{\#(A_1 \cup \cdots \cup A_{r-1})+r+1} A_1|\cdots|A_{r-1}|\partial A_r|A_{r+1}|\cdots|A_k.$$

Then $(C_*(P_n); \partial, \Delta_P)$ is a (non-coassociative) DG coalgebra.

Now identify the vertices of P_{n+1} with the permutations in S_{n+1} and extend the *weak order* on S_{n+1} given by $\cdots|x_i|x_{i+1}|\cdots < \cdots|x_{i+1}|x_i|\cdots$ if $x_i < x_{i+1}$ to a partial order (p-o). Then the associated Hasse diagram orients the 1-skeleton of P_{n+1} [6]. Denote the minimal and maximal vertices of a face e of P_{n+1} by $\min e$ and $\max e$, respectively, and define $e \leq e'$ if there exists an oriented edge-path in P_{n+1} from $\max e$ to $\min e'$.

When X is an n-dimensional polytope as above, the projection $p : P_{n+1} \to X$ induces a p-o on the cells of X. For example, when the faces of P_{n+1} are indexed by planar leveled trees (PLTs) with $n+2$ leaves and the faces of K_{n+2} are indexed by planar rooted trees (PRTs) with $n+2$ leaves (without levels), Tonks' projection $p = \theta$ given by forgetting levels [35] induces the *Tamari order* on the faces $\{\theta(T_i)\}$ of K_{n+2} given by $\theta(T_i) \leq \theta(T_j)$ if $T_i \leq T_j$.

Let e be a cell of X and let $|e|$ denote its dimension. A *k-subdivision cube of e* is a set of faces of e whose union is a k-subcube of I^n for some $k \leq n$. For example, when e is the top dimensional cell of P_4, the facets in $\{2|134, 24|13\}$ and $\{2|134, 24|13, 23|14, 234|1\}$ form 2-subdivision cubes of e, but any three in the latter do not (see Figure 2). Denote the set of vertices of e by $\mathcal{V}_e$ (when $e = X$ we suppress the subscript e). Given a vertex $v \in \mathcal{V}_e$, let $I^{k_1}_{v,1}$ and $I^{k_2}_{v,2}$ be k_i-subdivision cubes of e such that $\max I^{k_1}_{v,1} = \min I^{k_2}_{v,2} = v$ and $k_1 + k_2 = |e|$; then $\left(I^{k_1}_{v,1}, I^{k_2}_{v,2}\right)$ is a *pair of* (k_1, k_2)*-subdivision cubes of* e. Denote the set of all such pairs by e_v and let $(\mathbf{I}^{k_1}_{v,1}, \mathbf{I}^{k_2}_{v,2})_e$ denote its unique maximal element; then $\left(I^{k_3}_{v,3}, I^{k_4}_{v,4}\right) \subseteq (\mathbf{I}^{k_1}_{v,1}, \mathbf{I}^{k_2}_{v,2})_e$ for all $(I^{k_3}_{v,3}, I^{k_4}_{v,4}) \in e_v$. For example, when e is the top dimensional cell of P_4 and $v = 4|3|2|1$, we have $(\mathbf{I}^2_{v,1}, \mathbf{I}^1_{v,2})_e = (\{2|134, 24|13\}, \{4|23|1\})$. For an explicit description of $\left(\mathbf{I}^{k_1}_{v,1}, \mathbf{I}^{k_2}_{v,2}\right)_e$ when $e \subseteq P_n$ see (2.15) below.

If in addition, the cellular projection $p : P_{n+1} \to X$ preserves maximal pairs of (k_1, k_2)-subdivision cubes, i.e., for every cell e of P_{n+1} we have

$$p\left(\mathbf{I}^{k_1}_{v,1}, \mathbf{I}^{k_2}_{v,2}\right)_e = \left(\mathbf{I}^{k_1}_{p(v),1}, \mathbf{I}^{k_2}_{p(v),2}\right)_{p(e)},$$

the components of the induced diagonal Δ_X on a cell f of X form the set of product cells

$$\Delta_X(f) := \bigcup_{\substack{\left(e^{k_1}, e^{k_2}\right) \in \left(\mathbf{I}^{k_1}_{v,1}, \mathbf{I}^{k_2}_{v,2}\right)_f \\ v \in \mathcal{V}_f}} \{e^{k_1} \times e^{k_2}\}. \tag{2.14}$$

In particular, $p = \theta$ preserves maximal pairs of (k_1, k_2)-subdivision cubes and $\Delta_K(e)$ is given by setting $X = K_{n+2}$ (see (2.16) below). Note that $\left(e^{k_1}, e^{k_2}\right) \in \left(\mathbf{I}^{k_1}_{v,1}, \mathbf{I}^{k_2}_{v,2}\right)_X$ implies $e^{k_1} \leq e^{k_2}$. Thus $e^{k_1} \times e^{k_2}$ is a "matching pair" in the sense of Masuda, Thomas, Tonks, and Vallette in [26] (see Definition 2.9). Furthermore, since $f = p(e)$ for some $e = P_{n_1} \times \cdots \times P_{n_s}$ and $p(e) = p(P_{n_1}) \times \cdots \times p(P_{n_s})$, the diagonal $\Delta_X(f)$ is automatically the comultiplicative extension of its values on the factors of f, i.e., $\Delta_X(f) = \Delta_X(p(P_{n_1})) \times \cdots \times \Delta_X(p(P_{n_s}))$.

When $X = P_{n+1}$, Formulas (2.11) and (2.14) are equivalent. The maximal (k_1, k_2)-subdivision pair with respect to a vertex σ of P_{n+1} is

$$\left(\mathbf{I}^{k_1}_{\sigma,1}, \mathbf{I}^{k_2}_{\sigma,2}\right) = \left(\bigcup_{e_1 \in A_\sigma} e_1, \bigcup_{e_2 \in B_\sigma} e_2\right). \tag{2.15}$$

Definition 2.8. A positive dimensional face e of P_n is **non-degenerate** if $|\theta(e)| = |e|$. A positive dimensional partition $a = A_1|\cdots|A_p \in P(\mathfrak{n})$ is **degenerate** if for some j and some $k > 0$, there exist $x, z \in A_j$ and $y \in A_{j+k}$ such that $x < y < z$; otherwise a is **non-degenerate**. A CP $\alpha \times \beta$ is **non-degenerate** if α and β are non-degenerate.

Define $\Delta_K(K_{n+1}) = \Delta_K(\theta(P_n)) := (\theta\times\theta)\Delta_P(P_n)$; then

$$\Delta_K(e^{n-1}) = \bigcup_{\substack{\text{non-degenerate CPs}\\ \alpha\times\beta\in A_\sigma\times B_\sigma\\ \sigma\in S_n}} \{\theta(\alpha)\times\theta(\beta)\}. \tag{2.16}$$

Definition 2.9. A pair of faces $a\times b\subseteq K_{n+1}\times K_{n+1}$ is a **Matching Pair** (MP) if $a\leq b$ and $|a|+|b| = n-1$.

J.-L. Loday proposed the following "magical formula" for a diagonal Δ'_K on associahedra:

$$\Delta'_K\left(e^{n-1}\right) = \bigcup_{\substack{\text{MPs of faces}\\ a\times b\subseteq K_{n+1}\times K_{n+1}}} \{a\times b\}. \tag{2.17}$$

Formula (2.17), derived by Markl and Shnider in [24] and more recently by Masuda, Thomas, Tonks, and Vallette in [26], agrees with Formula (2.16) (see [32]).

2.5 The Subdivision Complex of a Diagonal Approximation

Recall that every map of CW-complexes is homotopic to a cellular map and a cellular map of CW-complexes induces a chain map of cellular chains. Let X be a CW complex, let $\Delta: X\to X\times X$ be the geometric diagonal, and let $C_*(X)$ denote the cellular chains of X. A *diagonal approximation of* Δ is a cellular map Δ_X homotopic to Δ, and the induced chain map $\Delta_X: C_*(X)\to C_*(X\times X)\approx C_*(X)\otimes C_*(X)$ is a *diagonal on* $C_*(X)$.

Let X be a polytope. For each $k\geq 1$, there is a diagonal approximation Δ_X whose *(left) k-fold iterate*

$$\Delta_X^{(k)} := \left(\Delta_X\times \mathbf{Id}^{\times(k-1)}\right)\cdots(\Delta_X\times\mathbf{Id})\,\Delta_X$$

is an embedding $X\hookrightarrow X^{\times(k+1)}$. Given a diagonal approximation Δ_X, define $\Delta_X^{(0)} :=\mathbf{Id}$. Then for each $k\geq 0$, there exists a unique cellular complex $X^{(k)}$, called the k^{th} *subdivision complex of* X (*with respect to* Δ_X), with the following property: Given a cell $e\sqsubseteq X$ and a subdivision cell $e'\sqsubseteq e$ in $X^{(k)}$, there exist unique cells $u_1,\ldots,u_{k+1}\sqsubseteq e$ such that $\Delta_X^{(k)}(e') = u_1\times\cdots\times u_{k+1}$. Thus $\Delta_X^{(k)}$ extends to a cellular inclusion $\Delta_X^{(k)}: X^{(k)}\hookrightarrow X^{\times(k+1)}$, and $X^{(k)}$ is the geometric representation of $\Delta_X^{(k)}(X)$ obtained by gluing the cells of $\Delta_X^{(k)}(X)$ together along their common boundaries in the only possible way.

Example 2.10. Let $I=[0,1]$ and consider the Serre diagonal Δ_I. For each $k\geq 0$ we have

$$\Delta_I^{(k)}(I) = \bigcup_{i=0}^{k} 0^{\times i}\times I\times 1^{\times(k-i)},$$

and the k^{th} subdivision complex of I is

$$I^{(k)} = \left[1-2^{-k},1\right]\cup\left[1-2^{1-k},1-2^{-k}\right]\cup\cdots\cup\left[0,1-2^{-1}\right].$$

A subdivision 1-cell

$$e_i = \begin{cases}\left[1-2^{-k},1\right], & i=0\\ \left[1-2^{i-k},1-2^{i-k-1}\right], & 0<i\leq k\end{cases}$$

is uniquely determined by the product cell $0^{\times i}\times I\times 1^{\times k-i}$ via $\Delta_I^{(k)}(e_i)=0^{\times i}\times I\times 1^{\times(k-i)}$ and a subdivision 0-cell $1-2^{i-k}$, $0\le i<k$, is uniquely determined by the product cell $0^{\times i}\times 1^{\times(k-i+1)}$ via $\Delta_I^{(k)}\left(1-2^{i-k}\right)=0^{\times i}\times 1^{\times(k-i+1)}$. Thus $\Delta_I^{(k)}\left(I^{(k)}\right)=\bigcup_{i=0}^{k}0^{\times i}\times I\times 1^{\times(k-i)}$ and $\Delta_I^{(k)}$ extends to a cellular inclusion $I^{(k)}\hookrightarrow I^{\times(k+1)}$. When $k=1$ we have

$$I^{(1)}=\left[\tfrac{1}{2},1\right]\cup\left[0,\tfrac{1}{2}\right]\overset{\Delta_I^{(1)}}{\mapsto} I\times 1\cup 0\times I\hookrightarrow I^2.$$

Consider the (left) k-fold iterated diagonal $\Delta_P^{(k)}$ on the permutahedron P_n. If e is a cell of P_n and $X\sqsubseteq\Delta_P^{(k)}(e)$, then $|X|\le|e|$. A *diagonal component of* $\Delta_P^{(k)}(e)$ is product cell $X\sqsubseteq\Delta_P^{(k)}(e)$ such that $|X|=|e|$, in which case we write $X\sqsubseteq_{diag}\Delta_P^{(k)}(e)$. For example, $1|3|2|4\times 13|24\sqsubseteq 1|23|4\times 13|24\sqsubseteq_{diag}\Delta_P(P_4)$ and $\left|1|3|2|4\times 13|24\right|<\left|1|23|4\times 13|24\right|=|P_4|$.

Let $\mathbf{x}=\alpha_1\times\cdots\times\alpha_q$ be a product of partitions and write $\alpha_i:=A_{i1}|\cdots|A_{ir_i}$. Given $i\in\mathfrak{q}$, $k,\ell<r_i$, and $M^k\subseteq A_{ik}$, define

$$\alpha'_j:=A'_{j1}|\cdots|A'_{jr'_j}=\begin{cases}\partial_{M^k}\alpha_i, & j=i\\ \alpha_j, & j\ne i,\end{cases}\quad\text{where } r'_j=\begin{cases}r_i+1, & j=i\\ r_j, & j\ne i,\end{cases}$$

$$\alpha''_j:=A''_{j1}|\cdots|A''_{jr''_j}=\begin{cases}\alpha_i[\ell], & j=i\\ \alpha_j, & j\ne i,\end{cases}\quad\text{where } r''_j=\begin{cases}r_i-1, & j=i\\ r_j, & j\ne i,\end{cases}$$

$$\partial^i_{M^k}\mathbf{x}:=\alpha'_1\times\cdots\times\alpha'_q\quad\text{and}\quad\mathbf{x}_{i\ell}:=\alpha''_1\times\cdots\times\alpha''_q.$$

Then $\left(\partial^j_{M^k}\mathbf{x}\right)_{jk}=\partial^j_{A_{jk}}\mathbf{x}_{jk}=\mathbf{x}$ for all j, k, and M^k. Note that if e is a cell of P_n, $n\ge 2$, and $\mathbf{x}\sqsubseteq_{diag}\Delta_P^{(q)}(\partial e)$, these equations (with non-extreme M^k) determine the cell structure of the boundary subdivision complex $\partial P_n^{(q)}$.

Proposition 2.11. Given a cell $e\sqsubseteq P_n$, $n\ge 2$, and a product cell $\mathbf{e}:=e_1\times\cdots\times e_q\sqsubseteq e^{\times q}$, write $e_i:=B_{i1}|\cdots|B_{ir_i}$.

1. Let $\mathbf{e}\sqsubseteq_{diag}\Delta_P^{(q-1)}(e)$ and $\mathbf{x}:=\partial^i_{M^k}\mathbf{e}\not\sqsubseteq_{diag}\Delta_P^{(q-1)}(\partial e)$ for some i and some non-extreme $M^k\subset B_{ik}$; write $\mathbf{x}=e'_1\times\cdots\times e'_q$ and $e'_j:=B'_{j1}|\cdots|B'_{jr'_j}$. There exist unique positive integers $j\le q$ and $\ell<r'_j$ such that $\mathbf{x}_{j\ell}\ne\mathbf{e}$ and $\mathbf{x}_{j\ell}\sqsubseteq_{diag}\Delta_P^{(q-1)}(e)$.

$$\begin{array}{ccccc}\exists!\ e_j & \overset{proj}{\longleftarrow} & \mathbf{e}=e_1\times\cdots\times e_q & \overset{\partial^i_{M^k}}{\longrightarrow} & \mathbf{x}=\partial^j_{B'_{j\ell}}\mathbf{x}_{j\ell}\ .\\ & & \mid\!\sqcap_{diag} & & \nearrow\partial^j_{B'_{j\ell}}\\ \mathbf{x}\not\sqsubseteq_{diag}\Delta_P^{(q-1)}(\partial e) & \overset{\partial}{\longleftarrow} & \Delta_P^{(q-1)}(e)\sqsupseteq_{diag} & & \mathbf{x}_{j\ell}\end{array}$$

2. If $\mathbf{x}:=\mathbf{e}\sqsubseteq_{diag}\Delta_P^{(q-1)}(\partial e)$, there exist unique positive integers $j\le q$ and $\ell<r_j$ such that $\mathbf{x}_{j\ell}\sqsubseteq_{diag}\Delta_P^{(q-1)}(e)$.

$$\begin{array}{ccccc} & \mathbf{x}=e_1\times\cdots\times e_q & \overset{proj}{\longrightarrow} & e_j=\partial_{B_{j\ell}}e_j[\ell] \\ & \| \prod_{diag} & & \uparrow\partial_{B_{j\ell}} \\ \mathbf{x}_{j\ell}\sqsubseteq_{diag}\ \Delta_P^{(q-1)}(e) \overset{\partial}{\longrightarrow} & \Delta_P^{(q-1)}(\partial e) & & \exists!\ e_j[\ell]. \end{array}$$

Proof. The proof follows immediately from the proof of Theorem 1 in [29]. Q.E.D.

Example 2.12. Consider the top dimensional cell $e = 123 \sqsubseteq P_3$ and its diagonal component $\mathbf{e} = e_1 \times e_2 = 13|2 \times 3|12$. Set $\mathbf{x} = \partial^1_{\{1\}}\mathbf{e} = 1|3|2 \times 3|12 \not\sqsubseteq_{diag} \Delta_P(\partial e)$ in Proposition 2.11, part (1). By uniqueness, $\mathbf{x}_{12} = 1|23 \times 3|12$ is the unique diagonal component of e distinct from $\mathbf{e}$ that can be obtained from $\mathbf{x}$ by a single factor replacement. On the other hand, set $\mathbf{x} = 3|1|2 \times 3|12 \sqsubseteq_{diag} \Delta_P(\partial e)$ in Proposition 2.11, part (2). By uniqueness, $\mathbf{x}_{11} = 13|2 \times 3|12$ is the unique diagonal component of e that can be obtained from $\mathbf{x}$ by a single factor replacement.

3 Bipartition Matrices

3.1 Bipartition Matrices Defined

Let $\mathbf{a}$ and $\mathbf{b}$ be ordered sets. A *bipartition on* $(\mathbf{a}, \mathbf{b})$ *of length* r is a pair $(\alpha, \beta) \in P'_r(\mathbf{a}) \times P'_r(\mathbf{b})$; it is *elementary* when $r = 1$. We will often denote a bipartition (α, β) as the fraction β/α and its length r by $l(\beta/\alpha)$. The k^{th} *biblock* of a bipartition $\frac{B_1|\cdots|B_r}{A_1|\cdots|A_r}$ is the pair (A_k, B_k); we refer to the components A_k and B_k as the k^{th} *input block* and *the* k^{th} *output block*, respectively. Denote the i^{th} row and j^{th} column of a matrix C by C_{i*} and C_{*j}.

Definition 3.1. Given ordered sets $\mathbf{a}$ and $\mathbf{b}$, let $(\mathbf{a}_*, \mathbf{b}_*) := (\mathbf{a}_1|\cdots|\mathbf{a}_p, \mathbf{b}_1|\cdots|\mathbf{b}_q) \in P'(\mathbf{a}) \times P'(\mathbf{b})$, let (r_{ij}) be a $q \times p$ matrix of positive integers, and choose a bipartition $c_{ij} \in P'_{r_{ij}}(\mathbf{a}_j) \times P'_{r_{ij}}(\mathbf{b}_i)$ for each (i, j); then $C = (c_{ij})$ is a $q \times p$ **bipartition matrix over** $(\mathbf{a}_*, \mathbf{b}_*)$. The sets $\mathbf{is}(C) := \mathbf{a}$ and $\mathbf{os}(C) := \mathbf{b}$ are the **input and output sets of** C, respectively. A bipartition matrix $C = (c_{ij})$ is **elementary** if $c_{ij} = \frac{\mathbf{b}_i}{\mathbf{a}_j}$ for all (i, j), **null** if $\mathbf{is}(C) = \mathbf{os}(C) = \varnothing$, and **semi-null** if either

1. $\mathbf{is}(C) = \varnothing$, $\mathbf{os}(C) \neq \varnothing$, and $C_{*1} = \cdots = C_{*p}$ or
2. $\mathbf{is}(C) \neq \varnothing$, $\mathbf{os}(C) = \varnothing$, and $C_{1*} = \cdots = C_{q*}$.

Note that if C is a $q \times p$ bipartition matrix over $(\mathbf{a}_*, \mathbf{b}_*)$, all numerators in C_{i*} lie in $P'(\mathbf{b}_i)$ and all denominators in C_{*j} lie in $P'(\mathbf{a}_j)$. Thus, an elementary matrix has constant denominators in each column and constant numerators in each row.

3.2 Products of Bipartition Matrices

Before we can impose a binary product on the set of bipartition matrices we need some definitions that apply the λ-projection map μ_λ defined in (2.2).

Definition 3.2. Let $C = \left(\frac{\beta_{ij}}{\alpha_{ij}}\right)$ be a $q \times p$ bipartition matrix and let $r_{ij} = l\left(\frac{\beta_{ij}}{\alpha_{ij}}\right)$. A **row equalizer of** C is a $q \times p$ matrix of ordered sets $\lambda^{row}(C) = \left(\lambda^{row}_{ij}\right)$, $\lambda^{row}_{ij} \subseteq \mathfrak{r}_{ij} \smallsetminus \{r_{ij}\}$, such that for each i, $\#\lambda^{row}_{ij} = s_i$ is constant for all j, and

$$\mu_{\lambda^{row}_{i1}}(\beta_{i1}) = \cdots = \mu_{\lambda^{row}_{ip}}(\beta_{ip}) \quad \text{for each} \quad i = 1, 2, \ldots, q; \tag{3.1}$$

it is **maximal** if s_i is maximal for all i. A **column equalizer of** C is a $q \times p$ matrix of ordered sets $\lambda^{col}(C) = \left(\lambda_{ij}^{col}\right), \lambda_{ij}^{col} \subseteq \mathfrak{r}_{\mathrm{ij}} \smallsetminus \{r_{ij}\}$, such that for each j, $\#\lambda_{ij}^{col} = t_j$ is constant for all i, and

$$\mu_{\lambda_{1j}^{col}}(\alpha_{1j}) = \cdots = \mu_{\lambda_{qj}^{col}}(\alpha_{qj}) \quad \text{for each} \quad j = 1, 2, \ldots, p; \tag{3.2}$$

it is **maximal** if t_j is maximal for all j. An **equalizer of** C is a $q \times p$ matrix of ordered sets $\lambda(C) = (\lambda_{ij}), \lambda_{ij} \subseteq \mathfrak{r}_{\mathrm{ij}} \smallsetminus \{r_{ij}\}$, such that $\#\lambda_{ij} = r$ is constant for all (i,j), and

$$\mu_{\lambda_{i1}}(\beta_{i1}) = \cdots = \mu_{\lambda_{ip}}(\beta_{ip}) \text{ and } \mu_{\lambda_{1j}}(\alpha_{1j}) = \cdots = \mu_{\lambda_{qj}}(\alpha_{qj}) \tag{3.3}$$

for each (i,j); it is **maximal** if r is maximal.

The **row and column equalizations of** C **with respect to** $\lambda^{row}(C)$ **and** $\lambda^{col}(C)$ are the bipartition matrices

$$\left(\frac{\mu_{\lambda_{ij}^{row}}(\beta_{ij})}{\mu_{\lambda_{ij}^{row}}(\alpha_{ij})}\right) \text{ and } \left(\frac{\mu_{\lambda_{ij}^{col}}(\beta_{ij})}{\mu_{\lambda_{ij}^{col}}(\alpha_{ij})}\right).$$

The **equalization of** C **with respect to** $\lambda(C)$ is the bipartition matrix

$$\left(\frac{\mu_{\lambda_{ij}}(\beta_{ij})}{\mu_{\lambda_{ij}}(\alpha_{ij})}\right).$$

Maximal equalizers are particularly important. However, when the entries of a bipartition matrix contain null bipartition blocks, multiple maximal equalizers can exist and produce different equalizations. For example, the bipartition matrix

$$C = \begin{pmatrix} \frac{0|0|1}{1|2|0} & \frac{0|1}{0|3} \end{pmatrix}$$

has two maximal row equalizers

$$\lambda_1^{row}(C) = \begin{pmatrix} \{1\} & \{1\} \end{pmatrix} \text{ and } \lambda_2^{row}(C) = \begin{pmatrix} \{2\} & \{1\} \end{pmatrix}$$

with respective equalizations

$$\begin{pmatrix} \frac{0|1}{1|2} & \frac{0|1}{0|3} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{0|1}{12|0} & \frac{0|1}{0|3} \end{pmatrix}.$$

Thus we distinguish one particular maximal (row/column) equalizer in the following way: Given a $q \times p$ bipartition matrix C, denote its class of maximal equalizers by $\mathcal{E}(C)$ and note that $\lambda_1 \neq \lambda_2 \in \mathcal{E}(C)$ implies $r > 0$. So assume $\lambda_1 \neq \lambda_2$, list the entries of λ_i in row order, and write $\lambda_1 = \left(\lambda_1^1, \ldots, \lambda_{pq}^1\right)$ and $\lambda_2 = \left(\lambda_1^2, \ldots, \lambda_{pq}^2\right)$. Define $\lambda_1 < \lambda_2$ if and only if for some i and all $j < i$, $\lambda_j^1 = \lambda_j^2$ and $\lambda_i^1 < \lambda_i^2$ with respect to the lexicographic ordering. Order the classes $\mathcal{E}^{row}(C)$ and $\mathcal{E}^{col}(C)$ of maximal row and maximal column equalizers in like manner and denote the minimal elements of $\mathcal{E}(C)$, $\mathcal{E}^{row}(C)$, and $\mathcal{E}^{col}(C)$, by $\Lambda(C) = (\Lambda_{ij})$, $\Lambda^{row}(C) = \left(\Lambda_{ij}^{row}\right)$, and $\Lambda^{col}(C) = \left(\Lambda_{ij}^{col}\right)$, respectively.

Definition 3.3. Let $C = \left(\frac{\beta_{ij}}{\alpha_{ij}}\right)$ be a bipartition matrix. The **(canonical) maximal row and column equalizers** of C are $\Lambda^{row}(C)$ and $\Lambda^{col}(C)$; the **(canonical) maximal row and column equalizations** of C are

$$C^{req} := \left(\frac{\mu_{\mathbf{\Lambda}_{ij}^{row}}(\beta_{ij})}{\mu_{\mathbf{\Lambda}_{ij}^{row}}(\alpha_{ij})}\right) \quad \text{and} \quad C^{ceq} := \left(\frac{\mu_{\mathbf{\Lambda}_{ij}^{col}}(\beta_{ij})}{\mu_{\mathbf{\Lambda}_{ij}^{col}}(\alpha_{ij})}\right).$$

The **(canonical) maximal equalizer of** C is $\Lambda(C)$; the **(canonical) maximal equalization of** C is

$$C^{eq} := \left(\frac{\mu_{\mathbf{\Lambda}_{ij}}(\beta_{ij})}{\mu_{\mathbf{\Lambda}_{ij}}(\alpha_{ij})}\right).$$

Example 3.4. For

$$C = \begin{pmatrix} \frac{1|2|3}{1|2|3} & \frac{1|2|3}{4|5|6} \\ \frac{4|5|6}{1|2|3} & \frac{45|6}{45|6} \end{pmatrix}$$

we have

$$\Lambda^{row}(C) = \begin{pmatrix} \{1,2\} & \{1,2\} \\ \{2\} & \{1\} \end{pmatrix} \text{ and } \Lambda^{col}(C) = \begin{pmatrix} \{1,2\} & \{2\} \\ \{1,2\} & \{1\} \end{pmatrix}$$

so that

$$C^{req} = \begin{pmatrix} \frac{1|2|3}{1|2|3} & \frac{1|2|3}{4|5|6} \\ \frac{45|6}{12|3} & \frac{45|6}{45|6} \end{pmatrix} \text{ and } C^{ceq} = \begin{pmatrix} \frac{1|2|3}{1|2|3} & \frac{12|3}{45|6} \\ \frac{4|5|6}{1|2|3} & \frac{45|6}{45|6} \end{pmatrix}.$$

Furthermore,

$$\Lambda(C) = \begin{pmatrix} \{2\} & \{2\} \\ \{2\} & \{1\} \end{pmatrix} \text{ and } C^{eq} = \begin{pmatrix} \frac{12|3}{12|3} & \frac{12|3}{45|6} \\ \frac{45|6}{12|3} & \frac{45|6}{45|6} \end{pmatrix}.$$

Remark 3.5. Since a maximal equalizer $\Lambda(C)$ is simultaneously a row and column equalizer, $\Lambda_{ij} \subseteq \Lambda^{row}_{ij} \cap \Lambda^{col}_{ij}$ for all (i,j). Furthermore, when an equalizer $\lambda(C) = (\lambda_{ij})$ is non-null, all entries in the corresponding equalization have constant length greater than 1, and consequently, $\lambda^{row}_{ij} \cap \lambda^{col}_{ij} \neq \varnothing$ for all (i,j). However, as our next example demonstrates, $\lambda^{row}_{ij} \cap \lambda^{col}_{ij} \neq \varnothing$ for all (i,j) does not imply the existence of a non-null equalizer.

Example 3.6. Although the maximal equalizer $\Lambda(C)$ of the bipartition matrix

$$C = \begin{pmatrix} \frac{1|2|3}{1|2|3} & \frac{1|3|2}{4|5|6} \\ \frac{4|5|6}{1|2|3} & \frac{5|4|6}{4|5|6} \end{pmatrix}$$

is null, we have

$$\Lambda^{row}(C) = \begin{pmatrix} \{1\} & \{1\} \\ \{2\} & \{2\} \end{pmatrix} \text{ and } \Lambda^{col}(C) = \begin{pmatrix} \{1,2\} & \{1,2\} \\ \{1,2\} & \{1,2\} \end{pmatrix}$$

so that $\Lambda^{row}_{ij} \cap \Lambda^{col}_{ij} \neq \varnothing$ for all (i,j).

Let C be a $q \times p$ bipartition matrix C with a non-null maximal equalizer $\Lambda(C) = (\Lambda_{ij})$. Write $\Lambda_{ij} = \{\Lambda^1_{ij} < \cdots < \Lambda^r_{ij}\}$, $\Lambda^{row}_{ij} = \{(\Lambda^{row}_{ij})^1 < \cdots < (\Lambda^{row}_{ij})^{s_i}\}$, and $\Lambda^{col}_{ij} = \{(\Lambda^{col}_{ij})^1 < \cdots < (\Lambda^{col}_{ij})^{t_j}\}$. By Remark 3.5, there is a subsequence $\{v^1_i, \ldots, v^{k_i}_i\} \subseteq \{1,2,\ldots,s_i\}$ such that $\Lambda_{i1} = \{(\Lambda^{row}_{i1})^{v^1_i} < \cdots < (\Lambda^{row}_{i1})^{v^{k_i}_i}\}$ for each $i \leq q$, and a subsequence $\{w^1_j, \ldots, w^{l_j}_j\} \subseteq \{1,2,\ldots,l_j\}$ such that $\Lambda_{1j} = \{(\Lambda^{col}_{1j})^{w^1_j} < \cdots < (\Lambda^{col}_{1j})^{w^{l_j}_j}\}$ for each $j \leq p$. The following proposition gives a method for constructing the maximal equalizer from the maximal row and maximal column equalizers:

Proposition 3.7. Given a $q \times p$ bipartition matrix C with a non-null maximal equalizer $\Lambda(C) = (\Lambda_{ij})$, write $\Lambda_{ij} = \{\Lambda_{ij}^1 < \cdots < \Lambda_{ij}^r\}$, $\Lambda_{i1} = \{(\Lambda_{i1}^{row})^{v_i^1} < \cdots < (\Lambda_{i1}^{row})^{v_i^{k_i}}\}$ for each $i \leq q$, and $\Lambda_{1j} = \{(\Lambda_{1j}^{col})^{w_j^1} < \cdots < (\Lambda_{1j}^{col})^{w_j^{l_j}}\}$ for each $j \leq p$. Then for all (i,j),

1. $r := k_i = l_j$ is constant and
2. $\Lambda_{ij} = \{(\Lambda_{ij}^{row})^{v_i^1} < \cdots < (\Lambda_{ij}^{row})^{v_i^r}\} = \{(\Lambda_{ij}^{col})^{w_j^1} < \cdots < (\Lambda_{ij}^{col})^{w_j^r}\}$.

Proof. First, $\Lambda_{ij}^{row} \cap \Lambda_{ij}^{col} \neq \varnothing$ for all (i,j) by Remark 3.5. Then $\Lambda_{11}^{row} \cap \Lambda_{11}^{col} = \{(\Lambda_{11}^{row})^{m_1} < \cdots < (\Lambda_{11}^{row})^{m_s}\} = \{(\Lambda_{11}^{col})^{n_1} < \cdots < (\Lambda_{11}^{col})^{n_s}\}$. Consider the corresponding subsets $\{(\Lambda_{1j}^{row})^{m_1} < \cdots < (\Lambda_{1j}^{row})^{m_s}\} \subseteq \Lambda_{1j}^{row}$ for each j and $\{(\Lambda_{i1}^{col})^{n_1} < \cdots < (\Lambda_{i1}^{col})^{n_s}\} \subseteq \Lambda_{i1}^{col}$ for each i, and let r_1 be the smallest positive integer such that

- $(\Lambda_{1j}^{row})^{m_{r_1}} \in \Lambda_{1j}^{col}$ for all j,
- $(\Lambda_{i1}^{col})^{n_{r_1}} \in \Lambda_{i1}^{row}$ for all i, and
- $(\Lambda_{ij}^{col})^{m_{r_1}} = (\Lambda_{ij}^{row})^{n_{r_1}}$ for all (i,j).

Then $n_{r_1} = v_i^1$ for all i, $m_{r_1} = w_j^1$ for all j, and $\Lambda_{ij}^1 = (\Lambda_{ij}^{row})^{v_i^1} = (\Lambda_{ij}^{col})^{w_j^1}$ for all (i,j).

Inductively, assume that for some $k > 1$, the set $\{\Lambda_{ij}^1 < \cdots < \Lambda_{ij}^{k-1}\}$ has been constructed for each (i,j). Let $r_k > r_{k-1}$ be the smallest integer such that

- $(\Lambda_{1j}^{row})^{m_{r_{k-1}}} < (\Lambda_{1j}^{row})^{m_{r_k}} \in \Lambda_{1j}^{col}$ for all j,
- $(\Lambda_{i1}^{col})^{n_{r_{k-1}}} < (\Lambda_{i1}^{col})^{n_{r_k}} \in \Lambda_{i1}^{row}$ for all i, and
- $(\Lambda_{ij}^{col})^{m_{r_k}} = (\Lambda_{ij}^{row})^{n_{r_k}}$ for all (i,j).

Then $n_{r_k} = v_i^k$ for all i, $m_{r_k} = w_j^k$ for all j, and $\Lambda_{ij}^k = (\Lambda_{ij}^{row})^{v_i^k} = (\Lambda_{ij}^{col})^{w_j^k}$ for all (i,j). The induction terminates after r steps and produces (2). Q.E.D.

Definition 3.8. Given a bipartition $c = \frac{B_1|\cdots|B_r}{A_1|\cdots|A_r}$ over $(\mathbf{a},\mathbf{b})$ with $r > 1$, let $\lambda \in \{1,2,\ldots,r-1\}$, let $\mathbf{a}_1|\cdots|\mathbf{a}_p := EP_{\mathbf{a}}(A_{\lambda+1} \cup \cdots \cup A_r)$, and let $\mathbf{b}_1|\cdots|\mathbf{b}_q := EP_{\mathbf{b}}(B_1 \cup \cdots \cup B_\lambda)$. For each $(i,j) \in \mathfrak{q} \times \mathfrak{p}$, let $B_1^j|\cdots|B_\lambda^j := (B_1 \cap \mathbf{b}_j)|\cdots|(B_\lambda \cap \mathbf{b}_j)$ and let $A_{\lambda+1}^i|\cdots|A_r^i := (A_{\lambda+1} \cap \mathbf{a}_i)|\cdots|(A_r \cap \mathbf{a}_i)$. The **transverse decomposition of** c **with respect to** λ is the formal product

$$A \cdot B = \begin{pmatrix} \frac{B_1^1|\cdots|B_\lambda^1}{A_1|\cdots|A_\lambda} \\ \vdots \\ \frac{B_1^q|\cdots|B_\lambda^q}{A_1|\cdots|A_\lambda} \end{pmatrix} \begin{pmatrix} \frac{B_{\lambda+1}|\cdots|B_r}{A_{\lambda+1}^1|\cdots|A_r^1} & \cdots & \frac{B_{\lambda+1}|\cdots|B_r}{A_{\lambda+1}^p|\cdots|A_r^p} \end{pmatrix}. \tag{3.4}$$

It follows immediately that

$$\begin{aligned} B_1|\cdots|B_\lambda &= B_1^1|\cdots|B_\lambda^1 \Cup \cdots \Cup B_1^q|\cdots|B_\lambda^q \text{ and} \\ A_{\lambda+1}|\cdots|A_r &= A_{\lambda+1}^1|\cdots|A_r^1 \Cup \cdots \Cup A_{\lambda+1}^p|\cdots|A_r^p. \end{aligned} \tag{3.5}$$

Definition 3.9. A pair of bipartition matrices $\left(A^{q\times 1}, B^{1\times p}\right)$ is a **Transverse Pair** (TP) if $A\cdot B$ is the transverse decomposition of some bipartition. A pair of bipartition matrices $(A^{q\times s}, B^{t\times p})$ is a **Block Transverse Pair** (BTP) if there exist $t\times s$ block decompositions $A=(A_{ij})$ and $B=(B_{ij})$ such that (A_{ij}, B_{ij}) is a TP for all (i,j). When (A,B) is a BTP, the value of the **formal product** $A\cdot B$ is the **formal matrix** $AB := (A_{ij}B_{ij})$, which is a bipartition matrix if $A_{ij}B_{ij}$ is a bipartition for all (i,j). When $C_1\cdots C_r$ is a formal product of bipartition matrices, (C_k, C_{k+1}) is a BTP for each k. A bipartition matrix C is **indecomposable** if $\Lambda(C)$ is null; otherwise C is **decomposable.** A **factorization** of C is a formal product $C_1\cdots C_r$ such that $C=C_1\cdots C_r$ and C_k is a bipartition matrix for all k; it is **indecomposable** if C_k is indecomposable for all k.

If $\left(A^{q\times 1}, B^{1\times p}\right)$ is a TP, formula (2.5) implies $(\#\mathbf{is}(A), \#\mathbf{os}(B)) = (p-1, q-1)$. Furthermore, if $(A^{q\times s}, B^{t\times p})$ is a BTP with $A=\left(A_{ij}^{q_{ij}\times 1}\right)$ and $B=\left(B_{ij}^{1\times p_{ij}}\right)$, then $(\#\mathbf{is}(A_{ij}), \#\mathbf{os}(B_{ij})) = (p_{ij}-1, q_{ij}-1)$ so that $\#\mathbf{is}(A) = \sum_j \#\mathbf{is}(A_{ij}) = \sum_j p_{ij} - s = p-s$ and $\#\mathbf{os}(B) = \sum_i \#\mathbf{os}(B_{ij}) = \sum_i q_{ij} - t = q-t$. Thus

$$(\#\mathbf{is}(A^{q\times s}), \#\mathbf{os}(B^{t\times p})) = (p-s, q-t). \tag{3.6}$$

When $C=C_1\cdots C_r$ define

$$\mathbf{is}(C) := \bigcup_{k\in\mathfrak{r}} \mathbf{is}(C_k) \quad\text{and}\quad \mathbf{os}(C) := \bigcup_{k\in\mathfrak{r}} \mathbf{os}(C_k).$$

Our next proposition is immediate.

Proposition 3.10. If a bipartition matrix C has a non-null equalizer $\lambda(C)=(\lambda_{ij})$, there is a factorization $C=C_1\cdots C_{r+1}$, where $r=\#\lambda_{ij}$. If $\lambda(C)=\Lambda(C)$, the factorization $C=C_1\cdots C_{r+1}$ is unique and indecomposable.

Example 3.11. The bipartition matrix

$$C=(c_{ij})=\begin{pmatrix} \frac{2|3|0|4}{1|3|2|0} & \frac{23|0|4}{7|5|6} \\ \frac{7|56|0}{13|0|2} & \frac{7|0|56}{5|7|6} \end{pmatrix}$$

over $(\mathbf{a}_1=\{1,2,3\}, \mathbf{a}_2=\{5,6,7\}, \mathbf{b}_1=\{2,3,4\}, \mathbf{b}_2=\{5,6,7\})$ has maximal equalizer

$$(\Lambda_{ij})=\begin{pmatrix} \{2\} & \{2\} \\ \{1\} & \{2\} \end{pmatrix}.$$

To compute the unique indecomposable factorization, apply the formulas in Definition 3.8 with $c=c_{ij}$ and $\lambda=\Lambda_{ij}$ for each (i,j):

$$c_{11}: \begin{cases} \mathbf{b}_1^1|\mathbf{b}_2^1 = 23|0 \\ \mathbf{a}_1^1|\mathbf{a}_2^1|\mathbf{a}_3^1 = 0|2|0 \end{cases} \Rightarrow \begin{array}{l} B_1^1|B_2^1 = 2|3,\ B_1^2|B_2^2 = 0|0 \\ A_3^1|A_4^1 = 0|0,\ A_3^2|A_4^2 = 2|0,\ A_3^3|A_4^2 = 0|0, \end{array}$$

$$c_{12}: \begin{cases} \mathbf{b}_1^1|\mathbf{b}_2^1 = 23|0 \\ \mathbf{a}_1^2|\mathbf{a}_2^2|\mathbf{a}_3^2 = 0|6|0 \end{cases} \Rightarrow \begin{array}{l} B_1^1|B_2^1 = 23|0,\ B_1^2|B_2^2 = 0|0 \\ A_3^1 = 0,\ A_3^2 = 6,\ A_3^3 = 0, \end{array}$$

$$c_{21}:\left\{\begin{array}{l}\mathbf{b}_1^2|\mathbf{b}_2^2|\mathbf{b}_3^2=0|07\\ \mathbf{a}_1^1|\mathbf{a}_2^1|\mathbf{a}_3^1=0|2|0\end{array}\right.\quad\Rightarrow\quad\begin{array}{l}B_1^1=0,\ B_1^2=0,\ B_1^3=7\\ A_2^1|A_3^1=0|0,\ A_2^2|A_3^2=0|2,\ A_2^3|A_3^3=0|0,\end{array}$$

$$c_{22}:\left\{\begin{array}{l}\mathbf{b}_1^2|\mathbf{b}_2^2|\mathbf{b}_3^2=0|0|7\\ \mathbf{a}_1^2|\mathbf{a}_2^2|\mathbf{a}_3^2=0|6|0\end{array}\right.\quad\Rightarrow\quad\begin{array}{l}B_1^1|B_2^1=0|0,\ B_1^2|B_2^2=0|0,\ B_1^3|B_2^3=7|0\\ A_3^1=0,\ A_3^2=6,\ A_3^3=0.\end{array}$$

Then

$$(A_{ij}B_{ij})=\left(\begin{array}{cc}\begin{pmatrix}\frac{2|3}{1|3}\\ \frac{0|0}{1|3}\end{pmatrix}\begin{pmatrix}\frac{0|4}{0|0}&\frac{0|4}{2|0}&\frac{0|4}{0|0}\end{pmatrix} & \begin{pmatrix}\frac{23|0}{7|5}\\ \frac{0|0}{7|5}\end{pmatrix}\begin{pmatrix}\frac{4}{0}&\frac{4}{6}&\frac{4}{0}\end{pmatrix}\\ \begin{pmatrix}\frac{0}{13}\\ \frac{0}{13}\\ \frac{7}{13}\end{pmatrix}\begin{pmatrix}\frac{56|0}{0|0}&\frac{56|0}{0|2}&\frac{56|0}{0|0}\end{pmatrix} & \begin{pmatrix}\frac{0|0}{5|7}\\ \frac{0|0}{5|7}\\ \frac{7|0}{5|7}\end{pmatrix}\begin{pmatrix}\frac{56}{0}&\frac{56}{6}&\frac{56}{0}\end{pmatrix}\end{array}\right)$$

and

$$C=AB=\begin{pmatrix}\frac{2|3}{1|3}&\frac{23|0}{7|5}\\ \frac{0|0}{1|3}&\frac{0|0}{7|5}\\ \frac{0}{13}&\frac{0|0}{5|7}\\ \frac{0}{13}&\frac{0|0}{5|7}\\ \frac{7}{13}&\frac{7|0}{5|7}\end{pmatrix}\begin{pmatrix}\frac{0|4}{0|0}&\frac{0|4}{2|0}&\frac{0|4}{0|0}&\frac{4}{0}&\frac{4}{6}&\frac{4}{0}\\ \frac{56|0}{0|0}&\frac{56|0}{0|2}&\frac{56|0}{0|0}&\frac{56}{0}&\frac{56}{6}&\frac{56}{0}\end{pmatrix}.$$

The matrix dimensions $(q,s)=(5,2)$ and $(t,p)=(2,6)$ together with $(\#\mathbf{is}(A),\ \#\mathbf{os}(B))=(4,3)$ verify Formula (3.6). To recover the matrix C, apply the formulas in (3.5) and obtain $c_{ij}=A_{ij}B_{ij}$. Then $\mathbf{is}(C)=\{1,2,3\}\cup\{5,6,7\}=\{1,3,5,7\}\cup\{2,6\}=\mathbf{is}(A)\cup\mathbf{is}(B)$, and $\mathbf{os}(C)=\{2,3,4\}\cup\{5,6,7\}=\{2,3,7\}\cup\{4,5,6\}=\mathbf{os}(A)\cup\mathbf{os}(B)$ as required. Note that the second row of A factors as

$$\begin{pmatrix}\frac{0|0}{1|3}&\frac{0|0}{7|5}\end{pmatrix}=\begin{pmatrix}\frac{0}{1}&\frac{0}{7}\end{pmatrix}\begin{pmatrix}\frac{0}{0}&\frac{0}{3}&\frac{0}{5}&\frac{0}{0}\end{pmatrix}.$$

Thus the rows (and columns) of an indecomposable matrix may be decomposable.

The indecomposable factorization of a bipartition is given by

Algorithm 1. Let $\frac{B_1|\cdots|B_r}{A_1|\cdots|A_r}$ be a bipartition with $r>1$.
For $k=1$ to r :
Let $\mathbf{a}_{k1}|\cdots|\mathbf{a}_{kp_k}:=EP_{A_1\cup\cdots\cup A_k}A_k$.
Let $\mathbf{b}_{k1}|\cdots|\mathbf{b}_{kq_k}:=EP_{B_k\cup\cdots\cup B_r}B_k$.
Form the $q_k\times p_k$ elementary matrix $C_k=\left(\frac{\mathbf{b}_{ki}}{\mathbf{a}_{kj}}\right)$.
Obtain the unique factorization

$$\frac{B_1|\cdots|B_r}{A_1|\cdots|A_r}=C_1\cdots C_r. \tag{3.7}$$

Example 3.12. To compute the indecomposable factorization of the bipartition $\frac{56|7|8}{1|23|4}$, set

$$\begin{array}{cc} \mathbf{a}_{11}=1 & \mathbf{b}_{11}|\mathbf{b}_{12}|\mathbf{b}_{13}=56|0|0 \\ \mathbf{a}_{21}|\mathbf{a}_{22}=0|23 & \mathbf{b}_{21}|\mathbf{b}_{22}=7|0 \\ \mathbf{a}_{31}|\mathbf{a}_{32}|\mathbf{a}_{33}|\mathbf{a}_{34}=0|0|0|4 & \mathbf{b}_{31}=8. \end{array}$$

Then

$$\frac{56|7|8}{1|23|4}=\begin{pmatrix} \frac{56}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix}\begin{pmatrix} \frac{7}{0} & \frac{7}{23} \\ \frac{0}{0} & \frac{0}{23} \end{pmatrix}\begin{pmatrix} \frac{8}{0} & \frac{8}{0} & \frac{8}{0} & \frac{8}{4} \end{pmatrix}=\mathbf{C}_1\mathbf{C}_2\mathbf{C}_3.$$

The fact that $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{C}_3$ are elementary matrices illustrates the fact that the indecomposable factorization of a bipartition is an elementary product.

3.3 Partitioning the Entries of a Bipartition Matrix

An essential action on a bipartition matrix C is to partition its entries. Such a partitioning is called a *partitioning action on* C. Let $\frac{\beta}{\alpha}=\frac{B_1|\cdots|B_r}{A_1|\cdots|A_r}$ be a bipartition and let $\left(M^k,N^k\right)\subseteq(A_k,B_k)$ for some k. The *(partitioning) action of* $\left(M^k,N^k\right)$ *on* $\frac{\beta}{\alpha}$ is the bipartition

$$\partial_{M^k,N^k}\left(\frac{\beta}{\alpha}\right):=\frac{\partial_{N^k}\beta}{\partial_{M^k}\alpha} \tag{3.8}$$

(see 2.1). The pair $\left(M^k,N^k\right)$ is *extreme* if M^k and N^k are extreme, and is *strongly extreme* when $\left(M^k,N^k\right)=(\varnothing,\varnothing)$ or $\left(M^k,N^k\right)=(A_k,B_k)$. When $\left(M^1,N^1\right)=(\varnothing,\varnothing)$ or $(M^r,N^r)=(A_r,B_r)$, we denote the special strongly extreme cases ∂_{M^1,N^1} and ∂_{M^r,N^r} by η_1 and η_2 respectively; thus

$$\eta_1\left(\frac{\beta}{\alpha}\right)=\frac{0|B_1|\cdots|B_r}{0|A_1|\cdots|A_r}\quad\text{and}\quad\eta_2\left(\frac{\beta}{\alpha}\right)=\frac{B_1|\cdots|B_r|0}{A_1|\cdots|A_r|0}.$$

Given a positive integer $k<r$, let

$$\frac{\beta}{\alpha}[k]:=\frac{\beta[k]}{\alpha[k]}=\frac{B_1|\cdots|B_k\cup B_{k+1}|\cdots|B_r}{A_1|\cdots|A_k\cup A_{k+1}|\cdots|A_r};$$

then $\partial_{A_k,B_k}\left(\frac{\beta}{\alpha}[k]\right)=\frac{\beta}{\alpha}$.

Definition 3.13. Let $C=(c_{ij})=\left(\frac{B_{ij}^1|\cdots|B_{ij}^{r_{ij}}}{A_{ij}^1|\cdots|A_{ij}^{r_{ij}}}\right)$ be a $q\times p$ bipartition matrix, let $\mathcal{U}\subseteq\mathfrak{q}\times\mathfrak{p}$, choose a family of pairs

$$(\mathbf{M},\mathbf{N})=\left\{\left(M_{ij}^{k_{ij}},N_{ij}^{k_{ij}}\right)\subseteq\left(A_{ij}^{k_{ij}},B_{ij}^{k_{ij}}\right)\right\}_{(i,j)\in\mathcal{U}},$$

and define

$$\partial_{ij}\left(c_{ij}\right):=\begin{cases}\partial_{M_{ij}^{k_{ij}},N_{ij}^{k_{ij}}}\left(c_{ij}\right), & \text{if } (i,j)\in\mathcal{U},\\ \mathbf{Id}, & \text{otherwise.}\end{cases} \tag{3.9}$$

The **(partitioning) action of** $(\mathbf{M},\mathbf{N})$ **on** C is the matrix $\partial_{\mathbf{M},\mathbf{N}}(C)=(\partial_{ij}(c_{ij}))$. When $\mathcal{U}=\varnothing$ there is the **trivial partitioning action** $\mathbf{Id}(C)=C$. Given $(i,j)\in\mathfrak{q}\times\mathfrak{p}$, let $\mathcal{U}_{i*}:=\{i\}\times\mathfrak{p}$, $\mathcal{U}_{*j}:=\mathfrak{q}\times\{j\}$, and $\mathcal{V}_{ij}\subseteq\mathfrak{q}\times\mathfrak{p}\setminus(\mathcal{U}_{i*}\cup\mathcal{U}_{*j})$. Then $\partial_{\mathbf{M},\mathbf{N}}(C)$ is a/an

- (i,j) **entry action** if $\mathcal{U}=\{(i,j)\}$, M_{ij}^{k} is not extreme when $q>1$, and N_{ij}^{k} is not extreme when $p>1$.
- **left (respt. right) row** i **action** if $\mathcal{U}=\mathcal{U}_{i*}$, $p\geq 2$, $N_{ij}^{k_{ij}}=\varnothing$ for each j (respt. $N_{ij}^{k_{ij}}=B_{ij}^{k_{ij}}$ for each j), and M_{ij}^{k} is not extreme for each j when $q>1$.
- **row** i **action** if $\partial_{\mathbf{M},\mathbf{N}}(C)$ is a left or right row i action.
- **left (respt. right) column** j **action** if $\mathcal{U}=\mathcal{U}_{*j}$, $q\geq 2$, $M_{ij}^{k_{ij}}=\varnothing$ for each i (respt. $M_{ij}^{k_{ij}}=A_{ij}^{k_{ij}}$ for each i), and N_{ij}^{k} is not extreme for each i when $p>1$.
- **column** j **action** if $\partial_{\mathbf{M},\mathbf{N}}(C)$ is a left or right column j action.
- **row** i/**column** j **action** if $\partial_{\mathbf{M},\mathbf{N}}(C)$ is a row i action for $\mathcal{U}=\mathcal{U}_{i*}$, $\partial_{\mathbf{M},\mathbf{N}}(C)$ is a column j action for $\mathcal{U}=\mathcal{U}_{*j}$, the **pivoting pair** $(M_{ij}^{k_{ij}},N_{ij}^{k_{ij}})$ common to both actions is not strongly extreme, and for some $\mathcal{V}_{ij}$ there exist strongly extreme pairs $\{(M_{st}^{k_{st}},N_{st}^{k_{st}})\}_{(s,t)\in\mathcal{V}_{ij}}$ such that $\partial_{\mathbf{M},\mathbf{N}}(C)$ is decomposable for $\mathcal{U}=\mathcal{U}_{i*}\cup\mathcal{U}_{*j}\cup\mathcal{V}_{ij}$.

Hence, a left (respt. right) row i action inserts a null partition block to the left (respt. right) of $B_{ij}^{k_{ij}}$ for each j, and dually for column j actions. Thus left (and right) row or column actions are always decomposable and the pivoting pair in a row i/column j action is extreme.

Example 3.14. Let

$$C=\begin{pmatrix}\frac{0|1}{0|1} & \frac{0|1}{3|0}\\ \frac{3}{1} & \frac{3|0}{3|0}\end{pmatrix},$$

$(\mathbf{M}_1,\mathbf{N}_1)=\{(\{1\}_{11},\varnothing_{11}),(\{1\}_{21},\varnothing_{21})\}$, and $(\mathbf{M}_2,\mathbf{N}_2)=\{(\{1\}_{21},\varnothing_{21}),(\{3\}_{22},\varnothing_{22})\}$. Then $\partial_{\mathbf{M}_1,\mathbf{N}_1}(C)$ is the right column action

$$\begin{pmatrix}\frac{0|0|1}{0|1|0}\\ \frac{0|3}{1|0}\end{pmatrix}=\begin{pmatrix}\frac{0|0}{0|1}\\ \frac{0|0}{0|1}\\ \frac{0}{1}\\ \frac{0}{1}\end{pmatrix}\begin{pmatrix}\frac{1}{0} & \frac{1}{0}\\ \frac{3}{0} & \frac{3}{0}\end{pmatrix},$$

$\partial_{\mathbf{M}_2,\mathbf{N}_2}(C)$ is the left row action

$$\begin{pmatrix}\frac{0|3}{1|0} & \frac{0|3|0}{3|0|0}\end{pmatrix}=\begin{pmatrix}\frac{0}{1} & \frac{0}{3}\\ \frac{0}{1} & \frac{0}{3}\end{pmatrix}\begin{pmatrix}\frac{3}{0} & \frac{3}{0} & \frac{3|0}{0|0} & \frac{3|0}{0|0}\end{pmatrix},$$

and $\partial_{(\mathbf{M}_1,\mathbf{N}_1)\cup(\mathbf{M}_2,\mathbf{N}_2)}(C)$ with $\mathcal{V}_{21}=\varnothing$ is the row 2/column 1 action

$$\begin{pmatrix}\frac{0|0|1}{0|1|0} & \frac{0|1}{3|0}\\ \frac{0|3}{1|0} & \frac{0|3|0}{3|0|0}\end{pmatrix}=\begin{pmatrix}\frac{0|0}{0|1} & \frac{0}{3}\\ \frac{0|0}{0|1} & \frac{0}{3}\\ \frac{0}{1} & \frac{0}{3}\\ \frac{0}{1} & \frac{0}{3}\end{pmatrix}\begin{pmatrix}\frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0}\\ \frac{3}{0} & \frac{3}{0} & \frac{3|0}{0|0} & \frac{3|0}{0|0}\end{pmatrix}.$$

3.4 Coherent Bipartition Matrices

In this subsection we define the *coherence* of a bipartition matrix in terms of $\Delta_P^{(k)}$. Coherence is fundamentally important in our development because the *dimension* of a coherent bipartition matrix is under control.

Let $C = (\beta_{ij}/\alpha_{ij})$ be a $q \times p$ bipartition matrix. Recall that a row equalization of C has equal numerators in each row (3.1) and a column equalization of C has equal denominators in each column (3.2). Consider the maximal row and column equalizers $\Lambda^{row}(C) = \left(\Lambda_{ij}^{row}\right)$ and $\Lambda^{col}(C) = \left(\Lambda_{ij}^{col}\right)$. The i^{th} *input and* j^{th} *output partitions of* C are

$$\overset{\wedge}{\alpha}_i(C) := \mu_{\Lambda_{i1}^{row}}(\alpha_{i1}) \Cup \cdots \Cup \mu_{\Lambda_{ip}^{row}}(\alpha_{ip}) \in P'(\mathbf{is}(C))$$
$$\overset{\vee}{\beta}_j(C) := \mu_{\Lambda_{1j}^{col}}(\beta_{1j}) \Cup \cdots \Cup \mu_{\Lambda_{qj}^{col}}(\beta_{qj}) \in P'(\mathbf{os}(C)),$$

which may vary in length with i and j. Consider the maximal equalization C^{eq} with respect to the maximal equalizer $\Lambda(C) = (\Lambda_{ij})$. The *input and output partitions of* C^{eq} are

$$\overset{\wedge}{eq}(C) := \mu_{\Lambda_{11}}(\alpha_{11}) \Cup \cdots \Cup \mu_{\Lambda_{1p}}(\alpha_{1p}) \in P'(\mathbf{is}(C))$$
$$\overset{\vee}{eq}(C) := \mu_{\Lambda_{11}}(\beta_{11}) \Cup \cdots \Cup \mu_{\Lambda_{q1}}(\beta_{q1}) \in P'(\mathbf{os}(C)).$$

Then $\overset{\wedge}{eq}(C)$ is obtained by merging the denominators in the first (or any) row of C^{eq} and $\overset{\wedge}{\alpha}_i(C)$ is obtained by appropriately partitioning $\overset{\wedge}{eq}(C)$ (and dually for $\overset{\vee}{eq}(C)$). When $\mathbf{is}(C)$ and $\mathbf{os}(C)$ are non-empty, $\pi\overset{\wedge}{eq}(C)$ is identified with a cell of $P_{\#\mathbf{is}(C)}$. In particular, when C is indecomposable (in which case $\Lambda(C)$ is null), $\pi\overset{\wedge}{eq}(C)$ is identified with the top dimensional cell of $P_{\#\mathbf{is}(C)}$.

The *input and output products of* C are

$$\overset{\wedge}{\mathbf{e}}(C) := \pi\overset{\wedge}{\alpha}_q(C) \times \cdots \times \pi\overset{\wedge}{\alpha}_1(C) \sqsubseteq P(\mathbf{is}(C))^{\times q}$$
$$\overset{\vee}{\mathbf{e}}(C) := \pi\overset{\vee}{\beta}_1(C) \times \cdots \times \pi\overset{\vee}{\beta}_p(C) \sqsubseteq P(\mathbf{os}(C))^{\times p}.$$

Example 3.15. Let $C = \begin{pmatrix} \frac{1|2|3|4|5}{0|0|0|1|2} & \frac{1|2|34|5}{0|0|3|4} & \frac{12|4|3|5}{0|0|5|6} \end{pmatrix}$. Then $\Lambda(C) = (\{2,4\}\ \{2,3\}\ \{1,3\})$ and $C^{eq} = \begin{pmatrix} \frac{12|34|5}{0|1|2} & \frac{12|34|5}{0|3|4} & \frac{12|34|5}{0|5|6} \end{pmatrix}$ so that $\overset{\wedge}{eq}(C) = \overset{\wedge}{\alpha}_1(C) = 0|135|246$, $\overset{\wedge}{\mathbf{e}}(C) = \pi\overset{\wedge}{\alpha}_1(C) = 135|246$, $\overset{\vee}{eq}(C) = 12|34|5$, and $\overset{\vee}{\mathbf{e}}(C) = 1|2|3|4|5 \times 1|2|34|5 \times 12|4|3|5 \sqsubseteq_{diag} \Delta_P^{(2)}(\overset{\vee}{eq}(C))$.

Consider the matrix $D = \begin{pmatrix} \frac{1|2|3|4|5}{0|0|0|1|2} & \frac{1|2|34|5}{0|0|3|4} & \frac{12|4|35}{0|0|56} \end{pmatrix}$ obtained from C via the replacement $\frac{12|4|3|5}{0|0|5|6} \leftarrow \frac{12|4|35}{0|0|56}$; then $C = \partial_{\{5\}_{13},\{3\}_{13}}(D)$. Furthermore, $\Lambda(D) = (\{2\}\ \{2\}\ \{1\})$ and $D^{eq} = \begin{pmatrix} \frac{12|345}{0|12} & \frac{12|345}{0|34} & \frac{12|345}{0|56} \end{pmatrix}$ so that $\overset{\wedge}{eq}(D) = 0|123456$, $\overset{\wedge}{\mathbf{e}}(D) = 123456$, $\overset{\vee}{eq}(D) = 12|345$, and $\overset{\vee}{\mathbf{e}}(D) = 1|2|3|4|5 \times 1|2|34|5 \times 12|4|35 \sqsubseteq_{diag} \Delta_P^{(2)}(\overset{\vee}{eq}(D))$. Let $e = \overset{\vee}{eq}(D)$; then $\overset{\vee}{eq}(C) \sqsubseteq \partial e$. Set $\mathbf{x} = \overset{\vee}{\mathbf{e}}(C)$; by uniqueness in Proposition 2.11, $\mathbf{x}_{33} = \overset{\vee}{\mathbf{e}}(D)$ is the unique diagonal component of $\overset{\vee}{eq}(D)$ that can be obtained from $\overset{\vee}{\mathbf{e}}(C)$ by a single factor replacement.

Definition 3.16. A $q \times p$ bipartition matrix C is

- **row precoherent** if $\mathbf{is}(C) \neq \varnothing$ and
$$\overset{\wedge}{\mathbf{e}}(C) \sqsubseteq \Delta_P^{(q-1)}(\pi \overset{\wedge}{eq}(C)); \tag{3.10}$$
- **maximally row precoherent** if $\mathbf{is}(C) \neq \varnothing$ and
$$\overset{\wedge}{\mathbf{e}}(C) \sqsubseteq_{diag} \Delta_P^{(q-1)}(\pi \overset{\wedge}{eq}(C)); \tag{3.11}$$
- **column precoherent** if $\mathbf{os}(C) \neq \varnothing$ and
$$\overset{\vee}{\mathbf{e}}(C) \sqsubseteq \Delta_P^{(p-1)}(\pi \overset{\vee}{eq}(C)); \tag{3.12}$$
- **maximally column precoherent** if $\mathbf{os}(C) \neq \varnothing$ and
$$\overset{\vee}{\mathbf{e}}(C) \sqsubseteq_{diag} \Delta_P^{(p-1)}(\pi \overset{\vee}{eq}(C)); \tag{3.13}$$
- **(maximally) precoherent** if C is (maximally) row and (maximally) column precoherent;
- **(maximally) row coherent** if $\mathbf{is}(C) = \varnothing$ or its indecomposable factors are (maximally) row precoherent;
- **(maximally) column coherent** if $\mathbf{os}(C) = \varnothing$ or its indecomposable factors are (maximally) column precoherent;
- **(maximally) coherent** if C is (maximally) row and (maximally) column coherent;
- **(maximally) totally coherent** if C is (maximally) coherent and its indecomposable factors have (maximally) coherent rows, columns, and entries.

Null bipartition matrices are maximally coherent. An indecomposable bipartition matrix is coherent if and only if it is precoherent, but a precoherent decomposable bipartition matrix is not necessarily coherent.

Example 3.17. The maximally precoherent matrix

$$\begin{pmatrix} \frac{0|1}{12|0} & \frac{0|1}{45|0} \end{pmatrix} = \begin{pmatrix} \frac{0}{12} & \frac{0}{45} \\ \frac{0}{12} & \frac{0}{45} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} \end{pmatrix} = C_1 C_2$$

is incoherent because its factor C_1 is incoherent. Since C_1 has coherent entries but fails to be coherent, coherence is a global property of bipartition matrices.

Example 3.18. The matrices

$$C = \begin{pmatrix} \frac{1|2|3|4|5}{0|0|0|1|2} & \frac{1|2|34|5}{0|0|3|4} & \frac{12|4|3|5}{0|0|5|6} \end{pmatrix} \text{ and } D = \begin{pmatrix} \frac{1|2|3|4|5}{0|0|0|1|2} & \frac{1|2|34|5}{0|0|3|4} & \frac{12|4|35}{0|0|56} \end{pmatrix}$$

discussed in Example 3.15 are maximally column precoherent since $\overset{\vee}{\mathbf{e}}(C) \sqsubseteq_{diag} \Delta_{P_5}^{(2)}(\overset{\vee}{eq}(C))$ and $\overset{\vee}{\mathbf{e}}(D) \sqsubseteq_{diag} \Delta_P^{(2)}(\overset{\vee}{eq}(D))$. Furthermore, by Proposition 2.11, part (2), D is the unique maximally column precoherent matrix such that $\partial_{\mathbf{M},\mathbf{N}}(D) = C$ for some $(\mathbf{M}, \mathbf{N})$.

Proposition 3.19. A coherent bipartition matrix is precoherent.

Proof. When i/o sets are empty, the proof is trivial. Let C be a $q \times p$ coherent bipartition matrix with indecomposable factorization $C = C_1 \cdots C_r$ and let $\mathbf{is}(C) \neq \varnothing$. We claim that C is row precoherent. If $r = 1$ or $q = 1$ there is nothing to prove. So assume $r, q > 1$. By hypothesis C_k is coherent for each k. The input product $\overset{\wedge}{\mathbf{e}}(C_k)$, which has the form

$$\overset{\wedge}{\mathbf{e}}(C_k) = (e_{k_q} \times \cdots \times e_{k_{q-1}+1}) \times \cdots \times (e_{k_1} \times \cdots \times e_1),$$

is a subcomplex of $\Delta_P^{(k_q-1)}\left(P_{\#\mathbf{is}(C_k)}\right)$ by the row coherence of C_k. On the other hand, $\overset{\wedge}{\mathbf{e}}(C)$ has the form

$$\overset{\wedge}{\mathbf{e}}(C) = E_q \times \cdots \times E_1,$$

where $E_i = \bar{e}_{i,1} \times \cdots \times \bar{e}_{i,r}$ and $\bar{e}_{i,k} \sqsubseteq P_{\#\mathbf{is}(C_k)}$. Now for all i and k, each factor e_ν of $e_{k_i} \times \cdots \times e_{k_{i-1}+1}$ is a cell of $\bar{e}_{i,k}$, and an analysis of Δ_P reveals that $e_{k_i} \times \cdots \times e_{k_{i-1}+1} \sqsubseteq \Delta_P^{(k_i - k_{i-1}-1)}(\bar{e}_{i,k})$ and

$$\bar{e}_{q,k} \times \cdots \times \bar{e}_{1,k} \sqsubseteq \Delta_P^{(q-1)}(P_{\#\mathbf{is}(C_k)}). \tag{3.14}$$

Since Δ_p acts multiplicatively on product cells, the inclusion in (3.14) yields

$$\begin{aligned}\overset{\wedge}{\mathbf{e}}(C) &= E_q \times \cdots \times E_1 = (\bar{e}_{q,1} \times \cdots \times \bar{e}_{q,r}) \times \cdots \times (\bar{e}_{1,1} \times \cdots \times \bar{e}_{1,r}) \\ &\sqsubseteq \Delta_P^{(q-1)}\left(P_{\#\mathbf{is}(C_1)} \times \cdots \times P_{\#\mathbf{is}(C_r)}\right) = \Delta_P^{(q-1)}\left(\pi \overset{\wedge}{eq}(C)\right),\end{aligned}$$

and it follows that C is row precoherent. A dual argument for column precoherence completes the proof. Q.E.D.

The class of coherent elementary matrices is highly constrained.

Proposition 3.20. Let $\mathbf{C}$ be a $q \times p$ elementary matrix with $pq > 1$.

1. If $\mathbf{is}(\mathbf{C}) \neq \varnothing$ and $q \geq 2$, then $\mathbf{C}$ is row coherent if and only if $\mathbf{C}$ is maximally row coherent if and only if $\#\mathbf{is}(\mathbf{C}) = 1$;

2. If $\mathbf{os}(\mathbf{C}) \neq \varnothing$ and $p \geq 2$, then $\mathbf{C}$ is column coherent if and only if $\mathbf{C}$ is maximally column coherent if and only if $\#\mathbf{os}(\mathbf{C}) = 1$;

3. If $\mathbf{is}(\mathbf{C}), \mathbf{os}(\mathbf{C}) \neq \varnothing$, and $p, q \geq 2$, then $\mathbf{C}$ is coherent if and only if $\mathbf{C}$ is maximally coherent if and only if $\#\mathbf{is}(\mathbf{C}) = \#\mathbf{os}(\mathbf{C}) = 1$.

Proof. (1) Since $\mathbf{C}$ is elementary, $\pi \overset{\wedge}{eq}(\mathbf{C}) = \pi \overset{\wedge}{\alpha}_i(\mathbf{C}) = \mathbf{is}(\mathbf{C})$ for all i. But $q \geq 2$ implies $\#\mathbf{is}(\mathbf{C}) = 1$ if and only if

$$\overset{\wedge}{\mathbf{e}}(\mathbf{C}) = \underbrace{\mathbf{is}(\mathbf{C}) \times \cdots \times \mathbf{is}(\mathbf{C})}_{q \text{ factors}} = \Delta_P^{(q-1)}(\mathbf{is}(\mathbf{C}))$$

if and only if $\mathbf{C}$ is maximally row coherent. The proof of (2) is completely dual and the proof of (3) follows immediately from (1) and (2). Q.E.D.

In fact, every coherent elementary matrix is maximally totally coherent.

Now if $\mathbf{C}$ is a $q \times p$ (maximally) coherent elementary matrix such that $q \geq 2$ and $\mathbf{is}\,(\mathbf{C}) \neq \varnothing$, then $\#\mathbf{is}\,(\mathbf{C}) = 1$ by Proposition 3.20. Since $\mathbf{C}$ has constant denominators in each column, row coherence and $\#\mathbf{is}\,(\mathbf{C}) = 1$ imply that denominators in exactly one column of $\mathbf{C}$ are non-empty constant singletons and all other denominators are null. Dually, if $p \geq 2$ and $\mathbf{os}\,(\mathbf{C}) \neq \varnothing$, numerators in exactly one row of $\mathbf{C}$ are non-empty constant singletons and all other numerators are null. Thus if $p, q \geq 2$ and $\mathbf{is}\,(\mathbf{C}) = \mathbf{os}\,(\mathbf{C}) = \{1\}$, exactly one row of $\mathbf{C}$ has the form $\left(\frac{1}{0} \cdots \frac{1}{0}\ \frac{1}{1}\ \frac{1}{0} \cdots \frac{1}{0}\right)$, exactly one column of $\mathbf{C}$ has the form $\left(\frac{0}{1} \cdots \frac{0}{1}\ \frac{1}{1}\ \frac{0}{1} \cdots \frac{0}{1}\right)^T$, and all other entries are null. For example, when $(p, q) = (3, 4)$ we have

$$\mathbf{C} = \begin{pmatrix} \frac{0}{0} & \frac{0}{0} & \frac{0}{1} & \frac{0}{0} \\ \frac{1}{0} & \frac{1}{0} & \frac{1}{1} & \frac{1}{0} \\ \frac{0}{0} & \frac{0}{0} & \frac{0}{1} & \frac{0}{0} \end{pmatrix}.$$

By similar calculations, if $p, q \geq 2$, $\mathbf{is}\,(\mathbf{C}) = \varnothing$ and $\mathbf{os}\,(\mathbf{C}) = \{1\}$, exactly one row of $\mathbf{C}$ has the form $\left(\frac{1}{0} \cdots \frac{1}{0}\right)$ and all other entries are null.

Bipartitions viewed as 1×1 matrices are (trivially) precoherent but not necessarily coherent. Indeed, the set of coherent bipartitions is highly constrained.

Proposition 3.21. A bipartition $c = \frac{B_1|\cdots|B_r}{A_1|\cdots|A_r}$ is coherent whenever one of the following conditions is satisfied:

1. $r = 1$.
2. $A_1 = \cdots = A_{i-1} = B_{i+1} = \cdots = B_r = \varnothing$ for $r \geq 2$ and some $i \leq r$.
3. $\#A_1, \ldots, \#A_{r-1}, \#B_2, \ldots, \#B_r \in \{0, 1\}$ for $r \geq 2$.

Proof. When $r = 1$ there is nothing to prove. When $r = 2$, consider the indecomposable factorization $c = C_1 C_2$. Since C_1 and C_2 are elementary column and row matrices, respectively, their coherence follows easily from (2) and (3) independently. Hence c is coherent. Conversely, if $\#A_1 \geq 2$ and $\#B_2 \geq 1$, the partition $EP_{B_1 \cup B_2} B_1$ has $k \geq 2$ blocks so that C_1 has k rows. Since the k-fold product cell $\overset{\wedge}{\mathbf{e}}\,(C_1) = A_1 \times \cdots \times A_1 \not\sqsubseteq \Delta_P^{(k-1)}(\pi \overset{\wedge}{eq}\,(C_1))$, it follows that C_1 is incoherent and so is c. On the other hand, $\#A_1 \geq 1$ and $\#B_2 \geq 2$ imply that C_2 is incoherent by a symmetric argument. Cases with $r > 2$ follow by similar arguments. Q.E.D.

Note that Condition (2) reduces to $A_1 = \cdots = A_{r-1} = \varnothing$ when $i = r$, and in particular, to $A_k = \varnothing$ for all k when $\mathbf{is}\,(c) = \varnothing$; dually, Condition (2) reduces to $B_2 = \cdots = B_r = \varnothing$ when $i = 1$, and in particular, to $B_k = \varnothing$ for all k when $\mathbf{os}\,(c) = \varnothing$. Thus every bipartition with a null input set or a null output set is coherent. A coherent bipartition satisfying Condition (2) has the form

$$\frac{B_1|\cdots|B_i|0|\cdots|0}{0|\cdots|0|A_i|\cdots|A_r}.$$

An example of a coherent bipartition satisfying Condition (3) is

$$\frac{123|0|0|4|5}{1|0|2|0|345}.$$

3.5 Coheretization

In this subsection we discuss conditions under which a sequence of partitioning actions transforms a bipartition matrix into a precoherent matrix.

Definition 3.22. A bipartition matrix C is **(pre)coheretizable** if there exists a sequence of partitioning operators $\partial_{\mathbf{M_1},\mathbf{N_1}},\ldots,\partial_{\mathbf{M}_k,\mathbf{N}_k}$ such that

$$C' = \partial_{\mathbf{M}_k,\mathbf{N}_k}\cdots\partial_{\mathbf{M_1},\mathbf{N_1}}(C)$$

is (pre)coherent; the matrix C' is a **(pre)coheretization of** C.

Coherent matrices C are trivially coheretizable via the identity partitioning action $\mathbf{Id}(C)$. An indecomposable precoheretization is a coheretization.

Consider a $q\times p$ semi-null elementary matrix C with $\mathbf{is}(C)\neq\varnothing$, an augmented partition $\alpha\in P'(\#\mathbf{is}(C))$, and a diagonal component $X_q := x_q\times\cdots\times x_1 \sqsubseteq_{diag} \Delta_P^{(q-1)}\left(P_{\#\mathbf{is}(C)}\right)$. Since the rows of C are equal and elementary, there is a row i partitioning action $\partial_{\mathbf{M}_i,\mathbf{N}_i}(C)$ for each i such that $\pi\overset{\wedge}{\alpha}_i(\partial_{\mathbf{M}_i,\mathbf{N}_i}(C)) = x_i$. Hence $C' := \partial_{\mathbf{M}_q,\mathbf{N}_q}\cdots\partial_{\mathbf{M_1},\mathbf{N_1}}(C)$ is a precoheretization such that $\overset{\wedge}{\mathbf{e}}(C') = X_q$. In fact, C' is a coheretization: When $q=2$, the partition factors of $X_2 = x_2\times x_1$ are distinct and $\Lambda(C')$ is null. When $q=3$, consider $X_3 = x_3\times x_2\times x_1 \sqsubseteq_{diag} (\Delta_P\times\mathbf{1})\,\Delta_P\left(P_{\#\mathbf{is}(C)}\right)$. Let D be the matrix obtained from C by deleting the first row, and let $D' = \partial_{\mathbf{M}_3,\mathbf{N}_3}\partial_{\mathbf{M_2},\mathbf{N_2}}(D)$. Then $\overset{\wedge}{\mathbf{e}}(D') = x_3\times x_2$ and $\Lambda(D')$ is null. Hence $\Lambda(C')$ is null, and so on inductively. It follows that C' is indecomposable. The dual statement with $\mathbf{os}(C)\neq\varnothing$ follows by a similar argument.

Example 3.23. Consider the semi-null incoherent matrix

$$C = \begin{pmatrix} \frac{0}{12} & \frac{0}{45} \\ \frac{0}{12} & \frac{0}{45} \end{pmatrix}.$$

To construct the coheretization C' such that $\overset{\wedge}{\mathbf{e}}(C') = 1|24|5\times 14|25 \sqsubseteq_{diag} \Delta_P(P_4)$, let $(\mathbf{M_1},\mathbf{N_1}) = \{(\{1\}_{i1},\varnothing_{i1}),(\{4\}_{i2},\varnothing_{i2})\}_{i=1,2}$ and $(\mathbf{M_2},\mathbf{N_2}) = \{(\{2\}_{21},\varnothing_{21}^2),\ (\varnothing_{22}^1,\varnothing_{22}^1)\}$. Then

$$C' = \partial_{\mathbf{M_2},\mathbf{N_2}}\partial_{\mathbf{M_1},\mathbf{N_1}}(C) = \begin{pmatrix} \frac{0|0}{1|2} & \frac{0|0}{4|5} \\ \frac{0|0|0}{1|2|0} & \frac{0|0|0}{0|4|5} \end{pmatrix}.$$

When $p\geq 2$, a coherent elementary row matrix has the form $C = \left(\frac{\mathbf{b}}{\mathbf{a}_1}\cdots\frac{\mathbf{b}}{\mathbf{a}_p}\right)$, where $\#\mathbf{b}\in\{0,1\}$. When $\#\mathbf{b}=1$, a family of pairs $(\mathbf{M},\mathbf{N}) = \left\{\left(M^j,\varnothing\right)\subseteq(\mathbf{a}_j,\mathbf{b})\right\}_{j\in\mathbf{p}}$ defines a row action

$$\begin{aligned}\partial_{\mathbf{M},\mathbf{N}}(C) &= \left(\frac{0|\mathbf{b}}{M^1|\mathbf{a}_1\smallsetminus M^1}\ \cdots\ \frac{0|\mathbf{b}}{M^p|\mathbf{a}_p\smallsetminus M^p}\right)\\ &= \begin{pmatrix} \frac{0}{M^1} & \cdots & \frac{0}{M^p} \\ \frac{0}{M^1} & \cdots & \frac{0}{M^p}\end{pmatrix}\left(\frac{\mathbf{b}}{A_{11}}\frac{\mathbf{b}}{A_{12}}\ \cdots\ \frac{\mathbf{b}}{A_{p1}}\frac{\mathbf{b}}{A_{p2}}\right) = C_1C_2, \qquad (3.15)\end{aligned}$$

where $A_{j1}|A_{j2} := EP_{\mathbf{a}_j}\left(\mathbf{a}_j\smallsetminus M^j\right)$. For example, when $\#M^j=1$ for all j, the factor C_1 is incoherent whenever $\#\mathbf{is}(C_1)\geq 2$. However, given a *primitive* component $X\sqsubseteq_{diag}\Delta_P(P_{\cup M^j})$, a row action $C_1' = \partial_{\mathbf{M},\mathbf{N}}(C_1)$ such that $\overset{\wedge}{\mathbf{e}}(C_1') = X$ can be expressed as compositions of η_* actions on the entries of C_1.

Example 3.24. Let $C = \left(\frac{1}{12}\ \frac{1}{45}\ \frac{1}{78}\right)$ and $(\mathbf{M},\mathbf{N}) = \{(\{j\},\varnothing) : j = 1,4,7\}$; then

$$\partial_{\mathbf{M},\mathbf{N}}(C) = \begin{pmatrix} \frac{0|1}{1|2} & \frac{0|1}{4|5} & \frac{0|1}{7|8} \end{pmatrix} = \begin{pmatrix} \frac{0}{1} & \frac{0}{4} & \frac{0}{7} \\ \frac{0}{1} & \frac{0}{4} & \frac{0}{7} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{2} & \frac{1}{0} & \frac{1}{5} & \frac{1}{0} & \frac{1}{8} \end{pmatrix} = C_1C_2.$$

The coheretization C_1' of C_1 such that $\overset{\wedge}{\mathbf{e}}(C_1') = 1|4|7 \times 147 \sqsubseteq_{diag} \Delta_P^{(1)}(P_3)$ is given by

$$C_1' = \begin{pmatrix} \frac{0}{1} & \frac{0}{4} & \frac{0}{7} \\ \eta_2\eta_2\left(\frac{0}{1}\right) & \eta_1\eta_2\left(\frac{0}{4}\right) & \eta_1\eta_1\left(\frac{0}{7}\right) \end{pmatrix} = \begin{pmatrix} \frac{0}{1} & \frac{0}{4} & \frac{0}{7} \\ \frac{0|0|0}{1|0|0} & \frac{0|0|0}{0|4|0} & \frac{0|0|0}{0|0|7} \end{pmatrix}.$$

More generally, let $\mathbf{C} = (\mathbf{b}_i/\mathbf{a}_j)$ be a $q \times p$ elementary matrix over $(\mathbf{a}_*, \mathbf{b}_*)$. If $\mathbf{a}_j \neq \varnothing$, write $\mathbf{a}_j = \{a^j_{j_1} < \cdots < a^j_{j_s}\}$; if $\mathbf{b}_i \neq \varnothing$, write $\mathbf{b}_i = \{b^i_{i_1} < \cdots < b^i_{i_t}\}$. There exists a unique $q \times p$ totally coherent bipartition matrix $prim(\mathbf{C})$, called the *primitive coheretization of* $\mathbf{C}$, with the following properties:

- $\overset{\wedge}{\alpha}_1(prim(\mathbf{C})) = \mathbf{a}_1 \cup \cdots \cup \mathbf{a}_p$ and $\overset{\wedge}{\alpha}_i(prim(\mathbf{C})) = a^1_1|\cdots|a^1_{u_i} \Cup \cdots \Cup a^p_1|\cdots|a^p_{u_i}$ for $2 \leq i \leq q$, where $\pi(a^j_1|\cdots|a^j_{u_i}) = a^j_{j_1}|\cdots|a^j_{j_s}$ when $\mathbf{a}_j \neq \varnothing$,
- $\overset{\vee}{\beta}_1(prim(\mathbf{C})) = \mathbf{b}_1 \cup \cdots \cup \mathbf{b}_q$ and $\overset{\vee}{\beta}_j(prim(\mathbf{C})) = b^1_{v_j}|\cdots|b^1_1 \Cup \cdots \Cup b^q_{v_j}|\cdots|b^q_1$ for $2 \leq j \leq p$, where $\pi(b^i_{v_j}|\cdots|b^i_1) = b^i_{i_t}|\cdots|b^i_{i_1}$ when $\mathbf{b}_i \neq \varnothing$,
- non-empty partition blocks in the entries of $prim(\mathbf{C})$ are in left-most positions,
- and $|prim(\mathbf{C})| = \#\mathbf{is}(\mathbf{C}) + \#\mathbf{os}(\mathbf{C}) - 1$.

Example 3.25. The primitive coheretization of the elementary matrix

$$\mathbf{C} = \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{0} & \frac{1}{3} \\ \frac{23}{1} & \frac{23}{2} & \frac{23}{0} & \frac{23}{3} \end{pmatrix} \text{ is } prim(\mathbf{C}) = \begin{pmatrix} \frac{1}{1} & \frac{0|0|1}{0|2|0} & \frac{0|0|1}{0|0|0} & \frac{0|0|1}{0|0|3} \\ \frac{3|2|0}{1|0|0} & \frac{3|2|0}{0|2|0} & \frac{3|2|0}{0|0|0} & \frac{3|2|0}{0|0|3} \end{pmatrix}$$

so that $\overset{\wedge}{\mathbf{e}}(prim(\mathbf{C})) = 1|2|3 \times 123$ and $\overset{\vee}{\mathbf{e}}(prim(\mathbf{C})) = 123 \times 3|2|1 \times 3|2|1 \times 3|2|1$.

3.6 Generalized Bipartition Matrices

Given an elementary bipartition $\mathbf{c} = \mathbf{b}/\mathbf{a}$, choose a bipartition $c^1 \in P_r'(\mathbf{a}) \times P_r'(\mathbf{b})$ for some $r \geq 1$ (the *trivial choice* is $c^1 = \mathbf{c}$) and let $c^1 = \mathbf{C}^1_1 \cdots \mathbf{C}^1_r$ be the elementary factorization. The 1-*formal product* $T^1(\mathbf{c}) := \mathbf{C}^1_1 \cdots \mathbf{C}^1_r$ is a 1-*formal bipartition on* $\mathbf{c}$.

For each k, write $\mathbf{C}^1_k = (\mathbf{c}^{ij}_k)$, choose a 1-formal bipartition $T^1(\mathbf{c}^{ij}_k)$ for each (i,j), and form the 1-*formal (bipartition) matrix* $C^2_k = T^1\left(\mathbf{C}^1_k\right) := (T^1(\mathbf{c}^{ij}_k))$. The 2-*formal product* $c^2 = T^2(\mathbf{c}) := C^2_1 \cdots C^2_r$ of 1-formal matrices is a 2-*formal bipartition on* $\mathbf{c}$.

For each (i,j,k), choose a 2-formal bipartition $T^2(\mathbf{c}^{ij}_k)$ and form the 2-*formal matrix* $C^3_k = T^2\left(\mathbf{C}^1_k\right) := (T^2(\mathbf{c}^{ij}_k))$. Given an elementary matrix $\mathbf{C}$ and a 2-formal matrix $C = T^2(\mathbf{C})$, define

is $(C) :=$ **is** $(\mathbf{C})$ and **os** $(C) :=$ **os** $(\mathbf{C})$. Recall that if $\left(A^{q\times 1}, B^{1\times p}\right)$ is a TP of bipartition matrices, then $(\#\mathbf{is}(A), \#\mathbf{os}(B)) = (p-1, q-1)$. A pair of 2-formal matrices $\left(A^{q\times 1}, B^{1\times p}\right)$ is a *(formal) Transverse Pair (TP)* if $(\#\mathbf{is}(A), \#\mathbf{os}(B)) = (p-1, q-1)$; then the notion of a formal product of bipartition matrices extends to 2-formal matrices in the obvious way. The 3-*formal product* $c^3 = T^3(\mathbf{c}) := C_1^3 \cdots C_r^3$ of 2-formal matrices is a 3-*formal bipartition on* $\mathbf{c}$.

Continue inductively. After h steps we obtain a sequence $\left(c^1, \ldots, c^h\right)$ of i-formal bipartitions $c^i = T^i(\mathbf{c}) = C_1^i \cdots C_r^i$, where $C_k^i = T^{i-1}\left(\mathbf{C}_k^1\right)$ is an $(i-1)$-formal matrix for each k.

Definition 3.26. Let **a** and **b** be ordered sets. A **generalized bipartition on b/a** is an i-formal bipartition on $\mathbf{b}/\mathbf{a}$ for some $i \geq 1$. Let $\mathbf{C} = (\mathbf{b}_s/\mathbf{a}_t)$ be a $q\times p$ elementary matrix over $(\mathbf{a}_*, \mathbf{b}_*)$ with $pq \geq 1$, let c_{st} be a generalized bipartition on $\mathbf{b}_s/\mathbf{a}_t$ for each (s,t), and let $C = (c_{st})$. If $B = (b_{ij})$ is a $v\times u$ matrix within c_{st}, and $b_{ij} = B_1 \cdots B_{k-1}\cdot \mathbf{0} \cdot B_{k+1} \cdots B_r$ for some k, identify b_{ij} with $B_1 \cdots B_{k-1} B_{k+1} \cdots B_r$ if either $uv = 1$ and $B = C$ or $uv > 1$ and B_{i*} and B_{*j} are indecomposable. Then C with this identification is a **Generalized BiPartition Matrix** (GBPM) **over** $(\mathbf{a}_*, \mathbf{b}_*)$.

Given a bipartition matrix $C^1 = T^1(\mathbf{C})$ and its indecomposable factorization $C^1 = C_1^1 \cdots\ C_r^1$, let $C = \left(C^1, \ldots, C^h\right)$ be a sequence of GBPMs over $(\mathbf{a}_*, \mathbf{b}_*)$ such that $C^i = (c_{st}^i = T_{st}^{i-1}(C_1^1) \cdots T_{st}^{i-1}(C_r^1))$ for $2 \leq i \leq h$; then C is an h-**level path (of GBPMs) from** C^1 **to** C^h with **initial bipartition matrix** C^1 and i^{th} **level** C^i. The sequence $\tilde{C} = \left(\mathbf{C}, C^1, \ldots, C^h\right)$ is an **augmented** h-level path from C^1 to C^h with **augmentation C.**

Example 3.27. Let $\mathbf{C} = \begin{pmatrix} \frac{123}{123} \\ \frac{456}{123} \end{pmatrix}$ and define

$$C^1 = T^1(\mathbf{C}) := \begin{pmatrix} \frac{123|0}{1|23} = \left(\frac{123}{1}\right)\left(\frac{0}{0}\ \ \frac{0}{23}\right) \\ \frac{45|6}{12|3} = \begin{pmatrix} \frac{45}{12} \\ \frac{0}{12} \end{pmatrix}\left(\frac{6}{0}\ \ \frac{6}{0}\ \ \frac{6}{3}\right) \end{pmatrix},$$

$$C^2 = T^2(\mathbf{C}) := \begin{pmatrix} \left(\frac{12|3}{0|1}\right)\left(\frac{0}{0}\ \ \frac{0}{23}\right) = \begin{pmatrix} \frac{12}{0} \\ \frac{0}{0} \end{pmatrix}\left(\frac{3}{1}\right)\left(\frac{0}{0}\ \ \frac{0}{23}\right) \\ \begin{pmatrix} \frac{4|5}{1|2} = \begin{pmatrix} \frac{4}{1} \\ \frac{0}{1} \end{pmatrix}\left(\frac{5}{0}\ \ \frac{5}{2}\right) \\ \frac{0}{12} \end{pmatrix}\left(\frac{6}{0}\ \ \frac{6}{0}\ \ \frac{6}{3}\right) \end{pmatrix},$$

$$\text{and}\quad C^3 = T^3(\mathbf{C}) := \begin{pmatrix} \begin{pmatrix} \frac{12}{0} \\ \frac{0}{0} \end{pmatrix}\left(\frac{3}{1}\right)\left(\frac{0}{0}\ \ \frac{0}{23}\right) \\ \begin{pmatrix} \begin{pmatrix} \frac{4}{1} \\ \frac{0}{1} \end{pmatrix}\left(\frac{5}{0}\ \ \frac{5|0}{0|2}\right) \\ \frac{0}{12} \end{pmatrix}\left(\frac{6}{0}\ \ \frac{6}{0}\ \ \frac{6}{3}\right) \end{pmatrix}.$$

Then $\left(\mathbf{C}, C^1, C^2, C^3\right)$ is an augmented 3-level path of GBPMs from C^1 to C^3.

3.7 The Dimension of a Generalized Bipartition Matrix

Given a bipartition matrix C, let $\pi(C)$ denote the matrix obtained from C by discarding empty biblocks $0/0$ in non-null entries and by reducing null entries to $0/0$.

Definition 3.28. Let C be a $q \times p$ GBPM with $pq \geq 1$. Denote the dimension of C by $|C|$. If C is null, define $|C| := 0$. Otherwise, define

$$|C| := |C|^{row} + |C|^{col} + |C|^{ent}, \tag{3.16}$$

where the **row dimension** $|C|^{row}$**, column dimension** $|C|^{col}$, and **entry dimension** $|C|^{ent}$ are independent and given by the following recursive algorithms**:**

Row Dimension Algorithm. Let C be a $q \times p$ GBPM over $(\mathbf{a}_*, \mathbf{b}_*)$ with $pq \geq 1$ and indecomposable factorization $C = C_1 \cdots C_r$.

If $r > 1$, define $|C|^{row} := \sum_{k \in \mathfrak{r}} |C_k|^{row}$, where $|C_k|^{row}$ is given by setting $C = C_k$ and continuing recursively.

If $q > 1$, define $|C|^{row} := \sum_{i \in \mathfrak{q}} |C_{i*}|^{row}$, where $|C_{i*}|^{row}$ is given by setting $C = C_{i*}$ and continuing recursively.

Otherwise, write $C = (c_1 \cdots c_p)$.

If C is a bipartition matrix, set $C = \pi(C)$.

If $C = \left(\frac{\mathbf{b}}{\mathbf{a}}\right)$ with $\mathbf{b} \neq \varnothing$, define $|C|^{row} := 0$.

If $C = \left(\frac{0}{\mathbf{a}_1} \cdots \frac{0}{\mathbf{a}_p}\right)$, define $|C|^{row} := \begin{cases} 0, & C \text{ is null} \\ \#\mathbf{is}(C) - 1, & \text{otherwise.} \end{cases}$

Otherwise, define $|C|^{row} := \sum_{j \in \mathfrak{p}} |c_j|^{row}$, where $|c_j|^{row}$ is given by setting $C = c_j$ and continuing recursively.

Column Dimension Algorithm. Let C be a $q \times p$ GBPM over $(\mathbf{a}_*, \mathbf{b}_*)$ with $pq \geq 1$ and indecomposable factorization $C = C_1 \cdots C_r$.

If $r > 1$, define $|C|^{col} := \sum_{k \in \mathfrak{r}} |C_k|^{col}$, where $|C_k|^{col}$ is given by setting $C = C_k$ and continuing recursively.

If $p > 1$, define $|C|^{col} := \sum_{j \in \mathfrak{p}} |C_{*j}|^{col}$, where $|C_{*j}|^{col}$ is given by setting $C = C_{*j}$ and continuing recursively.

Otherwise, write $C = (c_1 \cdots c_q)^T$.

If C is a bipartition matrix, set $C = \pi(C)$.

If $C = \left(\frac{\mathbf{b}}{\mathbf{a}}\right)$ with $\mathbf{a} \neq \varnothing$, define $|C|^{col} := 0$.

If $C = \left(\frac{\mathbf{b}_1}{0} \cdots \frac{\mathbf{b}_q}{0}\right)^T$, define $|C|^{col} := \begin{cases} 0, & C \text{ is null} \\ \#\mathbf{os}(C) - 1, & \text{otherwise.} \end{cases}$

Otherwise, define $|C|^{col} := \sum_{i \in \mathfrak{q}} |c_i|^{col}$, where $|c_i|^{col}$ is given by setting $C = c_i$ and continuing recursively.

Entry Dimension Algorithm. Let $C = (c_{ij})$ be a $q \times p$ GBPM over $(\mathbf{a}_*, \mathbf{b}_*)$ with $pq \geq 1$ and indecomposable factorization $C = C_1 \cdots C_r$.

If $r > 1$, define $|C|^{ent} := \sum_{k\in\mathfrak{r}} |C_k|^{ent}$, where $|C_k|^{ent}$ is given by setting $C = C_k$ and continuing recursively.

Otherwise, define $|C|^{ent} := \sum_{(i,j)\in\mathfrak{q}\times\mathfrak{p}} |c_{ij}|^{ent}$, where $|c_{ij}|^{ent}$ is given by setting $C = c_{ij}$ and continuing recursively unless $c_{ij} = \frac{\mathbf{b}_i}{\mathbf{a}_j}$, in which case define

$$|c_{ij}|^{ent} := \begin{cases} \#\mathbf{a}_j + \#\mathbf{b}_i - 1, & \mathbf{a}_j, \mathbf{b}_i \neq \varnothing \\ 0, & \text{otherwise.} \end{cases}$$

We highlight several important facts:

1. If $C = C_1 \cdots C_r$, then

$$|C| = \sum_{k\in\mathfrak{r}} \left(|C_k|^{row} + |C_k|^{col} + |C_k|^{ent} \right) = \sum_{k\in\mathfrak{r}} |C_k| . \tag{3.17}$$

2. $|C|$ *is the sum of the dimensions of all terminating matrices in the recursion*, i.e., elementary bipartitions $\frac{\mathbf{b}}{\mathbf{a}}$ with $\mathbf{a}, \mathbf{b} \neq \varnothing$, elementary row matrices $\left(\frac{0}{\mathbf{a}_1} \cdots \frac{0}{\mathbf{a}_p} \right)$, and elementary column matrices $\left(\frac{\mathbf{b}_1}{0} \cdots \frac{\mathbf{b}_q}{0} \right)^T$. Indeed, if C is an elementary matrix, $|C_{i*}|^{row} = 0$ when the denominators in C_{i*} are empty or the (equal) numerators in C_{i*} are non-empty; dually, $|C_{*j}|^{col} = 0$ when the numerators in C_{*j} are empty or the (equal) denominators in C_{*j} are non-empty. Thus if C is an elementary matrix over $(\mathbf{a}_*, \mathbf{b}_*)$ with $\mathbf{a}_j, \mathbf{b}_i \neq \varnothing$ for all (i, j), then $|C|^{row} = |C|^{col} = 0$ and $|C| = |C|^{ent}$.

3. $|C|$ *is not the sum of the dimensions of its entries.* For an underlying geometric example, consider a facet $A|B \sqsubset P_n$. Let $\mathbf{a}_1 | \cdots | \mathbf{a}_p := EP_{\mathfrak{n}}B$; then $A|B$ corresponds to the monomial $C_1C_2 = \left(\frac{0}{A}\right) \left(\frac{0}{\mathbf{a}_1} \cdots \frac{0}{\mathbf{a}_p} \right) \in \mathfrak{n} \circledast \mathfrak{o}$. By the Row Dimension Algorithm, $|C_1C_2| = |C_1| + |C_2| = \#\mathbf{is}(C_1) - 1 + \#\mathbf{is}(C_2) - 1 = n - 2$ as required, but if $\mathbf{a}_i, \mathbf{a}_j \neq \varnothing$ for some $i \neq j$, then $\left| \frac{0}{\mathbf{a}_1} \right| + \cdots + \left| \frac{0}{\mathbf{a}_p} \right| < |C_2|$.

4. When C is an indecomposable row or column bipartition matrix, the row and column dimension algorithms reduce the calculation to that of $|\pi(C)|$. This reduction is critical. For example, if $C = \begin{pmatrix} \frac{0}{1} & \frac{0|0}{0|2} \end{pmatrix}$, then $|C| = |\pi(C)| = \left| \begin{pmatrix} \frac{0}{1} & \frac{0}{2} \end{pmatrix} \right| = 1$, but without reduction to $|\pi(C)|$, the computed dimension of C would be 0.

Example 3.29. Let us compute the dimension of the indecomposable bipartition matrix

$$C = \begin{pmatrix} \frac{0}{12} & \frac{0}{0} \\ \frac{3|4}{1|2} & \frac{34}{0} \end{pmatrix}.$$

Then $|C| = |C|^{row} + |C|^{col} + |C|^{ent}$, where

$$|C|^{row} = \left|\begin{pmatrix} \frac{0}{12} & \frac{0}{0} \end{pmatrix}\right|^{row} + \left|\begin{pmatrix} \left(\frac{34}{12}, \begin{pmatrix} \frac{3}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{4}{0} & \frac{4}{2} \end{pmatrix}\right) & \left(\frac{34}{0}, \frac{34}{0}\right) \end{pmatrix}\right|^{row}$$

$$= 1 + \left|\begin{pmatrix} \frac{3}{1} \\ \frac{0}{1} \end{pmatrix}\right|^{row} + \left|\begin{pmatrix} \frac{4}{0} & \frac{4}{2} \end{pmatrix}\right|^{row} + 0 = 1;$$

$$|C|^{col} = \left|\begin{pmatrix} \frac{0}{12} \\ \frac{3|4}{1|2} \end{pmatrix}\right|^{col} + \left|\begin{pmatrix} \frac{0}{0} \\ \frac{34}{0} \end{pmatrix}\right|^{col} = 0 + \left|\begin{pmatrix} \frac{3}{1} \\ \frac{0}{1} \end{pmatrix}\right|^{col} + \left|\begin{pmatrix} \frac{4}{0} & \frac{4}{2} \end{pmatrix}\right|^{col} + 1 = 1;$$

$$|C|^{ent} = \left|\frac{0}{12}\right|^{ent} + \left|\frac{0}{0}\right|^{ent} + \left|\begin{pmatrix} \frac{3}{1} \\ \frac{0}{1} \end{pmatrix}\right|^{ent} + \left|\begin{pmatrix} \frac{4}{0} & \frac{4}{2} \end{pmatrix}\right|^{ent} + \left|\frac{34}{0}\right|^{ent} = 0 + 0 + 1 + 1 + 0 = 2.$$

Therefore $|C| = 4$.

Example 3.30. Let us compute the dimension of the indecomposable bipartition matrix

$$C = \begin{pmatrix} \frac{0|1}{1|0} & \frac{0|1}{2|0} \\ \frac{0|2}{1|0} & \frac{0|2|0}{0|2|0} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \left(\frac{1}{0} \;\; \frac{1}{0}\right) & \begin{pmatrix} \frac{0}{2} \\ \frac{0}{2} \end{pmatrix} \left(\frac{1}{0} \;\; \frac{1}{0}\right) \\ \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \left(\frac{2}{0} \;\; \frac{2}{0}\right) & \frac{0|2|0}{0|2|0} = \begin{pmatrix} \frac{0}{0} \\ \frac{0}{0} \end{pmatrix} \left(\frac{2|0}{0|0}\right) = \left(\frac{0|2}{0|2}\right) \left(\frac{0}{0} \;\; \frac{0}{0}\right) \end{pmatrix}.$$

Then

$$C_{1*} = \begin{pmatrix} \frac{0}{1} & \frac{0}{2} \\ \frac{0}{1} & \frac{0}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} \end{pmatrix}, \qquad C_{2*} = \begin{pmatrix} \frac{0}{1} & \frac{0}{0} \\ \frac{0}{1} & \frac{0}{0} \end{pmatrix} \begin{pmatrix} \frac{2}{0} & \frac{2}{0} & \frac{2|0}{2|0} \end{pmatrix},$$

$$C_{*1} = \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} \\ \frac{2}{0} & \frac{2}{0} \end{pmatrix}, \qquad \text{and} \qquad C_{*2} = \begin{pmatrix} \frac{0}{2} \\ \frac{0}{2} \\ \frac{0|2}{0|2} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} \\ \frac{0}{0} & \frac{0}{0} \end{pmatrix}$$

so that $|C|^{row} = \left|\begin{pmatrix} \frac{0}{1} & \frac{0}{2} \\ \frac{0}{1} & \frac{0}{2} \end{pmatrix}\right|^{row} = 1 + 1 = 2$, $|C|^{col} = \left|\begin{pmatrix} \frac{1}{0} & \frac{1}{0} \\ \frac{2}{0} & \frac{2}{0} \end{pmatrix}\right|^{col} = 1 + 1 = 2$, and $|C|^{ent} = \left|\frac{0|2|0}{0|2|0}\right| = 1$. Therefore $|C| = |C|^{row} + |C|^{col} + |C|^{ent} = 5$.

Example 3.31. Consider the indecomposable GBPM $C = C^3$ in Example 3.27 :

$$C = \begin{pmatrix} \begin{pmatrix} \frac{12}{0} \\ \frac{0}{0} \end{pmatrix} \left(\frac{3}{1}\right) \left(\frac{0}{0}\ \ \frac{0}{23}\right) \\ \begin{pmatrix} \begin{pmatrix} \frac{4}{1} \\ \frac{0}{1} \end{pmatrix} \left(\frac{5}{0}\ \ \left(\frac{5}{0}\right) \left(\frac{0}{2}\right)\right) \\ \frac{0}{12} \end{pmatrix} \left(\frac{6}{0}\ \ \frac{6}{0}\ \ \frac{6}{3}\right) \end{pmatrix}.$$

Then $|C|^{row} = \left|\left(\frac{0}{0}\ \ \frac{0}{23}\right)\right|^{row} + \left|\left(\frac{0}{12}\right)\right|^{row} = 2,\ \ |C|^{col} = \left|\begin{pmatrix} \frac{12}{0} \\ \frac{0}{0} \end{pmatrix}\right|^{col} = 1$, and

$|C|^{ent} = \left|\frac{3}{1}\right|^{ent} + \left|\frac{4}{1}\right|^{ent} + \left|\frac{6}{3}\right|^{ent} = 3$ imply $|C| = 6$.

Our next proposition follows immediately from Proposition 3.20.

Proposition 3.32. If $\mathbf{C}$ is a $q \times p$ coherent elementary matrix with $p, q \geq 2$, then

$$|\mathbf{C}| = \begin{cases} 1, & \mathbf{is}\,(\mathbf{C})\,, \mathbf{os}\,(\mathbf{C}) \neq \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

In the special case of bipartition matrices, computer implementations of the algorithms in Definition 3.28 were given by D. Freeman and the second author in [8].

4 Framed Matrices

4.1 The Framed Join $\mathfrak{m} \circledast \mathfrak{n}$

Given an elementary bipartition $\mathbf{c}$, choose a 1-formal bipartition $c^1 = T^1(\mathbf{c})$ with elementary factorization $c^1 = \mathbf{C}_1^1 \cdots \mathbf{C}_r^1$. Then $\left(c^1\right)$ is a 1-level path of generalized bipartitions on $\mathbf{c}$. A *structure element within* c^1 is an elementary bipartition entry of some $\mathbf{C}_k^1$. Define $\mathcal{T}_1(\mathbf{c}) :=$ {1-level paths of generalized bipartitions on $\mathbf{c}$}.

Let $\left(c^1, c^2 = C_1^2 \cdots C_r^2\right)$ be a 2-level path of generalized bipartitions on $\mathbf{c}$. A *structure element within* c^2 is a bipartition entry of some C_k^2. Define $\mathcal{T}_2(\mathbf{c}) :=$ {2-level paths of generalized bipartitions on $\mathbf{c}$}.

Let $\left(c^1, c^2, c^3 = C_1^3 \cdots C_r^3\right)$ be a 3-level path of generalized bipartitions on $\mathbf{c}$. A *structure element within* c^3 is a structure element within an entry of some C_k^3. Define $\mathcal{T}_3(\mathbf{c}) :=$ {3-level paths of generalized bipartitions on $\mathbf{c}$}.

Continuing inductively, let $\left(c^1, \ldots, c^h = C_1^h \cdots C_r^h\right)$, $h > 1$, be an h-level path of generalized bipartitions on $\mathbf{c}$. Then a structure element within c^h is a structure element within an entry of some C_k^h and $\mathcal{T}_h(\mathbf{c}) =$ {h-level paths of generalized bipartitions on $\mathbf{c}$}. Note that if $i < j \leq h$ and all choices from level i through level $j-1$ are trivial, then $c^i = c^{i+1} = \cdots = c^j$.

Definition 4.1. Let $\mathbf{a}$ and $\mathbf{b}$ be ordered sets and let $\mathbf{c} = \mathbf{b}/\mathbf{a}$. The **framed join of a with b** is defined and denoted by

$$\mathbf{a} \circledast \mathbf{b} := \bigcup_{h \geq 1} \mathcal{T}_h(\mathbf{c}) \Big/ \sim, \text{ where} \tag{4.1}$$

1. $\left(c^1,\ldots,c^h\right)\sim\left(d^1,\ldots,d^h\right)$ if $c^h=d^h$,
2. $\left(c^1,\ldots,c^h\right)\sim\left(c^1,\ldots,c^i,c^i,\ldots,c^h\right)$ for all i.

A class $[c]\in\mathbf{a}\circledast\mathbf{b}$ is a **framed element on c**. Let $c=\left(c^1,\ldots,c^h\right)\in[c]$. A **level** i **structure element within** c is a structure element within c^i. The **canonical path** in $[c]$, denoted by $\bar{c}=(\bar{c}^1,\ldots,\bar{c}^{\bar{h}})$, has a minimum number of levels $\bar{h}$ and components $\bar{c}^i$ of maximal (indecomposable) factorization length. The **height** $h[c]:=\bar{h}$**.**

Let $\mathbf{C}=(\mathbf{b}_s/\mathbf{a}_t)$ be a $q\times p$ elementary matrix with $pq\geq 1$. For each (s,t), let $[c_{st}]\in\mathbf{a}_t\circledast\mathbf{b}_s$; Let $h=max\left\{h\left[c_{st}\right]\right\}$, choose $(c_{st}^1,\ldots,c_{st}^{h_{st}})\in[c_{st}]$ such that $h_{st}\leq h$, and define

$$c_{st}:=(c_{st}^1,\ldots,c_{st}^{h_{st}},\underbrace{c_{st}^{h_{st}},\ldots,c_{st}^{h_{st}}}_{h-h_{st}}). \tag{4.2}$$

Then $[C]=([c_{st}])$ is a $q\times p$ **framed matrix on C**. Let $C^i=\left(c_{st}^i\right)$; then $C=\left(C^1,\ldots,C^h\right)\in[C]$, the **initial bipartition matrix** of C is C^1, the i^{th} **level of** C is C^i, and a **structure matrix within** C is an indecomposable factor of either C^1 or a structure element within some entry of some C^i, $i>1$. Set $c_{st}=\bar{c}_{st}$ in (4.2)**;** the **canonical path** in $[C]$ is $\bar{C}:=(\bar{c}_{st})$ and the **height** $h[C]:=\bar{h}$. The **input set is**$[C]:=$ **is**$(\mathbf{C})$, the **output set os**$[C]:=$ **os**$(\mathbf{C})$, and the **dimension** $|[C]|:=|C|:=\left|C^h\right|$.

In view of Definition 3.26, a framed matrix has finitely many representatives. A bipartition can be thought of as a framed element of height 1, and a bipartition matrix can be thought of as a framed matrix of height 1. In other words, the set of all bipartitions on $(\mathbf{a},\mathbf{b})$ is a subset of $\mathbf{a}\circledast\mathbf{b}$, and the inclusion is a bijection when $(\mathbf{a},\mathbf{b})=(\mathfrak{n},\mathfrak{o})$ (or $(\mathfrak{o},\mathfrak{n})$). On the other hand, such bipartitions can be identified with ordinary partitions in $P(\mathfrak{n})$ by discarding null numerators (or null denominators). Thus $\mathfrak{o}\circledast\mathfrak{n}=\mathfrak{n}\circledast\mathfrak{o}=P(\mathfrak{n})$.

Let $[c]$ be a framed element and let $c=\left(c^1,\ldots,c^h\right)\in[c]$. Definition 4.1, part (1), says that all paths in $[c]$ terminate at c^h, and part (2) says that repeated components in c can be suppressed. Thus the paths in $[c]$ form a connected digraph with maximal element c^h. The canonical path $\bar{c}\in[c]$ has minimal height $\bar{h}$ and components of maximal factorization length. Thus, if c is a non-canonical path of minimal height $\bar{h}$, some component c^i has non-maximal factorization length.

Example 4.2. Let $\mathbf{a}=\mathbf{b}=\{1,2,3,4\}$. The initial bipartition of the canonical path

$$\bar{c}=\left(\frac{1|2|34}{1|2|34},\ \begin{pmatrix}\frac{1}{1}\\ \frac{0}{1}\\ \frac{0}{1}\\ \frac{0}{1}\end{pmatrix}\begin{pmatrix}\frac{2}{0} & \frac{2}{2}\\ \frac{0}{0} & \frac{0}{2}\\ \frac{0}{0} & \frac{0}{2}\end{pmatrix}\begin{pmatrix}\frac{4|3}{0|0} & \frac{3|4}{0|0} & \frac{3|4}{4|3}\end{pmatrix}\right)$$

has factorization length 3. Since

$$\begin{pmatrix}\frac{1}{1}\\ \frac{0}{1}\\ \frac{0}{1}\\ \frac{0}{1}\end{pmatrix}\begin{pmatrix}\frac{2|4|3}{0|0|0} & \frac{2|3|4}{2|4|3}\end{pmatrix}=\begin{pmatrix}\frac{1}{1}\\ \frac{0}{1}\\ \frac{0}{1}\\ \frac{0}{1}\end{pmatrix}\begin{pmatrix}\frac{2}{0} & \frac{2}{2}\\ \frac{0}{0} & \frac{0}{2}\\ \frac{0}{0} & \frac{0}{2}\end{pmatrix}\begin{pmatrix}\frac{4|3}{0|0} & \frac{3|4}{0|0} & \frac{3|4}{4|3}\end{pmatrix}=\begin{pmatrix}\frac{1|2}{1|2}\\ \frac{0|0}{1|2}\\ \frac{0|0}{1|2}\end{pmatrix}\begin{pmatrix}\frac{4|3}{0|0} & \frac{3|4}{0|0} & \frac{3|4}{4|3}\end{pmatrix},$$

the framed element $[\bar{c}]$ is also represented by

$$\left(\frac{1|234}{1|234}, \begin{pmatrix} \frac{1}{1} \\ \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{2|4|3}{0|0|0} & \frac{2|3|4}{2|4|3} \end{pmatrix}\right)$$

and

$$\left(\frac{12|34}{12|34}, \begin{pmatrix} \frac{1|2}{1|2} \\ \frac{0|0}{1|2} \\ \frac{0|0}{1|2} \end{pmatrix} \begin{pmatrix} \frac{4|3}{0|0} & \frac{4|3}{0|0} & \frac{3|4}{4|3} \end{pmatrix}\right)$$

whose initial bipartitions have non-maximal factorization length 2.

4.2 The Structure Tree of a Framed Element

Let $[c]$ be a framed element, let $c = \left(c^1, \ldots, c^h\right) \in [c]$ and consider the augmented path $\tilde{c} = \left(\mathbf{c}_1^0, c^1, \ldots, c^h\right)$. Then $\left(\mathbf{c}_1^0\right)$ is an m_0-tuple with $m_0 = 1$. Inductively, if $i \geq 0$ and the m_i-tuple $c^i = \left(\mathbf{c}_1^i, \ldots, \mathbf{c}_{m_i}^i\right)$ has been constructed, write $T^1(\mathbf{c}_j^i) = \mathbf{C}_1^i \cdots \mathbf{C}_{r_{ij}}^i$. Identify $\mathbf{C}_k^i$ with the tuple of its entries listed in row order, identify $T^1(\mathbf{c}_j^i)$ with the concatenation of tuples $(\mathbf{C}_1^i, \ldots, \mathbf{C}_{r_{ij}}^i)$, and identify c^{i+1} with the m_{i+1}-tuple $\left(\mathbf{c}_1^{i+1}, \ldots, \mathbf{c}_{m_{i+1}}^{i+1}\right) := \left(T^1\left(\mathbf{c}_1^i\right), \ldots, T^1\left(\mathbf{c}_{m_i}^i\right)\right)$.

The *structure tree of* c, denoted by $\mathbb{T}(c)$, is the PLT with $h+1$ levels constructed as follows: If $h = 1$, then

$$\mathbb{T}(c) := \begin{array}{c} \mathbf{c}_1^1 \ \cdots \ \mathbf{c}_{m_1}^1 \\ \diagdown | \diagup \\ \mathbf{c}_1^0 \end{array}$$

is the 2-level tree with root in level 0 and vertices $\mathbf{c}_j^1$ in level 1. Inductively, if $1 \leq i < h$ and level i has been constructed, construct level $i+1$ by connecting each level i vertex $\mathbf{c}_j^i$ to the level $i+1$ vertices in $T^1(\mathbf{c}_j^i)$ (see Figure 3).

$$\begin{array}{c} \mathbf{c}_1^h \cdots\cdots\cdots \mathbf{c}_{m_h}^h \\ \vdots \qquad\quad \vdots \\ \diagdown | \diagup \cdots \diagdown | \diagup \\ \mathbf{c}_1^1 \ \cdots \ \mathbf{c}_{m_1}^1 \\ \diagdown | \diagup \\ \mathbf{c}_1^0 \end{array}$$

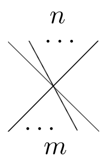
Figure 3. The structure tree $\mathbb{T}(c)$.

When the marked units on a measuring stick are levels, the measured height of $\mathbb{T}(c)$ is h. The *structure tree of* $[c]$ is defined and denoted by $\mathbb{T}[c] := \mathbb{T}(\bar{c})$.

Let $1 \leq i < h$. Given a bipartition b within c^{i+1}, let $b = \mathbf{B}_1 \cdots \mathbf{B}_r$ be the elementary factorization and let $b = T^1(\mathbf{b})$; note that $\mathbf{b}$ lies within c^i when b is a structure element. Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_t\}$

be the entries of $\mathbf{B}_1, \ldots, \mathbf{B}_r$ listed in row order as above. For each s, let $b_s = T^1(\mathbf{b}_s)$ and denote the maximal subtree of $\mathbb{T}(c)$ with root $\mathbf{b}_s$ by $\mathbb{T}(c|_{b_s})$. The *restriction of c to b* is the representative $c|_b := \left(T^1(\mathbf{b}), \ldots, T^{h-i}(\mathbf{b})\right)$ whose structure tree $\mathbb{T}(c|_b)$ is obtained by attaching the root $\mathbf{b}_s$ of each subtree $\mathbb{T}(c|_{b_s})$ to $\mathbf{b}$. The class $[c|_b]$ is a *subframed element in* $[c]$. Let $C = \left(C^1, \ldots, C^h\right)$ be a framed matrix representative and consider a bipartition matrix $B = (b_{uv})$ within C^{i+1}. The *restriction of C to B*, defined and denoted by $C|_B := (C|_{b_{uv}})$, represents the *subframed matrix* $[C|_B]$ *in* $[C]$.

Let Γ_m^n denote the upward directed double corolla with m incoming leaves (inputs) and n outgoing leaves (outputs):

$\Gamma_m^n =$ 

Given ordered sets $\mathbf{a}$ and $\mathbf{b}$, there is the projection

$$\frac{\mathbf{b}}{\mathbf{a}} \to \Gamma^{\#\mathbf{b}+1}_{\#\mathbf{a}+1}. \tag{4.3}$$

When the context is clear, we suppress the scripts and simply write Γ. The *graph structure tree of* $[c]$, denoted by $\Gamma[c]$, is the PLT given by replacing each vertex $\mathbf{c}_j^i$ in $\mathbb{T}_h[c]$ with the corresponding double corolla Γ_{ij}.

$$\begin{array}{c} \Gamma_{h1} \cdots\cdots\cdots \Gamma_{hm_h} \\ \vdots \qquad\quad \vdots \\ \diagdown|\diagup \cdots \diagdown|\diagup \\ \Gamma_{11} \cdots \Gamma_{1m_1} \\ \diagdown|\diagup \\ \Gamma_{01} \end{array} \tag{4.4}$$

Figure 4. The graph structure tree $\Gamma[c]$.

The *graph matrix* of an elementary matrix $\mathbf{C} = (\mathbf{c}_{ij})$ is the matrix $\mathbb{C} = (\Gamma_{ij})$ given by the replacement $\mathbf{c}_{ij} \leftarrow \Gamma_{ij}$ for all (i,j). The *graph* of a bipartition $\frac{\beta}{\alpha} = \mathbf{C}_1 \cdots \mathbf{C}_r$ is the evaluated formal product

$$\mathbb{G} := \mathbb{C}_1 \cdots \mathbb{C}_r, \tag{4.5}$$

i.e., the corresponding iterated elementary fraction constructed by M. Markl in [22]. The *graph matrix* of a bipartition matrix $C = (c_{ij})$ is the matrix $\mathbb{C} = (\mathbb{G}_{ij})$ given by the replacement $c_{ij} \leftarrow \mathbb{G}_{ij}$ for all (i,j).

4.3 Formal Decomposability

Definition 4.3. For $1 \le i \le r$, let $C_1, \ldots, C_r$ be framed matrix representatives on $\mathbf{C}_1, \ldots, \mathbf{C}_r$, respectively. The string $\rho := C_1 \cdots C_r$ is a **formal product** if $\rho^1 := C_1^1 \cdots C_r^1$ is a formal product of bipartition matrices, in which case

$$(\mathbf{is}(\rho), (\mathbf{os}(\rho)) := \left(\mathbf{is}(\rho^1), \mathbf{os}(\rho^1)\right) = (\mathbf{is}(C_r), \mathbf{os}(C_1)).$$

A framed matrix representative F is **formally decomposable** if there exist framed matrix representatives $F_1, \ldots, F_r$ such that $F = F_1 \cdots F_r$ is a formal product with $r > 1$; when r is maximal, the **length** $l(F) := r$.

For each $r \ge 1$, let $\mathbf{a} \circledast^r \mathbf{b} := \{[c] \in \mathbf{a} \circledast \mathbf{b} : l[c] = r\}$. There is the canonical decomposition

$$\mathbf{a} \circledast \mathbf{b} := \bigcup_{r \ge 1} \mathbf{a} \circledast^r \mathbf{b}.$$

Example 4.4. Let c_i^o and d_i^o denote representatives of framed elements with i inputs and o outputs. A matrix of the form

$$F = \begin{pmatrix} \begin{pmatrix} c_1^2 \\ c_1^1 \end{pmatrix} \begin{pmatrix} d_1^2 \end{pmatrix} & \begin{pmatrix} c_2^2 \\ c_2^1 \end{pmatrix} \begin{pmatrix} d_1^2 & d_2^2 \end{pmatrix} \\ \begin{pmatrix} c_1^1 \end{pmatrix} \begin{pmatrix} d_1^1 \end{pmatrix} & \begin{pmatrix} c_2^1 \end{pmatrix} \begin{pmatrix} d_1^1 & d_2^1 \end{pmatrix} \end{pmatrix}$$

is formally decomposable and factors as

$$F = \begin{pmatrix} c_1^2 & c_2^2 \\ c_1^1 & c_2^1 \\ c_1^1 & c_2^1 \end{pmatrix} \begin{pmatrix} d_1^2 & d_1^2 & d_2^2 \\ d_1^1 & d_1^1 & d_2^1 \end{pmatrix}.$$

When $r > 1$, a path $C = \left(C^1, \ldots, C^h\right) = \left(C_1^1 \cdots C_r^1, \ldots, C_1^h \cdots C_r^h\right)$ is formally decomposable as a matrix of formal products and factors as

$$C = C_1 \cdots C_r = \left(C_1^1, \ldots, C_1^h\right) \cdots \left(C_r^1, \cdots, C_r^h\right) = C|_{C_1^1} \cdots C|_{C_r^1}.$$

The k^{th} factor $C_k = C|_{C_k^1}$ represents the subframed matrix $[C|_{C_k^1}]$ in $[C]$. Moreover, if $1 \le k < \ell \le r$, the formal product $[C_k] \cdots [C_\ell]$ is a subframed matrix in $[C]$ as well. To see this, consider an entry AB of $C_k C_{k+1}$; then (A, B) is a formal TP. Let $A = (A^1, \ldots, A^h)$ and $B = (B^1, \ldots, B^h)$; then $\left(A^i, B^i\right)$ is also a formal TP since A^i and B^i have the same respective matrix dimensions and i/o sets as A and B. Furthermore, since $C^1 = C_1^1 \cdots C_r^1$ is a bipartition matrix, so is $C_k^1 C_{k+1}^1$, and it follows that $A^1 B^1$ is a bipartition. Let $T^1(\mathbf{AB}) = A^1 B^1$; then $T^i(\mathbf{AB}) = A^i B^i$ for all i and $AB = (A^1 B^1, \ldots, A^h B^h)$ represents the subframed element $[AB] = [A][B]$ in $[C]$. Thus $C_k C_{k+1}$ represents the subframed matrix $[C_k C_{k+1}] = [C_k][C_{k+1}]$ in $[C]$. Continue inductively.

4.4 Coherent Framed Matrices

Recall that if $\mathbf{C} = (\mathbf{c}_{ij})$ is an elementary matrix, then $T^k(\mathbf{c}_{ij})$ is a k-formal bipartition and $T^k(\mathbf{C}) = \left(T^k(\mathbf{c}_{ij})\right)$ is a k-formal matrix.

Definition 4.5. A k-formal matrix is

- **simple** if its indecomposable factors are bipartition matrices.
- **semi-simple** if its rows and columns are simple.

A 1-formal matrix is a bipartition matrix, hence simple.

Definition 4.6. A **(maximally) (totally) coherent simple matrix** has (maximally) (totally) coherent indecomposable factors. A **(maximally) coherent semi-simple matrix** has (maximally) coherent rows, columns, and entries. A **(maximally) totally coherent semi-simple matrix** is (maximally) coherent with (maximally) totally coherent entries.

A path (C^1, C^2) is **(maximally) (totally) coherent** if C^1 is (maximally) precoherent and C^2 is both semi-simple and (maximally) (totally) coherent.

Example 4.7. Continuing the discussion in Example 3.30, consider the indecomposable maximally precoherent bipartition matrix

$$C^1 = \begin{pmatrix} \frac{0|1}{1|0} & \frac{0|1}{2|0} \\ \frac{0|2}{1|0} & \frac{0|2|0}{0|2|0} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} \end{pmatrix} & \begin{pmatrix} \frac{0}{2} \\ \frac{0}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} \end{pmatrix} \\ \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{2}{0} & \frac{2}{0} \end{pmatrix} & \frac{0|2|0}{0|2|0} \end{pmatrix}.$$

The first row and first column of C^1 have the respective incoherent factors

$$\begin{pmatrix} \frac{0}{1} & \frac{0}{2} \\ \frac{0}{1} & \frac{0}{2} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{0} & \frac{1}{0} \\ \frac{2}{0} & \frac{2}{0} \end{pmatrix},$$

each of which can be coheretized two ways:

$$\left\{ \begin{pmatrix} \frac{0}{1} & \frac{0}{2} \\ \frac{0|0}{1|0} & \frac{0|0}{0|2} \end{pmatrix}, \begin{pmatrix} \frac{0|0}{0|1} & \frac{0|0}{2|0} \\ \frac{0}{1} & \frac{0}{2} \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} \frac{1|0}{0|0} & \frac{1}{0} \\ \frac{0|2}{0|0} & \frac{2}{0} \end{pmatrix}, \begin{pmatrix} \frac{1}{0} & \frac{0|1}{0|0} \\ \frac{2}{0} & \frac{2|0}{0|0} \end{pmatrix} \right\}.$$

Thus, there are four coherent choices for C^2. Consider the indecomposable choice

$$C^2 = \begin{pmatrix} \begin{pmatrix} \frac{0}{1} \\ \frac{0|0}{1|0} \end{pmatrix} \begin{pmatrix} \frac{1|0}{0|0} & \frac{1}{0} \end{pmatrix} & \begin{pmatrix} \frac{0}{2} \\ \frac{0|0}{0|2} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} \end{pmatrix} \\ \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{0|2}{0|0} & \frac{2}{0} \end{pmatrix} & \begin{pmatrix} \frac{0|2}{0|2} \end{pmatrix} \begin{pmatrix} \frac{0}{0} & \frac{0}{0} \end{pmatrix} = \begin{pmatrix} \frac{0}{0} \\ \frac{0}{0} \end{pmatrix} \begin{pmatrix} \frac{2|0}{2|0} \end{pmatrix} \end{pmatrix}.$$

Its rows factor as

$$\begin{pmatrix} \frac{0}{1} & \frac{0}{2} \\ \frac{0|0}{1|0} & \frac{0|0}{0|2} \end{pmatrix} \begin{pmatrix} \frac{1|0}{0|0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{0}{1} & \frac{0}{0} \\ \frac{0}{1} & \frac{0}{0} \end{pmatrix} \begin{pmatrix} \frac{0|2}{0|0} & \frac{2}{0} & \frac{2|0}{2|0} \end{pmatrix},$$

and its columns factor as

$$\begin{pmatrix} \frac{0}{1} \\ \frac{0|0}{1|0} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{1|0}{0|0} & \frac{1}{0} \\ \frac{0|2}{0|0} & \frac{2}{0} \end{pmatrix} \text{ and } \begin{pmatrix} \frac{0}{2} \\ \frac{0|0}{0|2} \\ \frac{0|2}{0|2} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} \\ \frac{0}{0} & \frac{0}{0} \end{pmatrix}.$$

Thus C^2 is semi-simple with maximally coherent rows, columns, and entries. Since the entries of C^2 are totally coherent, the path $C = (C^1, C^2)$ is maximally totally coherent. Note that

$$|C|^{row} + |C|^{col} + |C|^{ent} = 1 + 1 + 1 = 3 = \#\mathbf{is}\,(C) + \#\mathbf{os}\,(C) - 1,$$

which is a property of all "top dimensionally coherent framed matrices" (see Proposition 4.20).

Remark 4.8. Unless explicitly indicated otherwise, we henceforth identify a framed matrix $[C]$ with its canonical representative $\bar{C}$ and denote both by C (without the bar).

Definition 4.9. Let $pq \geq 1$. A $q \times p$ framed matrix $C = (C^1, \ldots, C^h)$ is **(maximally) coherent** if

1. $h = 1$ and C^1 is (maximally) totally coherent.
2. $h \geq 2$, (maximal) coherence has been defined for all framed matrices of height less than h, the path (C^1, C^2) is (maximally) coherent, and the entries of C are (maximally) coherent.

For example, the path (C^1, C^2) in Example 4.7 represents a 2×2 maximally coherent framed matrix of height 2.

Remark 4.10. In general, an empty biblock in a framed matrix C is *non unital* and deleting it may change the dimension and/or coherence properties of C. Henceforth, we will only display an empty biblock in a (coherent) C when deleting it changes the dimension (or coherence properties) of C (cf. Examples 17 and 21).

Proposition 4.11. If C is a (maximally) coherent $q \times p$ framed matrix and $pq \geq 1$, top level structure matrices within C are (maximally) totally coherent.

Proof. We proceed by induction on height.

Let $C = (C^1)$ be a (maximally) coherent framed matrix of height 1. Since C^1 is (maximally) totally coherent by Definition 4.9, part (1), the top level structure matrices within C, i.e., the indecomposable factors of C^1, are (maximally) totally coherent.

Let c be a (maximally) coherent framed element of height 2. Then c has (maximally) coherent formally indecomposable factors C_k by Definition 4.9, part (2a). But C_k is a (maximally) coherent framed matrix of height 1. Hence $C_k^1 = C_k$ is (maximally) totally coherent by Definition 4.9, part (1), and the top level structure matrices within c are the (maximally) totally coherent factors C_k.

Let $C = (c_{ij})$ be a $q \times p$ (maximally) coherent framed matrix of height 2 with $p + q > 2$. Since c_{ij} is a (maximally) coherent framed element of height 2, the top level structure matrices within C, which are top level structure matrices within some c_{ij}, are (maximally) totally coherent by the previous argument.

Let c be a (maximally) coherent framed element of height 3. Then c has (maximally) coherent indecomposable factors C_k by Definition 4.9, part (2a). Since C_k is a (maximally) coherent framed matrix of height 2, a top level structure matrix within c is a top level structure matrix within some C_k, which is (maximally) totally coherent by the previous argument.

Continue inductively. Q.E.D.

4.5 The Coherent Framed Join $\mathfrak{m} \circledast_{cc} \mathfrak{n}$

Definition 4.12. Let $\mathbf{a}$ and $\mathbf{b}$ be ordered sets. The **coherent framed join of a with b** is defined and denoted by

$$\mathbf{a} \circledast_{cc} \mathbf{b} : = \{\text{coherent framed elements in } \mathbf{a} \circledast \mathbf{b}\}.$$

The subset of length r coherent framed partitions in $\mathbf{a} \circledast_{cc} \mathbf{b}$ is denoted by $\mathbf{a} \circledast_{cc}^{r} \mathbf{b}$.

Remark 4.13. Since $\mathfrak{o} \circledast_{cc} \mathfrak{n} = \mathfrak{o} \circledast \mathfrak{n}$ and $\mathfrak{n} \circledast_{cc} \mathfrak{o} = \mathfrak{n} \circledast \mathfrak{o}$, we identify $\mathfrak{o} \circledast_{cc} \mathfrak{n}$ and $\mathfrak{n} \circledast_{cc} \mathfrak{o}$ with $P(\mathfrak{n})$.

Example 4.14. Let us determine the elements of $\mathfrak{2} \circledast_{cc} \mathfrak{1}$ with initial bipartition

$$c^1 = \frac{0|1}{12|0} = \begin{pmatrix} \frac{0}{12} \\ \frac{0}{12} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{0} \end{pmatrix}.$$

The replacements $\left\{ \frac{0}{12}, \frac{0|0}{1|2}, \frac{0|0}{2|1} \right\}$ of $\frac{0}{12}$ give rise to the following seven indecomposables, the last three of which are coherent:

$$\left\{ \begin{pmatrix} \frac{0}{12} \\ \frac{0}{12} \end{pmatrix}, \begin{pmatrix} \frac{0}{12} \\ \frac{0|0}{2|1} \end{pmatrix}, \begin{pmatrix} \frac{0|0}{1|2} \\ \frac{0}{12} \end{pmatrix}, \begin{pmatrix} \frac{0|0}{1|2} \\ \frac{0|0}{2|1} \end{pmatrix}, \begin{pmatrix} \frac{0}{12} \\ \frac{0|0}{1|2} \end{pmatrix}, \begin{pmatrix} \frac{0|0}{2|1} \\ \frac{0}{12} \end{pmatrix}, \begin{pmatrix} \frac{0|0}{2|1} \\ \frac{0|0}{1|2} \end{pmatrix} \right\},$$

Thus

$$c^2 \in \left\{ \begin{pmatrix} \frac{0}{12} \\ \frac{0|0}{1|2} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{0} \end{pmatrix}, \begin{pmatrix} \frac{0|0}{2|1} \\ \frac{0}{12} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{0} \end{pmatrix}, \begin{pmatrix} \frac{0|0}{2|1} \\ \frac{0|0}{1|2} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{0} \end{pmatrix} \right\}.$$

Example 4.15. Let us count the codimension 1 elements in $\mathfrak{3} \circledast_{cc} \mathfrak{1}$. Let $c^1 = \frac{B_1|B_2}{A_1|A_2} \in P_2'(\mathfrak{3}) \times P_2'(\mathfrak{1})$. Twelve bipartitions are produced when the six two-block partitionings of 123 are paired with $0|1$ and $1|0$. In addition, pre and post appending an empty block to 123 produces an additional four, of which $\frac{0|1}{0|123}$ and $\frac{123|0}{1|0}$ are discarded for dimensional reasons. Thus the codimension 1 elements in $\mathfrak{3} \circledast_{cc} \mathfrak{1}$ arise from 14 initial bipartitions c^1. Routine calculations show that $\left\{c^2\right\}$ is a singleton set unless $c^1 \in \left\{ \frac{0|1}{12|3}, \frac{0|1}{13|2}, \frac{0|1}{23|1}, \frac{0|1}{123|0} \right\}$. Of these, the first three give rise to two codimension 1 elements and the last gives rise to eight (corresponding to the eight terms of $\Delta_P(P_3)$). For example, if $c^1 = \frac{0|1}{12|3}$, then

$$c^2 \in \left\{ \begin{pmatrix} \frac{0}{12} \\ \frac{0|0}{1|2} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{0|0}{2|1} \\ \frac{0}{12} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{3} \end{pmatrix} \right\}.$$

Thus $\mathfrak{3} \circledast_{cc} \mathfrak{1}$ contains exactly 24 codimension 1 elements.

4.6 Top Dimensional Coherence

Define the *input and output vacuosities* of a $q \times p$ bipartition matrix C by $\hat{v}(C) := \sum_{i \in \mathbf{q}} v(\overset{\wedge}{\alpha}_i(C))$ and $\overset{\vee}{v}(C) := \sum_{j \in \mathbf{p}} v(\overset{\vee}{\beta}_j(C))$, respectively (see (2.4)). In particular, $\hat{v}(C) = 0$ if and only if all blocks in the input partitions of C are non-empty. This occurs, for example, when the columns of C are indecomposable, and dually for $\overset{\vee}{v}(C) = 0$. Of course, an elementary matrix has total vacuosity zero.

A framed matrix B *embeds* in a framed matrix C if there exists an injection $\iota : B \to C$ such that $\iota(B) = C|_{B^1}$.

Definition 4.16. A $q \times p$ framed matrix C with $pq > 1$ is **Dimensionally Complete** (**DC**) if one of the following conditions holds:

1. $\mathbf{is}(C) = \varnothing$ or $\mathbf{os}(C) = \varnothing$.
2. If a $t \times s$ framed matrix B embeds in C, then

$$|B|^{row} = \overset{\vee}{v}\left(B^1\right) + \sum_{j \in \mathfrak{s}} |B_{*j}|^{row} \quad \text{and} \quad |B|^{col} = \hat{v}\left(B^1\right) + \sum_{i \in \mathfrak{t}} |B_{i*}|^{col}.$$

A framed element c is **Dimensionally Complete** (**DC**) if all formally indecomposable factors of c are DC.

By definition of dimension, a framed row matrix C with $\mathbf{is}(C) \neq \varnothing$ and $\mathbf{os}(C) \neq \varnothing$ is DC if and only if $\overset{\vee}{v}\left(B^1\right) = 0$ for all subframed row matrices B in C, and dually for framed column matrices. Furthermore, elementary matrices, and hence bipartitions, are DC.

Definition 4.17. A maximally coherent DC matrix is **Top Dimensionally (TD) coherent**. A framed element c with TD coherent formally indecomposable factors is **TD coherent**. Denote the subset of TD coherent elements in $\mathbf{a} \circledast_{cc} \mathbf{b}$ by $\mathbf{a} \circledast_{td} \mathbf{b}$ and the subset of TD coherent formal products in $\mathbf{a} \circledast_{cc}^{r} \mathbf{b}$ by $\mathbf{a} \circledast_{td}^{r} \mathbf{b}$.

The definition of maximal coherence implies

Proposition 4.18. Coherent elementary matrices are TD coherent, and the rows and columns of a TD coherent matrix are TD coherent.

Example 4.19. The indecomposable maximally coherent matrix

$$C = \begin{pmatrix} \frac{1|0|2}{0|1|0} & \frac{12}{0} \end{pmatrix}$$

is not TD coherent because $|C|^{row} = 0 \neq \overset{\vee}{v}(C) = 1$, but the indecomposable maximally totally coherent matrix

$$D = \begin{pmatrix} \frac{1|2}{0|1} & \frac{12}{0} \end{pmatrix}$$

is TD coherent because $|D|^{row} = \overset{\vee}{v}(D) = |D|^{col} = \hat{v}(D) = 0$.

When verifying the TD coherence of a framed matrix $[C]$, it suffices to check that its canonical representative $\bar{C}$ is TD coherent.

4.7 The Dimension of a TD Coherent Matrix

Unlike the dimension of a general framed matrix, the dimension of a *TD coherent* framed matrix c is completely determined by the cardinalities of its i/o sets and its maximal factorization length $l(C)$.

Proposition 4.20. If C is a non-null TD coherent framed matrix, then

$$|C| = \#\mathbf{is}(C) + \#\mathbf{os}(C) - l\,(C)\,. \tag{4.6}$$

Proof. Assume C is an elementary matrix. Then C is a $q\times p$ maximally coherent elementary matrix and $l(C)=1$. If $p=q=1$, the conclusion follows by definition of dimension. If $p=1$, $q\geq 2$, and $\mathbf{is}\,(C)\neq\varnothing$, maximal row coherence implies $\#\mathbf{is}\,(C)=1$ by Proposition 3.20. Since C has equal singleton denominators, $|C|^{col}=|C|^{row}=0$ so that $|C|=|C|^{ent}=\#\mathbf{os}\,(C)$, and dually for $p\geq 2$, $q=1$, and $\mathbf{os}\,(C)\neq\varnothing$. If $p,q\geq 2$, then $|C|=1$ when $\mathbf{is}\,(C)\,,\mathbf{os}\,(C)\neq\varnothing$; otherwise $|C|=0$ (see Example 3.32). The conclusion follows in each case.

Let $r=l\,(C)$. Since the initial bipartition matrix C^1 is maximally precoherent,

$$\left|\overset{\wedge}{\mathbf{e}}(C^1)\right| = \left|\overset{\wedge}{eq}(C^1)\right| = \#\mathbf{is}(C)-r \quad \text{when} \quad \mathbf{is}(C)\neq\varnothing \quad\text{and} \tag{4.7}$$

$$\left|\overset{\vee}{\mathbf{e}}(C^1)\right| = \left|\overset{\vee}{eq}(C^1)\right| = \#\mathbf{os}(C)-r \quad \text{when} \quad \mathbf{os}(C)\neq\varnothing. \tag{4.8}$$

Case 4.21. $\mathbf{is}(C)=\varnothing$ or $\mathbf{os}(C)=\varnothing$. Then $h\,(C)=1$ and either $|C|=|C|^{col}=\#\mathbf{os}\,(C)-r$ or $|C|=|C|^{row}=\#\mathbf{is}\,(C)-r$ so that Formula 4.6 holds.

Case 4.22. $(\#\mathbf{in}(C),\#\mathbf{os}(C))=(m,n)\neq(0,0)$. Assume Formula 4.6 holds for all non-null TD coherent matrices B such that $\#\mathbf{in}(B)+\#\mathbf{os}(B)<m+n$ and $(\#\mathbf{in}(B),\,\#\mathbf{os}(B))\leq(m,n)$. Since C is DC,

$$\begin{aligned} |C| &= |C|^{row}+|C|^{col}+|C|^{ent} \\ &= \overset{\vee}{v}\left(C^1\right)+\sum_{j\in\mathfrak{p}}|C_{*j}|^{row}+\sum_{j\in\mathfrak{p}}|C_{*j}|^{col}+\sum_{j\in\mathfrak{p}}|C_{*j}|^{ent} = \overset{\vee}{v}\left(C^1\right)+\sum_{j\in\mathfrak{p}}|C_{*j}|\,. \end{aligned} \tag{4.9}$$

For each j, the framed column matrix C_{*j} is TD coherent by Proposition 4.18. Let $r_j := l(\overset{\vee}{\beta}_j(C^1))$ and let $C_{*j}=C_{*j}^1\cdots C_{*j}^{r_j}$; then $|C_{*j}|=\#\mathbf{is}(C_{*j})+\#\mathbf{os}(C_{*j})-l(\overset{\vee}{\beta}_j(C^1))$ by the induction hypotheses. But $\#\mathbf{os}(C_{*j})=|\overset{\vee}{\beta}_j(C^1)|+l(\pi(\overset{\vee}{\beta}_j(C^1)))$ implies

$$\begin{aligned} |C_{*j}| &= \#\mathbf{is}(C_{*j})+|\overset{\vee}{\beta}_j(C^1)|-l(\overset{\vee}{\beta}_j(C^1))+l(\pi(\overset{\vee}{\beta}_j(C^1))) \\ &= \#\mathbf{is}(C_{*j})+|\overset{\vee}{\beta}_j(C^1)|-\overset{\vee}{v}_j(C^1), \end{aligned} \tag{4.10}$$

and by combining Formulas (4.9), (4.10), and (4.8) we obtain

$$\begin{aligned} |C| &= \sum_{j\in\mathfrak{p}}|C_{*j}|+\overset{\vee}{v}\left(C^1\right) = \sum_{j\in\mathfrak{p}}\left[\#\mathbf{is}(C_{*j})+|\overset{\vee}{\beta}_j(C^1)|-\overset{\vee}{v}_j(C^1)\right]+\overset{\vee}{v}\left(C^1\right) \\ &= \sum_{j\in\mathfrak{p}}\left[\#\mathbf{is}(C_{*j})+|\overset{\vee}{\beta}_j(C^1)|\right] = \#\mathbf{is}(C)+\left|\overset{\vee}{\mathbf{e}}(C^1)\right| = \#\mathbf{is}(C)+\#\mathbf{os}(C)-r. \end{aligned}$$

Q.E.D.

Corollary 4.23. If c is a non-null TD coherent framed element on $\frac{\mathbf{b}}{\mathbf{a}}$, then

$$|c| = \#\mathbf{a} + \#\mathbf{b} - l\,(c)\,. \tag{4.11}$$

Proof. Apply Proposition 4.20 to c. Q.E.D.

Since a coherent bipartition is a TD coherent framed element we immediately obtain

Corollary 4.24. If a bipartition $\frac{\beta}{\alpha} \in P'_r\,(\mathbf{a}) \times P'_r\,(\mathbf{b})$ is coherent, then $\frac{\beta}{\alpha}$ is maximally totally coherent and

$$\left|\frac{\beta}{\alpha}\right| = \#\mathbf{a} + \#\mathbf{b} - r = |\alpha \Cup \beta|. \tag{4.12}$$

Corollary 4.25. If $c \in \mathbf{a} \circledast_{cc} \mathbf{b}$ with initial bipartition c^1, then $|c| \leq \left|c^1\right|$.

Proof. By Corollaries 4.23 and (4.12), $|c|$ attains its maximum value $\left|c^1\right|$ when c is TD coherent. Q.E.D.

5 Face Operators and Chain Complexes

5.1 The Face Operator $\tilde{\delta}$ on $\mathfrak{m}\circledast_{cc}\mathfrak{n}$

Let $m, n \geq 0$. In this subsection we define a face operator $\tilde{\delta}$ on $\mathfrak{m} \circledast_{cc} \mathfrak{n}$ in terms of partitioning actions on bipartition matrices. Incoherent matrices so obtained are either coheretized or discarded (cf. Example 5.3).

Recall Proposition 4.11, which asserts that a top level structure matrix within a coherent framed matrix is totally coherent.

Definition 5.1. Given $c \in \mathfrak{m} \circledast_{cc} \mathfrak{n}$, let $\tilde{\delta}(c)$ denote the set of **coherent codimension** 1 **faces of** c. If $c = \frac{\mathfrak{n}}{\mathfrak{m}}$ define

$$\tilde{\delta}\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) := \left\{ \begin{array}{cc} \varnothing, & m+n<2 \\ \mathfrak{m} \circledast_{td}^2 \mathfrak{n}, & m+n \geq 2. \end{array} \right. \tag{5.1}$$

Otherwise, let $c = C_1 \cdots C_r$ and define

$$\tilde{\delta}(c) := \bigcup_{\substack{1 \leq k \leq r \\ F \in \tilde{\delta}(C_k)}} \left\{ C_1 \cdots C_{k-1} \cdot F \cdot C_{k+1} \cdots C_r \right\},$$

where $\tilde{\delta}\,(C_k)$ denotes the set of all $|C_k| - 1$ dimensional coherent framed matrices that arise from C_k in the following way: Given a positive dimensional top level structure matrix E within C_k, let B be either E or a positive dimensional row, column, or elementary bipartition entry of E. If B is a/an

1. elementary bipartition $\frac{\mathbf{b}}{\mathbf{a}}$, let $D \in \{B_1 B_2 \in \tilde{\delta}\left(\frac{\mathbf{b}}{\mathbf{a}}\right)\}$.

2. $1 \times p$ matrix, $p \geq 2$, let $D \in \{$coheretizations $B_1' B_2$ of $B_1 B_2 = \partial_{\mathbf{M},\mathbf{N}}(B)$, where $\partial_{\mathbf{M},\mathbf{N}}(B)$ ranges over all row actions$\}$.

3. $q \times 1$ matrix, $q \geq 2$, let $D \in \{$coheretizations $B_1 B_2'$ of $B_1 B_2 = \partial_{\mathbf{M},\mathbf{N}}(B)$, where $\partial_{\mathbf{M},\mathbf{N}}(B)$ ranges over all column actions $\}$.

4. $q \times p$ matrix, $p, q \geq 2$, let $D \in \{$coheretizations $B_1' B_2'$ of $B_1 B_2 = \partial_{\mathbf{M},\mathbf{N}}(B)$, where $\partial_{\mathbf{M},\mathbf{N}}(B)$ ranges over all row i/column j actions $\}$.

Given D, let $t \times s$ be the matrix dimensions of C_k and let C_k^D be the $t \times s$ framed matrix representative obtained from C_k via the replacement $B \leftarrow D$. If C_k^D is coherent and formally decomposable, then $C_k^D \in \tilde{\delta}(C_k)$. If C_k^D is formally indecomposable and F is the canonical representative of $[C_k^D]$, then $F \in \tilde{\delta}(C_k)$ if either $s, t \geq 2$ or

(a) $s \geq 2$, $C_k = (c_1 \cdots c_s)$, $F = (f_1 \cdots f_s)$, and for some non-strongly extreme pair $(\mathbf{M}^i, \mathbf{N}^i)$ and some $j \in \mathfrak{s}$, the entry action $\partial_{\mathbf{M}^i,\mathbf{N}^i}(c_j^1) = f_j^1$.

(b) $t \geq 2$, $C_k = (c_1 \cdots c_t)^T$, $F = (f_1 \cdots f_t)^T$, and for some non-strongly extreme pair $(\mathbf{M}^j, \mathbf{N}^j)$ and some $i \in \mathfrak{t}$, the entry action $\partial_{\mathbf{M}^j,\mathbf{N}^j}(c_i^1) = f_i^1$.

When $m, n \geq 2$, the set $\tilde{\delta}\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right) = \mathfrak{m} \circledast_{td}^2 \mathfrak{n}$ consists of all framed partitions of $\frac{\mathfrak{n}}{\mathfrak{m}}$ of length 2.

Example 5.2. Consider the 5-dimensional coherent framed element

$$c = C_1 C_2 = \begin{pmatrix} \frac{1}{1} \\ \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{2|34}{0|0} & \frac{24|3}{2|34} \end{pmatrix}.$$

Write $C_2 = (c_1 \; c_2)$ and consider the 4-dimensional 1×1 (totally) coherent top level element

$$c_2^1 = B_1 B_2 = \frac{24|3}{2|34} = \begin{pmatrix} \frac{2}{2} \\ \frac{4}{2} \end{pmatrix} \begin{pmatrix} \frac{3}{0} & \frac{3}{34} \end{pmatrix}.$$

Six admissible formally indecomposable matrices F arise from two column actions and four row actions on the indecomposable factors B_i. For example, the replacement $B_1 \leftarrow D$, where D is the right column action

$$D = \partial_{\mathbf{M},\mathbf{N}}(B_1) = \begin{pmatrix} \frac{0|2}{2|0} \\ \frac{4|0}{2|0} \end{pmatrix} = \begin{pmatrix} \frac{0}{2} \\ \frac{0}{2} \\ \frac{4}{2} \end{pmatrix} \begin{pmatrix} \frac{2}{0} & \frac{2}{0} \\ \frac{0}{0} & \frac{0}{0} \end{pmatrix},$$

produces the 3-dimensional coherent bipartition

$$\frac{4|2|3}{2|0|34} = \begin{pmatrix} \frac{0}{2} \\ \frac{0}{2} \\ \frac{4}{2} \end{pmatrix} \begin{pmatrix} \frac{2}{0} & \frac{2}{0} \\ \frac{0}{0} & \frac{0}{0} \end{pmatrix} \begin{pmatrix} \frac{3}{0} & \frac{3}{34} \end{pmatrix}.$$

Alternatively, $\frac{4|2|3}{2|0|34}$ is obtained by appropriately partitioning the biblock $\frac{24}{2}$. The canonical representative

$$F = (f_1\ f_2) = \left(\left(\frac{2|34}{0|0} \right)\ \left(\frac{4|2|3}{2|0|34} \right) \right)$$

given by the replacement $c_2^1 \leftarrow \frac{4|2|3}{2|0|34}$ is coherent, indecomposable, and satisfies $f_2^1 = \partial_{(\{2\},\{4\})}(c_2^1)$, Since $(\{2\},\{4\})$ is not strongly extreme, F is admissible by Definition 5.1, condition (a), and $C_1 \cdot F \in \tilde{\delta}(c)$.

Our next example demonstrates that partitioning actions may fail to preserve coherence.

Example 5.3. Consider the dimension 4 coherent framed element

$$c = \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{1|2|3}{3|2|0} & \frac{123}{0} \end{pmatrix}.$$

The partitioning action given by the replacement $\frac{123}{0} \leftarrow \frac{12|3}{0|0}$ factors as

$$\begin{pmatrix} \frac{1|2|3}{3|2|0} & \frac{12|3}{0|0} \end{pmatrix} = \begin{pmatrix} \frac{1|2}{3|2} & \frac{12}{0} \\ \frac{0|0}{3|2} & \frac{0}{0} \end{pmatrix} \begin{pmatrix} \frac{3}{0} & \frac{3}{0} & \frac{3}{0} & \frac{3}{0} \end{pmatrix}.$$

Since the 2×2 (row) incoherent factor fails to admit a 3-dimensional coheretization, the given partitioning action fails to preserve coherence, and consequently, fails to produce an element of $\tilde{\delta}(c)$.

Proposition 5.4. If C is a formally indecomposable TD coherent matrix, then $F \in \tilde{\delta}(C)$ is TD coherent if and only if F is formally decomposable.

Proof. A formally indecomposable $F \in \tilde{\delta}(C)$ is not TD coherent for dimensional reasons by Proposition 4.20, and a formally decomposable $F \in \tilde{\delta}(C)$ is TD coherent by a straightforward check. Q.E.D.

Example 5.5. The 3-dimensional maximally coherent framed element

$$c = \left(\mathbf{C}_1\mathbf{C}_2\mathbf{C}_3 = \frac{0|13|2}{2|13|0},\ \begin{pmatrix} \frac{0}{2} \\ \frac{0}{2} \\ \frac{0}{2} \\ \frac{0}{2} \end{pmatrix} \begin{pmatrix} \frac{0|1}{0|1} & \frac{0|1}{3|0} \\ \frac{3}{1} & \frac{3|0}{3|0} \end{pmatrix} \begin{pmatrix} \frac{2}{0} & \frac{2}{0} & \frac{2}{0} & \frac{2}{0} \end{pmatrix} \right)$$

has exactly fourteen (14) codimension 1 (decomposable) faces. To see this, note that $c = C_1C_2C_3$, where $C_1 = \mathbf{C}_1$, $C_2 = \left(\mathbf{C}_2, C_2^1\right)$, and $C_3 = \mathbf{C}_3$. The only positive dimensional top level structure matrix within c is

$$C_2^1 = \begin{pmatrix} \frac{0|1}{0|1} & \frac{0|1}{3|0} \\ \frac{3}{1} & \frac{3|0}{3|0} \end{pmatrix}.$$

Thus a codimension 1 face in $\tilde{\delta}(c)$ has the form $C_1 \cdot F \cdot C_3$, where $F = \left(\mathbf{C}_2, F^1\right) \in \tilde{\delta}(C_2)$ and F^1 is a totally coherent coheretization of some row i/column j action $\partial_{\mathbf{M},\mathbf{N}}\left(C_2^1\right)$ given by Definition 5.1, item (4).

One codimension 1 face arises from the row 1/column 1 incoherent action with $\mathcal{V}_{11} = \varnothing$:

$$E = \begin{pmatrix} \frac{0|0|1}{0|1|0} & \frac{0|1|0}{3|0|0} \\ \frac{3|0}{1|0} & \frac{3|0}{3|0} \end{pmatrix} = \begin{pmatrix} \frac{0|0}{0|1} & \frac{0}{3} \\ \frac{0|0}{0|1} & \frac{0}{3} \\ \frac{3}{1} & \frac{3}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1|0}{0|0} & \frac{1|0}{0|0} \\ \frac{0}{0} & \frac{0}{0} & \frac{0}{0} & \frac{0}{0} \end{pmatrix}$$

the first factor of which coheretizes one way and gives

$$F^1 = \begin{pmatrix} \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{3}{1} & \frac{3}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1|0}{0|0} & \frac{1|0}{0|0} \\ \frac{0}{0} & \frac{0}{0} & \frac{0}{0} & \frac{0}{0} \end{pmatrix}.$$

Twelve codimension 1 faces arise from row 2/column 1 actions. Four are given by coherent actions with $\mathcal{V}_{21} = \varnothing$:

$$F^1 \in \left\{ \begin{pmatrix} \frac{0|1|0}{0|1|0} & \frac{0|1}{3|0} \\ \frac{3|0}{0|1} & \frac{3|0|0}{3|0|0} \end{pmatrix}, \begin{pmatrix} \frac{0|1|0}{0|1|0} & \frac{0|1}{3|0} \\ \frac{3|0}{0|1} & \frac{3|0|0}{0|3|0} \end{pmatrix}, \begin{pmatrix} \frac{0|1|0}{0|0|1} & \frac{0|1}{3|0} \\ \frac{3|0}{0|1} & \frac{3|0|0}{3|0|0} \end{pmatrix}, \begin{pmatrix} \frac{0|1|0}{0|0|1} & \frac{0|1}{3|0} \\ \frac{3|0}{0|1} & \frac{3|0|0}{0|3|0} \end{pmatrix} \right\}.$$

Four arise from the incoherent action with $\mathcal{V}_{21} = \{(1,2)\}$:

$$E \in \left\{ \begin{pmatrix} \frac{0|1|0}{0|1|0} & \frac{0|1|0}{3|0|0} \\ \frac{0|3}{1|0} & \frac{0|3|0}{3|0|0} \end{pmatrix}, \begin{pmatrix} \frac{0|0|1}{0|1|0} & \frac{0|0|1}{0|3|0} \\ \frac{0|3}{1|0} & \frac{0|3|0}{0|3|0} \end{pmatrix} \right\}.$$

The first factor of the first matrix coheretizes two ways and gives

$$F^1 \in \left\{ \begin{pmatrix} \frac{0|1}{0|1} & \frac{0|1}{3|0} \\ \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{0}{1} & \frac{0}{3} \end{pmatrix} \begin{pmatrix} \frac{0}{0} & \frac{0}{0} & \frac{0}{0} & \frac{0}{0} \\ \frac{3}{0} & \frac{3}{0} & \frac{3|0}{0|0} & \frac{3|0}{0|0} \end{pmatrix}, \begin{pmatrix} \frac{0|1}{0|1} & \frac{0|1}{3|0} \\ \frac{0}{1} & \frac{0}{3} \\ \frac{0|0}{1|0} & \frac{0|0}{0|3} \end{pmatrix} \begin{pmatrix} \frac{0}{0} & \frac{0}{0} & \frac{0}{0} & \frac{0}{0} \\ \frac{3}{0} & \frac{3}{0} & \frac{3|0}{0|0} & \frac{3|0}{0|0} \end{pmatrix} \right\};$$

the second factor of the second matrix also coheretizes two ways and gives

$$F^1 \in \left\{ \begin{pmatrix} \frac{0|0}{0|1} & \frac{0}{0} \\ \frac{0|0}{0|1} & \frac{0}{0} \\ \frac{0}{1} & \frac{0}{0} \\ \frac{0}{1} & \frac{0}{0} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{0|1}{0|0} & \frac{0|1}{3|0} \\ \frac{3}{0} & \frac{3|0}{0|0} & \frac{3|0}{0|0} \end{pmatrix}, \begin{pmatrix} \frac{0|0}{0|1} & \frac{0}{0} \\ \frac{0|0}{0|1} & \frac{0}{0} \\ \frac{0}{1} & \frac{0}{0} \\ \frac{0}{1} & \frac{0}{0} \end{pmatrix} \begin{pmatrix} \frac{1|0}{0|0} & \frac{1}{0} & \frac{0|1}{3|0} \\ \frac{0|3}{0|0} & \frac{3}{0} & \frac{3|0}{0|0} \end{pmatrix} \right\}.$$

And four arise from the incoherent action with $\mathcal{V}_{21} = \varnothing$:

$$E = \begin{pmatrix} \frac{0|0|1}{0|1|0} & \frac{0|1}{3|0} \\ \frac{0|3}{1|0} & \frac{0|3|0}{3|0|0} \end{pmatrix} = \begin{pmatrix} \frac{0|0}{0|1} & \frac{0}{3} \\ \frac{0|0}{0|1} & \frac{0}{3} \\ \frac{0}{1} & \frac{0}{3} \\ \frac{0}{1} & \frac{0}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} \\ \frac{3}{0} & \frac{3}{0} & \frac{3|0}{0|0} & \frac{3|0}{0|0} \end{pmatrix}$$

each factor of which coheretizes two ways and gives

$$F^1 \in \left\{ \begin{pmatrix} \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{0}{1} & \frac{0}{3} \\ \frac{0|0}{1|0} & \frac{0|0}{0|3} \end{pmatrix} \begin{pmatrix} \frac{1|0}{0|0} & \frac{1}{0} & \frac{0|1}{0|0} & \frac{0|1}{0|0} \\ \frac{0|3}{0|0} & \frac{3}{0} & \frac{3|0}{0|0} & \frac{3|0}{0|0} \end{pmatrix}, \begin{pmatrix} \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{0}{1} & \frac{0}{3} \end{pmatrix} \begin{pmatrix} \frac{1|0}{0|0} & \frac{1}{0} & \frac{0|1}{0|0} & \frac{0|1}{0|0} \\ \frac{0|3}{0|0} & \frac{3}{0} & \frac{3|0}{0|0} & \frac{3|0}{0|0} \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{0}{1} & \frac{0}{3} \\ \frac{0|0}{1|0} & \frac{0|0}{0|3} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{0|1}{0|0} & \frac{0|1}{0|0} & \frac{0|1}{0|0} \\ \frac{3}{0} & \frac{3|0}{0|0} & \frac{3|0}{0|0} & \frac{3|0}{0|0} \end{pmatrix}, \begin{pmatrix} \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{0|0}{0|1} & \frac{0|0}{3|0} \\ \frac{0}{1} & \frac{0}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{0|1}{0|0} & \frac{0|1}{0|0} & \frac{0|1}{0|0} \\ \frac{3}{0} & \frac{3|0}{0|0} & \frac{3|0}{0|0} & \frac{3|0}{0|0} \end{pmatrix} \right\}.$$

Finally, one codimension 1 face arises from the row 2/column 2 incoherent action with $\mathcal{V}_{22} = \varnothing$:

$$E = \begin{pmatrix} \frac{0|1}{0|1} & \frac{0|0|1}{0|3|0} \\ \frac{0|3}{0|1} & \frac{0|3|0}{3|0|0} \end{pmatrix} = \begin{pmatrix} \frac{0}{0} & \frac{0|0}{0|3} \\ \frac{0}{0} & \frac{0|0}{0|3} \\ \frac{0}{0} & \frac{0}{3} \\ \frac{0}{0} & \frac{0}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{1} & \frac{1}{0} & \frac{1}{0} \\ \frac{3}{1} & \frac{3|0}{0|0} & \frac{3|0}{0|0} \end{pmatrix}$$

the second factor of which coheretizes one way and gives

$$F^1 = \begin{pmatrix} \frac{0}{0} & \frac{0|0}{0|3} \\ \frac{0}{0} & \frac{0|0}{0|3} \\ \frac{0}{0} & \frac{0}{3} \\ \frac{0}{0} & \frac{0}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{1} & \frac{0|1}{0|0} & \frac{0|1}{0|0} \\ \frac{3}{1} & \frac{3|0}{0|0} & \frac{3|0}{0|0} \end{pmatrix}.$$

5.2 The Face Operator $\tilde{\partial}$ on $\mathfrak{m}\widetilde{\circledast}_{pp}\mathfrak{n}$

In this subsection we introduce three important concepts: the integrability of certain framed matrices, the "prebalanced framed join" $\mathfrak{m}\widetilde{\circledast}_{pp}\mathfrak{n} \subset \mathfrak{m}\circledast_{cc}\mathfrak{n}$, and a face operator $\tilde{\partial}$ on $\mathfrak{m}\widetilde{\circledast}_{pp}\mathfrak{n}$.

Definition 5.6. Let $\mathcal{M}$ denote the set of coherent framed matrices. A **TD quadratic matrix** has exactly two non-null TD coherent formally indecomposable factors**.** A TD quadratic matrix F is **integrable** if there exists a unique TD coherent $C \in \mathcal{M}$ such that $F \in \tilde{\delta}(C)$. When F is integrable, we refer to C as the **integral of** F and write $\int F = C$. An indecomposable or integrable framed matrix whose TD quadratic subframed matrices are integrable is **totally integrable**.

When $C = \int F$ we have $|C| = |F| + 1$ by the definition of $\tilde{\delta}(C)$. Furthermore, if $F = F_1F_2$, and its factors F_1 and F_2 are indecomposable TD coherent column and row matrices, respectively, then

$$\int F = \frac{\mathbf{os}(F_1)}{\mathbf{is}(F_2)}.$$

Example 5.7. (a) Consider the TD quadratic matrix

$$F = \begin{pmatrix} \frac{1|2}{2|1} & \frac{1|2}{0|0} \\ \frac{0|0}{2|1} & \frac{0|0}{0|0} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{0} \\ \frac{0}{2} & \frac{0}{0} \\ \frac{0}{2} & \frac{0}{0} \end{pmatrix} \begin{pmatrix} \frac{2}{1} & \frac{2}{0} & \frac{2}{0} \\ \frac{0}{1} & \frac{0}{0} & \frac{0}{0} \end{pmatrix}$$

and the matrix

$$C_1 = \begin{pmatrix} \frac{1|2}{2|1} & \frac{1|2}{0|0} \\ \frac{0}{12} & \frac{0|0}{0|0} \end{pmatrix} \in \mathcal{M}.$$

Set $e = \hat{eq}(C_1) = 12$ and $\mathbf{x} = \hat{\mathbf{e}}(F) \sqsubseteq_{diag} \Delta_P^{(1)}(\partial e)$. By Proposition 2.11, part (2), $\mathbf{x}_{11} = 12 \times 2|1$ is the unique diagonal component of e that can be obtained from $\mathbf{x}$ by a single factor replacement; consequently, C_1 is the unique maximally row coherent framed bipartition matrix such that $\partial_{\mathbf{M},\mathbf{N}}(C_1) = F$ for some $(\mathbf{M},\mathbf{N})$. By a dual argument,

$$C_2 = \begin{pmatrix} \frac{1|2}{2|1} & \frac{12}{0} \\ \frac{0|0}{2|1} & \frac{0|0}{0|0} \end{pmatrix} \in \mathcal{M}$$

is the unique maximally column coherent framed matrix such that $\partial_{\mathbf{M},\mathbf{N}}(C_2) = F$ for some $(\mathbf{M},\mathbf{N})$. Since $C_1 \neq C_2$, no $C \in \mathcal{M}$ with the required integrability property exists and F is not integrable.

(b) On the other hand, consider the TD quadratic matrix

$$F = \begin{pmatrix} \frac{2|1}{1|2} & \frac{2|1}{0|0} \\ \frac{0|0}{1|2} & \frac{0|0}{0|0} \end{pmatrix} = \begin{pmatrix} \frac{0}{1} & \frac{0}{0} \\ \frac{2}{1} & \frac{2}{0} \\ \frac{0}{1} & \frac{0}{0} \end{pmatrix} \begin{pmatrix} \frac{1}{0} & \frac{1}{2} & \frac{1}{0} \\ \frac{0}{0} & \frac{0}{2} & \frac{0}{0} \end{pmatrix}$$

and the matrix

$$C = \begin{pmatrix} \frac{12}{12} & \frac{2|1}{0|0} \\ \frac{0|0}{1|2} & \frac{0|0}{0|0} \end{pmatrix} \in \mathcal{M}.$$

Then $\partial_{\{1\}_{11},\{2\}_{11}}(C)=F$, and by Proposition 2.11, part (2), C is the unique element of $\mathcal{M}$ such that $\partial_{\mathbf{M},\mathbf{N}}(C)=F$ for some $(\mathbf{M},\mathbf{N})$. Hence $\int F=C$.

(c) In Example 3.17 we considered the incoherent matrix

$$F=\begin{pmatrix}\frac{0|1}{12|0} & \frac{0|1}{45|0}\end{pmatrix}=\begin{pmatrix}\frac{0}{12} & \frac{0}{45}\\ \frac{0}{12} & \frac{0}{45}\end{pmatrix}\begin{pmatrix}\frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0}\end{pmatrix}$$
$$=\partial_{(\{12\}_{11},\varnothing_{11}),(\{45\}_{12},\varnothing_{12})}\begin{pmatrix}\frac{1}{12} & \frac{1}{45}\end{pmatrix}.$$

Since $\overset{\wedge}{\mathbf{e}}(F)=1245\not\sqsubseteq_{diag}\Delta_P^{(0)}(\partial e)$ for any cell e, Proposition 2.11 does not apply. Nevertheless by inspection, exactly three bipartition matrices C_i satisfy $\partial_{\mathbf{M},\mathbf{N}}(C_i)=F$ for some $(\mathbf{M},\mathbf{N})$, namely,

$$C_1=\begin{pmatrix}\frac{1}{12} & \frac{1}{45}\end{pmatrix},\ C_2=\begin{pmatrix}\frac{0|1}{12|0} & \frac{1}{45}\end{pmatrix},\ \text{and } C_3=\begin{pmatrix}\frac{1}{12} & \frac{0|1}{45|0}\end{pmatrix}.$$

Of these, only C_1 is TD coherent and the coheretization

$$F'=\begin{pmatrix}\frac{0}{12} & \frac{0}{45}\\ \frac{0|0|0|0}{1|2|0|0} & \frac{0|0|0|0}{0|0|4|5}\end{pmatrix}\begin{pmatrix}\frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0}\end{pmatrix}\in\tilde{\delta}(C_1).$$

Thus $\int F'=C_1$; in fact, F' is totally integrable via the coheretizations $\begin{pmatrix}\frac{0}{12}\\ \frac{0|0}{1|2}\end{pmatrix}\begin{pmatrix}\frac{1}{0} & \frac{1}{0} & \frac{1}{0}\end{pmatrix}$ and $\begin{pmatrix}\frac{0}{45}\\ \frac{0|0}{4|5}\end{pmatrix}\begin{pmatrix}\frac{1}{0} & \frac{1}{0} & \frac{1}{0}\end{pmatrix}$ of the incoherent subframed elements $\left(\frac{0|1}{12|0}\right)$ and $\left(\frac{0|1}{45|0}\right)$.

Let $\mathcal{CM}$ denote the subset of all totally integrable matrices in $\mathcal{M}$. Given $C\in\mathcal{M}$, denote the subsets of formally indecomposable and formally decomposable elements of $\tilde{\delta}(C)\cap\mathcal{CM}$ by $\tilde{\delta}_b(C)$ and $\tilde{\delta}_d(C)$, respectively. Let $x\in\{b,d\}$; if $F_1F_2\in\tilde{\delta}(C)$, extend $\tilde{\delta}_x$ as a derivation so that

$$\tilde{\delta}_x(F_1F_2):=\tilde{\delta}_x(F_1)\,F_2\cup F_1\tilde{\delta}_x(F_2).$$

Definition 5.8. Let $m,n\geq 1$ and define $\mathcal{CM}_1(m,n):=\{\frac{\mathfrak{n}}{\mathfrak{m}}\}$. For $r\geq 2$ define

$$\mathcal{CM}_r(m,n):=\bigcup_{\substack{x_i\in\{b,d\}\\ 1\leq i\leq s}}\left\{F\in\tilde{\delta}_{x_s}\cdots\tilde{\delta}_{x_1}\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right):l(F)=r\right\}.$$

The **prebalanced framed join of $\mathfrak{m}$ and $\mathfrak{n}$** is the positively graded set

$$\mathfrak{m}\tilde{\circledast}_{pp}\mathfrak{n}:=\mathcal{CM}_*(m,n)$$

with **face operator** $\tilde{\partial}:=\tilde{\delta}_d\cup\tilde{\delta}_b$.

TD coherent row and column matrices of positive dimension in $\mathcal{CM}$ have the following key property:

Lemma 5.9. Given a positive dimensional TD coherent framed row (respt. column) matrix $C \in \mathcal{CM}$ with at least 2 entries and $F \in \delta_b(C)$, there exists a unique row (respt. column) matrix $D \in \mathcal{M} \diagdown \{C\}$ such that $F \in \delta_b(D)$.

Proof. Let $C = (c_1 \cdots c_p) \in \mathcal{CM}$ be a positive dimensional TD coherent row matrix and let $F = (f_1 \cdots f_p) \in \tilde{\delta}_b(C)$; then $|F| = |C| - 1$. Since F is formally indecomposable, so is F^1. If $h = 1$, the underlying structure matrix $B \neq C^1$ (otherwise, the row action on C^1 implies that F^1 is decomposable); hence B is an elementary bipartition entry of C^1. If $h > 1$, then B lies in some top level entry c_j^h. In either case, there is a j^{th} entry action $f_j^1 = \partial_{M^i,N^i}(c_j^1)$ such that $\left(M^i, N^i\right)$ is not strongly extreme by Definition 5.1. Furthermore, the indecomposability of F^1 implies $\#\mathbf{os}(C) \geq 2$. Let $c_j^1 = \left(A_j^1 | \cdots | A_j^{r_j}, B_j^1 | \cdots | B_j^{r_j}\right)$; then $\left(M^i, N^i\right) \subseteq (A_j^i, B_j^i)$. Let $f_j = C_1 \cdots C_{r_j+1}$. There are two cases:

Case 5.10. $N^i \subseteq B_j^i$ is extreme. Since F^1 is indecomposable and $\left(M^i, N^i\right)$ is not strongly extreme, either $\left(M^i, N^i\right) = (A_j^i, \varnothing)$ for some $i > 1$ or $\left(M^i, N^i\right) = (\varnothing, B_j^i)$ for some $i < r_j$. If $N^i = B_j^i$, set $\ell = i$. Since elements of $\mathcal{CM}$ are totally integrable, all TD quadratic subframed matrices of F are integrable, and in particular, $C_\ell C_{\ell+1}$ is integrable. Let $D' := \int C_\ell C_{\ell+1}$ and set

$$d_j = C_1 \cdots C_{\ell-1} \cdot D' \cdot C_{\ell+2} \cdots C_{r_j+1},$$

where $d_j^1 = f_j^1[\ell]$; then $f_j \in \tilde{\delta}(d_j)$. Obtain D from F via the replacement $f_j \leftarrow d_j$; then $F \in \delta_b(D)$. If $N^i = \varnothing$, set $\ell = i - 1$ and obtain D from F by a similar argument. In either case, $D \neq C$ since the denominators of d_j^1 and c_j^1 are distinct. On the other hand, the numerators of d_j^1 and c_j^1 (hence the numerators of all entries) are equal. But $\overset{\vee}{\mathbf{e}}(D^1) = \overset{\vee}{\mathbf{e}}(C^1)$ implies D^1 is coherent so that $D \in \mathcal{M} \diagdown \{C\}$ as claimed.

Case 5.11. $N^i \subseteq B_j^i$ is not extreme. First, $l\pi(\overset{\vee}{\beta}_j(F^1)) = l\pi(\overset{\vee}{\beta}_j(C^1)) + 1$ by the indecomposability of F^1. Set $e = \overset{\vee}{eq}(C^1)$, $\mathbf{e} = \overset{\vee}{\mathbf{e}}(C^1)$, and $\partial^j_{N^i}(\mathbf{e}) = \overset{\vee}{\mathbf{e}}(F^1)$; then by Proposition 2.11, part (1), there is a unique $\mathbf{x}_{j\ell} \neq \mathbf{e}$ such that $\mathbf{x}_{j\ell} \sqsubseteq \Delta_P^{(q-1)}(e)$. As in Case 5.10, let $D' := \int C_\ell C_{\ell+1}$ and set $d_j = C_1 \cdots C_{\ell-1} \cdot D' \cdot C_{\ell+2} \cdots C_{r_j+1}$, where $d_j^1 = f_j^1[\ell]$; then $f_j \in \tilde{\delta}(d_j)$. Let D be the matrix obtained from F via the replacement $f_j \leftarrow d_j$; then $F \in \delta_b(D)$. But the numerator of $d_j^1 = f_j^1[\ell]$ is the j^{th} factor of $\mathbf{x}_{j\ell}$. Hence $\overset{\vee}{\mathbf{e}}(D^1) = \mathbf{x}_{j\ell}$, the bipartition matrix D^1 is coherent, and $D \in \mathcal{M} \diagdown \{C\}$ as desired.

The proof for column matrices is entirely dual. Q.E.D.

Example 5.12. Consider the totally coherent indecomposable 1×2 matrix

$$C = \left(\begin{array}{cc} \frac{1|2}{1|2} & \frac{12}{0} \end{array} \right).$$

(a) Set $j = 1$ and $\left(M^1, N^1\right) = (\varnothing, \{1\})$. Then N^1 is extreme,

$$F = \left(\begin{array}{cc} \frac{1|0|2}{0|1|2} & \frac{12}{0} \end{array} \right) \in \partial_{(M^1,N^1)}(C), \text{ and } D = \left(\begin{array}{cc} \frac{1|2}{0|12} & \frac{12}{0} \end{array} \right).$$

(b) Set $j = 2$ and $\left(M^2, N^2\right) = (\varnothing, \{2\})$. Then N^2 is not extreme,

$$F = \left(\begin{array}{cc} \frac{1|2}{1|2} & \frac{2|1}{0|0} \end{array} \right) \in \partial_{(M^2,N^2)}(C), \text{ and } D = \left(\begin{array}{cc} \frac{12}{12} & \frac{2|1}{0|0} \end{array} \right).$$

Example 5.13. For the 2-dimensional TD quadratic matrix

$$C=\begin{pmatrix}\frac{0|1}{1|2}\\ \frac{2|0}{1|2}\end{pmatrix}=\begin{pmatrix}\frac{0}{1}\\ \frac{0}{1}\\ \frac{2}{1}\end{pmatrix}\begin{pmatrix}\frac{1}{0} & \frac{1}{2}\\ \frac{0}{0} & \frac{0}{2}\end{pmatrix}=C_1C_2$$

we have

$$\tilde{\delta}\left(C_1\right)=\left\{\begin{pmatrix}\frac{0|0}{1|0}\\ \frac{0|0}{1|0}\\ \frac{0|2}{1|0}\end{pmatrix},\begin{pmatrix}\frac{0|0}{0|1}\\ \frac{0|0}{0|1}\\ \frac{2|0}{0|1}\end{pmatrix}\right\}=\left\{\begin{pmatrix}\frac{0}{1}\\ \frac{0}{1}\\ \frac{0}{1}\\ \frac{0}{1}\end{pmatrix}\begin{pmatrix}\frac{0}{0} & \frac{0}{0}\\ \frac{0}{0} & \frac{0}{0}\\ \frac{2}{0} & \frac{2}{0}\end{pmatrix},\begin{pmatrix}\frac{0}{0}\\ \frac{0}{0}\\ \frac{2}{0}\end{pmatrix}\begin{pmatrix}\frac{0}{1}\\ \frac{0}{1}\\ \frac{0}{1}\end{pmatrix}\right\}$$

(cf. Example 4.19).

Example 5.14. (Cf. Example 7.4). Let

$$\rho=\frac{0|1}{2|13}=\begin{pmatrix}\frac{0}{2}\\ \frac{0}{2}\end{pmatrix}\begin{pmatrix}\frac{1}{1} & \frac{1}{3}\end{pmatrix}=C_1C_2\in\mathfrak{z}\widetilde{\circledast}_{pp}\mathfrak{1};$$

then $\tilde{\partial}(\rho)=\tilde{\partial}(C_1)C_2\cup C_1\tilde{\partial}(C_2)=C_1\tilde{\partial}(C_2)$. In turn, $\tilde{\partial}(C_2):=\{\tilde{\partial}_k(C_2)\}_{1\leq k\leq 7}$ is given by the following row actions $\partial_{\mathbf{M},\mathbf{N}}\left(C_2\right)$:

- $(\mathbf{M},\mathbf{N})=\{(\{1\},\varnothing),(\varnothing,\varnothing))\}$:

$$\tilde{\partial}_1(C_2)=\begin{pmatrix}\frac{0|1}{1|0} & \frac{0|1}{0|3}\end{pmatrix}=\begin{pmatrix}\frac{0}{1} & \frac{0}{0}\\ \frac{0}{1} & \frac{0}{0}\end{pmatrix}\begin{pmatrix}\frac{1}{0} & \frac{1}{0} & \frac{1}{3}\end{pmatrix}.$$

- $(\mathbf{M},\mathbf{N})=\{(\varnothing,\varnothing),(\{3\},\varnothing)\}$:

$$\tilde{\partial}_2(C_2)=\begin{pmatrix}\frac{0|1}{0|1} & \frac{0|1}{3|0}\end{pmatrix}=\begin{pmatrix}\frac{0}{0} & \frac{0}{3}\\ \frac{0}{0} & \frac{0}{3}\end{pmatrix}\begin{pmatrix}\frac{1}{1} & \frac{1}{0} & \frac{1}{0}\end{pmatrix}.$$

- $(\mathbf{M},\mathbf{N})=\{(\{1\},\varnothing),(\{3\},\varnothing)\}$ (cf. Example 3.14):

$$\partial_{\mathbf{M},\mathbf{N}}(C_2)=\begin{pmatrix}\frac{0|1}{1|0} & \frac{0|1}{3|0}\end{pmatrix}=\begin{pmatrix}\frac{0}{1} & \frac{0}{3}\\ \frac{0}{1} & \frac{0}{3}\end{pmatrix}\begin{pmatrix}\frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0}\end{pmatrix}$$

$$\xrightarrow{(\eta_1,\eta_2)}\quad\tilde{\partial}_3(C_2)=\begin{pmatrix}\frac{0|0}{0|1} & \frac{0|0}{3|0}\\ \frac{0}{1} & \frac{0}{3}\end{pmatrix}\begin{pmatrix}\frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0}\end{pmatrix}.$$

$$\xrightarrow{(\eta_2,\eta_1)}\quad\tilde{\partial}_4(C_2)=\begin{pmatrix}\frac{0}{1} & \frac{0}{3}\\ \frac{0|0}{1|0} & \frac{0|0}{0|3}\end{pmatrix}\begin{pmatrix}\frac{1}{0} & \frac{1}{0} & \frac{1}{0} & \frac{1}{0}\end{pmatrix}.$$

- $(\mathbf{M},\mathbf{N}) = \{(\varnothing,\{1\}),(\{3\},\{1\})\}$:

$$\tilde{\partial}_5(C_2) = \begin{pmatrix} \frac{1|0}{0|1} & \frac{1|0}{3|0} \end{pmatrix} = \begin{pmatrix} \frac{1}{0} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{0}{1} & \frac{0}{0} & \frac{0}{0} \end{pmatrix}.$$

- $(\mathbf{M},\mathbf{N}) = \{(\{1\},\{1\}),(\varnothing,\{1\})\}$:

$$\tilde{\partial}_6(C_2) = \begin{pmatrix} \frac{1|0}{1|0} & \frac{1|0}{0|3} \end{pmatrix} = \begin{pmatrix} \frac{1}{1} & \frac{1}{0} \end{pmatrix} \begin{pmatrix} \frac{0}{0} & \frac{0}{0} & \frac{0}{3} \end{pmatrix}.$$

- $(\mathbf{M},\mathbf{N}) = \{(\varnothing,\{1\}),(\varnothing,\{1\})\}$:

$$\tilde{\partial}_7(C_2) = \begin{pmatrix} \frac{1|0}{0|1} & \frac{1|0}{0|3} \end{pmatrix} = \begin{pmatrix} \frac{1}{0} & \frac{1}{0} \end{pmatrix} \begin{pmatrix} \frac{0}{1} & \frac{0}{3} \end{pmatrix}.$$

5.3 The Chain Complex $(\mathfrak{m}\widetilde{\circledast}_{pp}\mathfrak{n},\tilde{\partial})$

Let R be a commutative ring with unity, and let

$$\tilde{C}_{n,m} := \langle \mathfrak{m}\widetilde{\circledast}_{pp}\mathfrak{n} \rangle$$

denote the free R-module generated by the set $\mathfrak{m}\widetilde{\circledast}_{pp}\mathfrak{n}$. Define a degree -1 endomorphism $\tilde{\partial} : \tilde{C}_{n,m} \to \tilde{C}_{n,m}$ on a generator $\rho = C_1 \cdots C_r \in \tilde{C}_{n,m}$ by

$$\tilde{\partial}(\rho) := \sum_{1\leq s\leq r} (-1)^{|C_1|+\cdots+|C_{s-1}|}\, C_1 \cdots \tilde{\partial}(C_s) \cdots C_r, \tag{5.2}$$

where $\tilde{\partial}(C_s) := \sum_{F\in\tilde{\delta}(C)} (-1)^{\varepsilon_F} F$ and ε_F is defined as follows: Let $q \times p$ be the matrix dimensions of C_s and let $C = (c_{ij}) := C_s$.

(a) If $F = (f_{ij}) \in \delta_b(C)$, let $(i,j) \in \mathfrak{q}\times\mathfrak{p}$ be the smallest pair of positive integers such that $|f_{ij}| = |c_{ij}| - 1$; then

$$\varepsilon_F := \sum_{\substack{1\leq k<i \\ j\in\mathfrak{p}}} [\#\mathbf{is}(C_{*j}) - l(c_{kj})] + \sum_{1\leq \ell<j} [\#\mathbf{os}(C_{i\ell}) - l(f_{i\ell})].$$

(b) Let $F = F_1F_2 \in \delta_d(C)$. There are two cases.

Case 5.15. C is TD coherent. Let

$$D_1|D_2 = \mathbf{is}(F_1)|\mathbf{is}(F_2) \uplus \mathbf{os}(F_1)|\mathbf{os}(F_2) \in P(\mathbf{is}(C) \uplus \mathbf{os}(C)),$$

and let $\overset{\diamond}{\varepsilon}_F := (-1)^{\#D_1} \cdot \mathit{shuff}(D_1;D_2)$. Denote the signs of $\overset{\wedge}{\mathbf{e}}(C)$ in $\Delta^{(q-1)}(P_{\#\mathbf{is}(C)})$ and $\overset{\vee}{\mathbf{e}}(C)$ in $\Delta^{(p-1)}(P_{\#\mathbf{os}(C)})$ by $\overset{\wedge}{\varepsilon}_C$ and $\overset{\vee}{\varepsilon}_C$, respectively; then

$$\varepsilon_F := \begin{cases} \overset{\diamond}{\varepsilon}_F + \overset{\vee}{\varepsilon}_{F_1} + \overset{\wedge}{\varepsilon}_{F_2}, & C = \frac{\mathfrak{m}}{\mathfrak{n}} \\ \overset{\diamond}{\varepsilon}_F, & \text{otherwise.} \end{cases}$$

Case 5.16. C is not TD coherent. Let $C \in \delta_b(D)$ for some TD coherent D. Then $F \in \delta_b\delta_d(D)$; furthermore, if $E = E_1E_2 \in \delta_d(D)$ and $F_i \in \delta_b(E_i)$, then $\varepsilon_F = \varepsilon_C\varepsilon_E(-1)^{|E_{i-1}|}\varepsilon_{F_i}+1$.

Proposition 5.17. The map $\tilde{\partial}$ is a differential on $\tilde{C}_{n,m}$, i.e., $\tilde{\partial}^2 = 0$.

Proof. We only prove $\tilde{\partial}^2(\frac{\mathfrak{n}}{\mathfrak{m}}) = 0$, which illustrates the basic arguments. Non-trivial cases assume $m+n \geq 3$ and apply Lemma 5.9. Consider a component $\tilde{\partial}_{E_1E_2}(\frac{\mathfrak{n}}{\mathfrak{m}}) := E_1E_2 \in \tilde{\partial}(\frac{\mathfrak{n}}{\mathfrak{m}})$ and a face component $F \in \tilde{\partial}(E_1E_2)$. It suffices to show that there exists a unique component $E_1'E_2'$ of $\tilde{\partial}(\frac{\mathfrak{n}}{\mathfrak{m}})$ distinct from E_1E_2 such that F is a codimension 1 face of $E_1'E_2'$. There are two cases.

Case 5.18. $F = \rho\cdot E_2 \in \tilde{\partial}(E_1)E_2$. Then ρ is coherent by definition and may or may not be decomposable. There are two subcases.

Subcase 5.18.a. $\rho = E_1^1E_1^2$ is decomposable. Then by definition there is a unique coherent matrix C such that $E_1^2E_2 \in \tilde{\partial}(C)$. Thus $E_1^1C \in \tilde{\partial}\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right)$ and $E_1'E_2' := E_1^1C$ has the required properties.

Subcase 5.18.b. ρ is indecomposable. Let $(B,C) := (\rho, E_2)$; by Lemma 5.9, there is a unique coherent matrix D such that $\rho \in \tilde{\partial}(D)$. Then $DE_2 \in \tilde{\partial}\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right)$ and $E_1'E_2' := DE_2$ has the required properties.

Case 5.19. $F = E_1\cdot\rho \in E_1\tilde{\partial}(E_2)$. Then ρ is coherent and may or may not be decomposable. There are two subcases.

Subcase 5.19.a. $\rho = E_1^2E_2^2$ is decomposable. Then by definition there is a unique coherent matrix C such that $E_1E_2^1 \in \tilde{\partial}(C)$. Thus $CE_2^2 \in \tilde{\partial}\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right)$ and $E_1'E_2' := CE_2^2$ has the required properties.

Subcase 5.19.b. ρ is indecomposable. Let $(C,B) := (E_1, \rho)$; by Lemma 5.9, there is a unique coherent matrix D such that $\rho \in \tilde{\partial}(D)$. Then $E_1D \in \tilde{\partial}\left(\frac{\mathfrak{n}}{\mathfrak{m}}\right)$ and $E_1'E_2' := E_1$ has the required properties.

Q.E.D.

5.4 The Balanced Framed Join $\mathfrak{m}\circledast_{pp}\mathfrak{n}$

Definition 5.20. A coherent framed element c is **balanced** if each top level (totally coherent) structure matrix in c is primitively coherent (cf. Subsection 3.5). The **balanced framed join of $\mathfrak{m}$ and $\mathfrak{n}$** is the subset

$$(\mathfrak{m}\circledast_{pp}\mathfrak{n}, \tilde{\partial}) \subseteq (\mathfrak{m}\tilde{\circledast}_{pp}\mathfrak{n}, \tilde{\partial})$$

of balanced framed elements of height $h \leq 3$.

In fact, if $\mathbf{c} = \mathfrak{n}/\mathfrak{m}$, a balanced farmed element $c \in \mathfrak{m}\circledast_{pp}\mathfrak{n}$ is characterized by a 2-level path of generalized bipartitions $c = (T^1(\mathbf{c}), T^2(\mathbf{c}))$ and indexes some face of the polytope $PP_{n,m}$ defined in Section 8 below.

If $0 \leq m \leq 2$ or $0 \leq n \leq 2$, an element $c \in \mathfrak{m}\circledast_{cc}\mathfrak{n}$ is balanced and uniquely determined by a 2-level path for dimensional reasons. Thus we immediately obtain (also see Section 8)

Proposition 5.21. If $0 \leq m \leq 2$ or $0 \leq n \leq 2$, then

$$\mathfrak{m}\circledast_{pp}\mathfrak{n} = \mathfrak{m}\circledast_{cc}\mathfrak{n}. \tag{5.3}$$

Remark 5.22. Our earlier result in [31] agrees with Equality (5.3) in the indicated ranges, and Definition 5.20 extends that result to all $m, n \geq 0$.

5.5 The Reduced Balanced Framed Join $\mathfrak{m} \circledast_{kk} \mathfrak{n}$

Definition 5.23. A bipartition matrix C is **reduced** if it represents a class of bipartition matrices in $\mathfrak{m} \circledast \mathfrak{n}$ such that $C \sim C'$ if and only if C and C' differ only in the number of empty biblocks in their entries. The **reduced** framed join and the **reduced balanced** framed join are denoted by

$$\mathfrak{m} \circledast_{red} \mathfrak{n} := \mathfrak{m} \circledast \mathfrak{n}/\sim \quad \text{and} \quad \mathfrak{m} \circledast_{kk} \mathfrak{n} := \mathfrak{m} \circledast_{pp} \mathfrak{n}/\sim,$$

respectively. The canonical projections (denoted by the same symbol) are

$$\vartheta\vartheta : \mathfrak{m} \circledast \mathfrak{n} \longrightarrow \mathfrak{m} \circledast_{red} \mathfrak{n} \tag{5.4}$$

and

$$\vartheta\vartheta : \mathfrak{m} \circledast_{pp} \mathfrak{n} \longrightarrow \mathfrak{m} \circledast_{kk} \mathfrak{n}. \tag{5.5}$$

If $C = (c_{ij})$ is a $q \times p$ reduced framed matrix, the dimension formula in Definition 3.28 reduces to

$$|C| = \sum_{(i,j)\in\mathfrak{q}\times\mathfrak{p}} |c_{ij}|.$$

Consequently, if $\rho \in \mathfrak{m} \circledast_{kk} \mathfrak{n}$, then

$$|\rho| = \sum \left|\frac{\mathbf{b}}{\mathbf{a}}\right| = \sum(\#\mathbf{a} + \#\mathbf{b} - 1),$$

where $\frac{\mathbf{b}}{\mathbf{a}}$ ranges over all top level elementary non-null bipartitions in ρ.

The face operator $\tilde{\partial}$ induces a face operator

$$\partial : \mathfrak{m} \circledast_{kk} \mathfrak{n} \to \mathfrak{m} \circledast_{kk} \mathfrak{n}$$

of degree -1, which acts on a reduced balanced framed matrix $C = (c_{ij})$ as a derivation with respect to the entries c_{ij}. Identify $(\mathfrak{o} \circledast_{pp} \mathfrak{n}, \tilde{\partial})$ and $(\mathfrak{m} \circledast_{pp} \mathfrak{o}, \tilde{\partial})$ with the permutahedra P_n and P_m, identify $(\mathfrak{o} \circledast_{kk} \mathfrak{n}, \partial)$ and $(\mathfrak{m} \circledast_{kk} \mathfrak{o}, \partial)$ with the associahedra K_{n+1} and K_{m+1}, and denote the induced projections by

$$\vartheta : P_n \to K_{n+1} \quad \text{and} \quad \vartheta : P_m \to K_{m+1}.$$

Example 5.24. In P_3 we have

$$2|1|3 = \begin{pmatrix} \frac{0}{2} \end{pmatrix}\begin{pmatrix} \frac{0}{1} & \frac{0}{0} \end{pmatrix}\begin{pmatrix} \frac{0}{0} & \frac{0}{0} & \frac{0}{3} \end{pmatrix} = \begin{pmatrix} \frac{0}{2} \end{pmatrix}\left(\begin{pmatrix} \frac{0}{1} \end{pmatrix}\begin{pmatrix} \frac{0}{0} & \frac{0}{0} \end{pmatrix} \; \begin{pmatrix} \frac{0}{0} \end{pmatrix}\begin{pmatrix} \frac{0}{3} \end{pmatrix}\right)$$

and

$$2|3|1 = \begin{pmatrix} \frac{0}{2} \end{pmatrix}\begin{pmatrix} \frac{0}{0} & \frac{0}{3} \end{pmatrix}\begin{pmatrix} \frac{0}{1} & \frac{0}{0} & \frac{0}{0} \end{pmatrix} = \begin{pmatrix} \frac{0}{2} \end{pmatrix}\left(\begin{pmatrix} \frac{0}{0} \end{pmatrix}\begin{pmatrix} \frac{0}{1} \end{pmatrix} \; \begin{pmatrix} \frac{0}{3} \end{pmatrix}\begin{pmatrix} \frac{0}{0} & \frac{0}{0} \end{pmatrix}\right).$$

Thus, in K_4 we obtain

$$\vartheta(2|13) = \vartheta(2|1|3) = \vartheta(2|3|1).$$

In Example 5.14, $|\vartheta\vartheta(\tilde{\partial}_3(\rho))| = |\vartheta\vartheta(\tilde{\partial}_4(\rho))| = |\vartheta\vartheta(\tilde{\partial}_7(\rho))| = 0$. Consequently, the 2-dimensional element $\vartheta\vartheta(\rho) \in \mathfrak{3} \circledast_{kk} \mathfrak{1}$ has four 1-dimensional faces induced by $\tilde{\partial}_1(C_2)$, $\tilde{\partial}_2(C_2)$, $\tilde{\partial}_5(C_2)$, and $\tilde{\partial}_6(C_2)$.

5.6 The Chain Complex $(\mathfrak{m} \circledast_{kk} \mathfrak{n}, \partial)$

Let R be a commutative ring with unity, and let $C_*(\mathfrak{m} \circledast_{kk} \mathfrak{n})$ denote the free R-module generated by the set $\mathfrak{m} \circledast_{kk} \mathfrak{n}$. The map ∂ on $\mathfrak{m} \circledast_{kk} \mathfrak{n}$ induces a degree -1 operator

$$\partial : C_*(\mathfrak{m} \circledast_{kk} \mathfrak{n}) \to C_*(\mathfrak{m} \circledast_{kk} \mathfrak{n})$$

defined for $\rho = C_1 \cdots C_r \in C_*(\mathfrak{m} \circledast_{kk} \mathfrak{n})$ by

$$\partial(\rho) := \sum_{1 \leq s \leq r} (-1)^{\varepsilon_{C_s}} \; C_1 \cdots \partial(C_s) \cdots C_r. \tag{5.6}$$

By Proposition 5.17 we immediately obtain

Theorem 5.25. The map ∂ is a differential on $C_*(\mathfrak{m} \circledast_{kk} \mathfrak{n})$, i.e., $\partial^2 = 0$.

6 Premetrads

6.1 Prematrads Defined

In this subsection we review the notion of a *prematrad* introduced in [31]. Let R be a (graded or ungraded) commutative ring with unity 1_R and let $M = \{M_{n,m}\}_{mn \geq 1}$ be a bigraded module over R. Fix a set of bihomogeneous module generators $G = \{\alpha_x^y \in M_{y,x} : x, y \in \mathbb{N}\}$; then $M_{y,x} = \langle \alpha_x^y \rangle$ is the R-module generated by α_x^y. Identify a monomial $\alpha_{x_1}^{y_1} \cdots \alpha_{x_p}^{y_p} \in G^{\otimes p}$ with the $1 \times p$ matrix $\left[\alpha_{x_1}^{y_1} \;\cdots\; \alpha_{x_p}^{y_p}\right]$, and identify the monomial

$$(\alpha_{x_{1,1}}^{y_{1,1}} \cdots \alpha_{x_{1,p}}^{y_{1,p}}) \otimes \cdots \otimes (\alpha_{x_{q,1}}^{y_{q,1}} \cdots \alpha_{x_{q,p}}^{y_{q,p}}) \in \left(G^{\otimes p}\right)^{\otimes q}$$

with the $q \times p$ matrix

$$\begin{bmatrix} \alpha_{x_{1,1}}^{y_{1,1}} & \cdots & \alpha_{x_{1,p}}^{y_{1,p}} \\ \vdots & & \vdots \\ \alpha_{x_{q,1}}^{y_{q,1}} & \cdots & \alpha_{x_{q,p}}^{y_{q,p}} \end{bmatrix}.$$

Denote the set of $q \times p$ matrices over $\mathbb{N}$ by $\mathbb{N}^{q \times p}$, and denote the double tensor module of M by TTM. Given $X = [x_{i,j}]$, $Y = [y_{i,j}] \in \mathbb{N}^{q \times p}$, let $M_{Y,X} = \left(M_{y_{1,1},x_{1,1}} \otimes \cdots \otimes M_{y_{1,p},x_{1,p}}\right) \otimes \cdots \otimes \left(M_{y_{q,1},x_{q,1}} \otimes \cdots \otimes M_{y_{q,p},x_{q,p}}\right)$ and consider the *matrix submodule*

$$\overline{\mathbf{M}} := \bigoplus_{pq \geq 1} \left(M^{\otimes p}\right)^{\otimes q} = \bigoplus_{\substack{X,Y \in \mathbb{N}^{q \times p} \\ pq \geq 1}} M_{Y,X} \subset TTM.$$

A $q \times p$ *bisequence matrix* $B \in \overline{\mathbf{M}}$ has the form

$$B = \begin{bmatrix} \beta_{x_1}^{y_1} & \cdots & \beta_{x_p}^{y_1} \\ \vdots & & \vdots \\ \beta_{x_1}^{y_q} & \cdots & \beta_{x_p}^{y_q} \end{bmatrix};$$

B is *elementary* if $\beta_{x_j}^{y_i} \in G$ for all (i, j). Define the *input and output leaf sequences of* B by $ils\,(B) := \mathbf{x} = (x_1, \ldots, x_p)$ and $ols\,(B) := \mathbf{y} = (y_1, \ldots, y_q)^T$; define the $indeg(B) := \Sigma x_j$, the $outdeg(B) := \Sigma y_i$, and the $bideg(B) := (indeg\,(B),\; outdeg\,(B))$. Let

$$\mathbf{M}_{\mathbf{x}}^{\mathbf{y}} := \left\{\text{bisequence matrices } B \in \overline{\mathbf{M}} : \mathbf{x} \times \mathbf{y} = ils\,(B) \times ols\,(B) \in \mathbb{N}^{1 \times p} \times \mathbb{N}^{q \times 1}\right\};$$

the *bisequence submodule*

$$\mathbf{M} := \bigoplus_{\substack{\mathbf{x}\times\mathbf{y}\in\mathbb{N}^{1\times p}\times\mathbb{N}^{q\times 1}\\ pq\geq 1}} \mathbf{M}_{\mathbf{x}}^{\mathbf{y}} \subset TTM.$$

A *Transverse Pair* (TP) of bisequence matrices is a pair $A\times B\in \mathbf{M}_p^{\mathbf{y}}\times\mathbf{M}_{\mathbf{x}}^{q}$ such that $\mathbf{x}\times\mathbf{y}\in\mathbb{N}^{1\times p}\times\mathbb{N}^{q\times 1}$. A pair of matrices $A\times B\in\overline{\mathbf{M}}\times\overline{\mathbf{M}}$ is a *Block Transverse Pair* (BTP) if there exist block decompositions $A=[A_{ij}]$ and $B=[B_{ij}]$ such that A_{ij} is a $q_i\times 1$ block for each j, B_{ij} is a $1\times p_j$ block for each i, and $A_{ij}\times B_{ij}$ is a TP for all (i,j). When a BTP decomposition of $A\times B$ exists, it is unique. Note that a pair of bisequence matrices $A^{q\times s}\times B^{t\times p}$ is a BTP if and only if

$$indeg(A)=p \text{ and } outdeg(B)=q. \tag{6.1}$$

Example 6.1. A $(4\times 2, 2\times 3)$ monomial pair $A\times B\in\mathbf{M}_{21}^{\substack{1\\5\\4\\3}}\times\mathbf{M}_{123}^{\substack{3\\1}}$ is a 2×2 BTP per the block decompositions

$$A=\left[\begin{array}{c:c} \alpha_2^1 & \alpha_1^1\\ \alpha_2^5 & \alpha_1^5\\ \alpha_2^4 & \alpha_1^4\\ \hdashline \alpha_2^3 & \alpha_1^3\end{array}\right] \text{ and } B=\left[\begin{array}{cc:c}\beta_1^3 & \beta_2^3 & \beta_3^3\\ \hdashline \beta_1^1 & \beta_2^1 & \beta_3^1\end{array}\right].$$

Given $\mathbf{x}=(x_1,\ldots,x_p)$, define $\|\mathbf{x}\|:=\Sigma x_j$. A family of maps $\gamma=\{\gamma_{\mathbf{x}}^{\mathbf{y}}:\mathbf{M}_p^{\mathbf{y}}\otimes\mathbf{M}_{\mathbf{x}}^{q}\to\mathbf{M}_{\|\mathbf{x}\|}^{\|\mathbf{y}\|}$, $\mathbf{x}\times\mathbf{y}\in\mathbb{N}^{1\times p}\times\mathbb{N}^{q\times 1}$, $pq\geq 1\}$ extends to a global product $\Upsilon:\overline{\mathbf{M}}\otimes\overline{\mathbf{M}}\to\overline{\mathbf{M}}$ via

$$\Upsilon(A\otimes B)=\begin{cases}[\gamma(A_{ij}\otimes B_{ij})], & A\times B=[A_{ij}]\times[B_{ij}] \text{ is a BTP}\\ 0, & \text{otherwise.}\end{cases} \tag{6.2}$$

Juxtaposition AB denotes the product $\Upsilon(A\otimes B)$; it is easy to check that Υ is closed in $\mathbf{M}$. Let $\eta: R\to M_{1,1}$ be a map and let $\mathbf{1}:=\eta(1_R)$. The map η is a γ-*unit* if $\mathbf{1}\alpha=\alpha\mathbf{1}=\alpha$ for all $\alpha\in M$. A constant matrix with unital entries is a *unital matrix*. Note that if $\beta_p^q\in M_{q,p}$, the products

$$[\underbrace{\mathbf{1}\ \mathbf{1}\ \cdots\ \mathbf{1}}_{q}]^T\times\beta_p^q \text{ and } \beta_p^q\times[\underbrace{\mathbf{1}\ \mathbf{1}\ \cdots\ \mathbf{1}}_{p}]$$

are TPs. More generally, if A is a matrix in $\overline{\mathbf{M}}$, there exist unital matrices whose sizes are determined by A such that $\mathbf{1}\times A$ and $A\times\mathbf{1}$ are BTPs. Thus if η is a γ-unit and $\mathbf{1}=\eta(1_R)$, then $\mathbf{1}A=A\mathbf{1}=A$ for all $A\in\overline{\mathbf{M}}$.

Definition 6.2. Let M be a bigraded R-module together with a family of structure maps $\tilde{\gamma} = \{\tilde{\gamma}_{\mathbf{x}}^{\mathbf{y}} : \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q \to \mathbf{M}_{\|\mathbf{x}\|}^{\|\mathbf{y}\|},\ \mathbf{x} \times \mathbf{y} \in \mathbb{N}^{1\times p} \times \mathbb{N}^{q \times 1},\ pq \geq 1\}$ such that Υ is associative in $\mathbf{M}$. Then $(M, \tilde{\gamma})$ is a **non-unital prematrad.** Let η be a $\tilde{\gamma}$-unit and let γ be the family of structure maps induced by the unital action $\tilde{\gamma}(\mathbf{1} \otimes m) = \tilde{\gamma}(m \otimes \mathbf{1}) = m$. Then (M, γ, η) is a **prematrad**. Given prematrads (M, γ, η) and (M', γ', η'), a map $f : M \to M'$ is a **map of prematrads** if $f\gamma_{\mathbf{x}}^{\mathbf{y}} = \gamma'^{\mathbf{y}}_{\mathbf{x}}(f^{\otimes q} \otimes f^{\otimes p})$ for all $\mathbf{x} \times \mathbf{y}$ and $\eta' = f\eta$.

Example 6.3. Let H be a free R-module of finite type. View $M = \mathcal{E}nd_{TH}$ as the bigraded R-module $M = \{M_{n,m} = Hom(H^{\otimes m}, H^{\otimes n})\}_{mn \geq 1}$ and define $\eta(1_R) := \mathbf{Id}_H$. In [22], M. Markl defined the submodule of *special elements* in M whose additive generators are monomials expressed as *elementary fractions* of the form

$$\frac{\alpha_p^{y_1} \cdots \alpha_p^{y_q}}{\alpha_{x_1}^q \cdots \alpha_{x_p}^q} \in \mathbf{M}_{\|\mathbf{x}\|}^{\|\mathbf{y}\|},$$

where juxtaposition in the numerator and denominator denotes tensor product and the j^{th} output of $\alpha_{x_i}^q$ is linked to the i^{th} input of $\alpha_p^{y_j}$. The *fraction product* $\gamma : \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q \to \mathbf{M}_{\|\mathbf{x}\|}^{\|\mathbf{y}\|}$ given by $\gamma\left(\alpha_p^{y_1} \cdots \alpha_p^{y_q} \otimes \alpha_{x_1}^q \cdots \alpha_{x_p}^q\right) = \frac{\alpha_p^{y_1} \cdots \alpha_p^{y_q}}{\alpha_{x_1}^q \cdots \alpha_{x_p}^q}$ extends to an associative product Υ on $\mathbf{M}$, η is a γ-unit, and $(\mathcal{E}nd_{TH}, \gamma, \eta)$ is a prematrad.

If A is a matrix in $\overline{\mathbf{M}}$, the transpose $A \mapsto A^T$ induces an automorphism $\sigma = \sigma_{*,*}$ of $\overline{\mathbf{M}}$ whose component $\sigma_{p,q} : (M^{\otimes p})^{\otimes q} \overset{\approx}{\to} (M^{\otimes q})^{\otimes p}$ is the linear extension of the canonical permutation of tensor factors in $(G^{\otimes p})^{\otimes q}$, i.e.,

$$\sigma_{p,q} : \left(\alpha_{x_{1,1}}^{y_{1,1}} \otimes \cdots \otimes \alpha_{x_{1,p}}^{y_{1,p}}\right) \otimes \cdots \otimes \left(\alpha_{x_{q,1}}^{y_{q,1}} \otimes \cdots \otimes \alpha_{x_{q,p}}^{y_{q,p}}\right) \mapsto \left(\alpha_{x_{1,1}}^{y_{1,1}} \otimes \cdots \otimes \alpha_{x_{q,1}}^{y_{q,1}}\right) \otimes \cdots \otimes \left(\alpha_{x_{1,p}}^{y_{1,p}} \otimes \cdots \otimes \alpha_{x_{q,p}}^{y_{q,p}}\right).$$

The identification $(G^{\otimes p})^{\otimes q} \leftrightarrow (p,q)$ expresses a $q \times p$ bisequence matrix $A \in \mathbf{M}_{\mathbf{x}}^{\mathbf{y}}$ as an operator $A : \left(G^{\otimes \|\mathbf{x}\|}\right)^{\otimes q} \to (G^{\otimes p})^{\otimes \|\mathbf{y}\|}$ on $\mathbb{N}^2$ whose action is given by the composition

$$\left(G^{\otimes \|\mathbf{x}\|}\right)^{\otimes q} \approx \left(G^{\otimes x_1} \otimes \cdots \otimes G^{\otimes x_p}\right)^{\otimes q} \to \left(G^{\otimes y_1}\right)^{\otimes p} \otimes \cdots \otimes \left(G^{\otimes y_q}\right)^{\otimes p}$$

$$\overset{\sigma_{y_1,p} \otimes \cdots \otimes \sigma_{y_q,p}}{\longrightarrow} \left(G^{\otimes p}\right)^{\otimes y_1} \otimes \cdots \otimes \left(G^{\otimes p}\right)^{\otimes y_q} \approx \left(G^{\otimes p}\right)^{\otimes \|\mathbf{y}\|}.$$

Thus A can be thought of as an arrow $(\|\mathbf{x}\|, q) \mapsto (p, \|\mathbf{y}\|)$ in $\mathbb{N}^2$ (see Figure 5).

A TP $\alpha \times \beta \in \mathbf{M}_p^{\mathbf{y}} \times \mathbf{M}_{\mathbf{x}}^q$ can be represented by a 2-step path $(\|\mathbf{x}\|, 1) \overset{\beta}{\mapsto} (p,q) \overset{\alpha}{\mapsto} (1, \|\mathbf{y}\|)$. If $p = 1$, α is represented by a vertical arrow $(1, q) \mapsto (1, \|\mathbf{x}\|)$. If $q = 1$, β is represented by a horizontal arrow $(\|\mathbf{x}\|, 1) \mapsto (p, 1)$. If $\mathbf{x} = \mathbf{1}$, β is represented by a vertical arrow $(\|\mathbf{x}\|, 1) \mapsto (\|\mathbf{x}\|, q)$. If $\mathbf{y} = \mathbf{1}$, α is represented by a horizontal arrow $(p, \|\mathbf{y}\|) \mapsto (1, \|\mathbf{y}\|)$. If $\mathbf{x} \neq \mathbf{1}$ and $q \neq 1$, β is represented by a left-leaning arrow $(\|\mathbf{x}\|, 1) \mapsto (p, q)$. And if $\mathbf{y} \neq \mathbf{1}$ and $p \neq 1$, α is represented by a left-leaning arrow $(p,q) \mapsto (1, \|\mathbf{y}\|)$. A non-zero monomial $A = A_1 \cdots A_s$ with $A_i \in \mathbf{M}$ is represented by a directed k-step piece-wise linear path from the x-axis to the y-axis in $\mathbb{N}^2$, where $s - k$ is the number of evaluated subproducts in an association. Of course, each such path can represent multiple monomials.

$$A = \begin{bmatrix} \alpha_2^2 & \alpha_3^2 \\ \alpha_2^4 & \alpha_3^4 \end{bmatrix} \in \mathbf{M}^{\substack{2\\4}}_{23} \mapsto$$

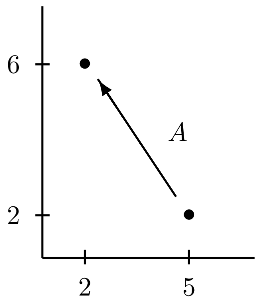


Figure 5. A 2×2 bisequence matrix A thought of as an arrow in $\mathbb{N}$.

Example 6.4. The unevaluated product

$$ABC = \begin{bmatrix} \alpha_2^2 \\ \alpha_2^2 \\ \alpha_2^2 \end{bmatrix} \begin{bmatrix} \beta_2^2 & \beta_1^2 \\ \beta_2^1 & \beta_1^1 \end{bmatrix} \begin{bmatrix} \gamma_2^2 & \gamma_2^2 & \gamma_2^2 \end{bmatrix} \in \mathbf{M}^{\substack{2\\2\\2}}_{2} \mathbf{M}^{\substack{2\\1}}_{21} \mathbf{M}^{2}_{222} \tag{6.3}$$

expressed as the composition

$$G^{\otimes 6} \overset{\xrightarrow{\qquad\qquad C \qquad\qquad}}{\longrightarrow \left(G^{\otimes 2}\right)^{\otimes 3} \overset{\sigma_{2,3}}{\longrightarrow}} \left(G^{\otimes 3}\right)^{\otimes 2} \overset{\xrightarrow{\qquad\qquad B \qquad\qquad}}{\longrightarrow \left(G^{\otimes 2}\right)^{\otimes 3} \overset{\sigma_{2,2} \otimes \mathbf{1}}{\longrightarrow}} \left(G^{\otimes 2}\right)^{\otimes 3} \overset{A}{\longrightarrow} G^{\otimes 6}$$

is represented by the 3-step path in Figure 6. If we evaluate AB or BC, the associations

$$(AB)\,C = \begin{bmatrix} \begin{bmatrix} \alpha_2^2 \\ \alpha_2^2 \end{bmatrix} \begin{bmatrix} \beta_2^2 & \beta_1^2 \end{bmatrix} \\ \begin{bmatrix} \alpha_2^2 \end{bmatrix} \begin{bmatrix} \beta_2^1 & \beta_1^1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \gamma_2^2 & \gamma_2^2 & \gamma_2^2 \end{bmatrix} \in \mathbf{M}^{\substack{4\\2}}_{3} \mathbf{M}^{2}_{222}$$

and

$$A\,(BC) = \begin{bmatrix} \alpha_2^2 \\ \alpha_2^2 \\ \alpha_2^2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \beta_2^2 \\ \beta_2^1 \end{bmatrix} \begin{bmatrix} \gamma_2^2 & \gamma_2^2 \end{bmatrix} & \begin{bmatrix} \beta_1^2 \\ \beta_1^1 \end{bmatrix} \begin{bmatrix} \gamma_2^2 \end{bmatrix} \end{bmatrix} \in \mathbf{M}^{\substack{2\\2\\2}}_{2} \mathbf{M}^{3}_{42}$$

expressed as the compositions

$$G^{\otimes 6} \overset{\xrightarrow{\qquad\qquad C \qquad\qquad}}{\longrightarrow \left(G^{\otimes 2}\right)^{\otimes 3} \overset{\sigma_{2,3}}{\longrightarrow}} \left(G^{\otimes 3}\right)^{\otimes 2} \overset{AB}{\longrightarrow} G^{\otimes 6},$$

and

$$G^{\otimes 6} \overset{\xrightarrow{\qquad\qquad BC \qquad\qquad}}{\rightarrow \left(G^{\otimes 3}\right)^{\otimes 2} \overset{\sigma_{3,2}}{\longrightarrow}} \left(G^{\otimes 2}\right)^{\otimes 3} \overset{A}{\longrightarrow} G^{\otimes 6}$$

are represented by the 2-step paths in Figure 6. Finally, the evaluated associations $(AB)\,C = A\,(BC) \in \mathbf{M}_6^6$ are represented by the 1-step path $G^{\otimes 6} \to G^{\otimes 6}$.

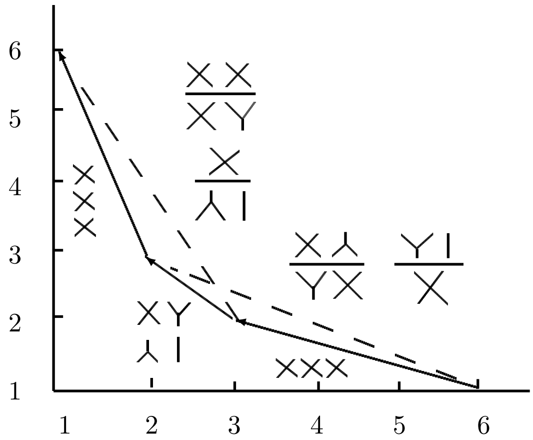


Figure 6. The polygonal paths of ABC.

6.2 Free Prematrads

In this subsection we review the fundamental notion of a *free prematrad*, but the development here differs somewhat from that in [31]. To begin, we introduce an auxiliary object, the free *non-unital prematrad.*

The notion of a bisequence matrix and its related constructions extend to matrices whose entries are general bigraded objects; we refer to such matrices as *Generalized BiSequence Matrices (GBSMs).* When A is a GBSM and

$$(indeg(A)), outdeg(A)) = (m,n),$$

we refer to A as an (m,n)-*matrix.*

Let $\xi = A_1 \cdots A_r$ be a monomial of GBSMs. The *bidegree of* ξ is defined and denoted by *bideg*$(\xi) :=$ (*outdeg*(A_1), *indeg*(A_r)). When *bideg*$(\xi) = (m,n)$, we refer to ξ as an (m,n)-*monomial.*

Definition 6.5. Given a bigraded set $Q = \{Q_{n,m}\}_{mn\geq 1}$ with based element $\mathbf{1} \in Q_{1,1}$, construct the set $\tilde{G}\,(Q) := \{\tilde{G}_{n,m}\,(Q)\}$ of **free bigraded monomials generated by** Q inductively as follows: Define $\tilde{G}_{1,1}\,(Q) := Q_{1,1}$. If $m+n \geq 3$, and $\tilde{G}_{j,i}\,(Q)$ has been constructed for all $(i,j) \leq (m,n)$ such that $i+j < m+n$, define

$$\tilde{G}_{n,m}\,(Q) := Q_{n,m} \cup \{A_1 \cdots A_s : s \geq 2\}\,, \tag{6.4}$$

where for all k,

(i) A_k is a GBSM over $\{\tilde{G}_{j,i}\,(Q) : (i,j) \leq (m,n) \,\text{and}\, i+j < m+n\}$,

(ii) A_1 is a column matrix with *outdeg*$(A_1) = n$,

(iii) A_s is a row matrix with $indeg(A_s) = m$,

(iv) if $B = (\xi_{uv})$ is a $q \times p$ GBSM within A_k and $\xi_{uv} = B_1 \cdots B_{\ell-1} \cdot \mathbf{1} \cdot B_{\ell+1} \cdots B_r$ for some ℓ, define $\xi_{uv} = B_1 \cdots B_{\ell-1} B_{\ell+1} \cdots B_r$ when either $pq = 1$ and $B = A_k$ or $pq > 1$ and B_{u*} and B_{*v} are indecomposable,

(v) $A_k \times A_{k+1}$ is a BTP,

(vi) $A_k A_{k+1}$ is the formal product, and

(vii) $(A_k A_{k+1}) A_{k+2} = A_k (A_{k+1} A_{k+2})$.

A Q-**monomial** is an elementary monomial in $\tilde{G}\,(Q)$.

Remark 6.6. Monomials in $\tilde{G}\,(Q)$ are generated by the component sets $Q_{n,m}$, and not by particular elements thereof. The double corolla Γ_m^n represents the set $Q_{n,m}$.

In view of item (iv), we henceforth assume that an (m,n)-monomial $\xi = A_1 \cdots A_r$ is a formal product of non-unital indecomposables unless explicitly indicated otherwise.

Let $\Theta = \{\Theta_{n,m}\}_{mn \geq 1}$, where $\Theta_{n,m} = \{\theta_m^n\}$ is a singleton set when $\Theta_{n,m} \neq \varnothing$. When this occurs, we identify the set $\Theta_{n,m}$ and its element θ_m^n.

Example 6.7. The monomial

$$\begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_2^1 & \mathbf{1} \\ \theta_2^1 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_1^2 \end{bmatrix} \in \tilde{G}(\Theta)$$

is a Θ-monomial, but

$$\begin{bmatrix} [\theta_2^1][\begin{array}{cc} \theta_2^1 & \mathbf{1} \end{array}] \\ [\theta_2^1][\begin{array}{cc} \mathbf{1} & \theta_2^1 \end{array}] \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_1^2 \end{bmatrix} \in \tilde{G}(\Theta)$$

is not.

Proposition 6.8. Let $M = \langle \widetilde{G}\,(\Theta) \rangle$ and let $\tilde{\gamma} = \{\tilde{\gamma}_{\mathbf{x}}^{\mathbf{y}} : \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q \to \mathbf{M}_{\|\mathbf{x}\|}^{\|\mathbf{y}\|}\}$ be the linear extension of juxtaposition. Then the induced global product $\widetilde{\Upsilon}$ acts associatively on $\mathbf{M}$.

Proof. Let $A \times B$ and $B \times C$ be BTPs of bisequence matrices over $\tilde{G}(\Theta)$ and consider the associated block decompositions $[A_{k',l}] \times [B_{i,j'}]$ and $[B_{i',j}] \times [C_{u,v'}]$. Let $B = [\xi_{i,j}]^{q \times p}$, let $B_{i,j'} = [\xi_{i,m_{j'-1}+1} \cdots \xi_{i,m_{j'}}]$, and let $B_{i',j} = [\xi_{n_{i'-1}+1,j} \cdots \xi_{n_{i'},j}]^T$, where $1 \leq j' \leq s$ indexes the blocks in the partition

$$m_0 + 1 < \cdots < m_1 | \cdots | m_{j'-1} + 1 < \cdots < m_{j'} | \cdots | m_{s-1} + 1 < \cdots < m_s \in P(\mathfrak{p})$$

and $1 \leq i' \leq t$ indexes the blocks in the partition

$$n_0 + 1 < \cdots < n_1 | \cdots | n_{i'-1} + 1 < \cdots < n_{i'} | \cdots | n_{t-1} + 1 < \cdots < n_t \in P(\mathfrak{q}).$$

Let $u_{j'} = m_{j'} - m_{j'-1}$ and $v_{i'} = n_{i'} - n_{i'-1}$, and consider the block decomposition $[B_{i',j'}]$ of B given by

$$B_{i',j'} = [\xi_{i,j}]^{v_{i'} \times u_{j'}}_{n_{i'-1} \leq i \leq n_{i'};\; m_{j'-1} \leq j \leq m_{j'}}.$$

Now consider the juxtaposition of bisequence matrices

$$A_{i',j}B_{i',j'}C_{i,j'} := \begin{bmatrix} A_{n_{i'-1}+1,j} \\ \vdots \\ A_{n_{i'},j} \end{bmatrix}^{v'_i \times 1} \cdot B_{i'j'}^{v_{i'} \times u_{j'}} \cdot [C_{i,m_{j'-1}+1} \;\cdots\; C_{i,m_{j'}}]^{1\times u_{j'}}.$$

It is easy to check that $A_{i',j} \times B_{i',j'}$ and $B_{i',j'} \times C_{i,j'}$ are BTPs so that $A_{i',j}B_{i',j'}C_{i,j'} \in \tilde{G}(\Theta)$ and axiom (vii) applies. Therefore $(A_{i',j}B_{i',j'})C_{i,j'} = A_{i',j}(B_{i',j'}C_{i,j'})$, and it follows that $(AB)C = A(BC)$ as claimed. Q.E.D.

Definition 6.9. Let $\tilde{F}^{pre}(\Theta) = \langle \widetilde{G}(\Theta) \rangle$ and let $\tilde{\gamma}$ be the product in Proposition 6.8. Then $(\tilde{F}^{pre}(\Theta), \tilde{\gamma})$ is the **free non-unital prematrad generated by** Θ.

Define $G_{n,m}(\Theta) := \widetilde{G}_{n,m}(\Theta)/\,(A \sim \mathbf{1}A \sim A\mathbf{1})$ and let $F^{pre}(\Theta) = \langle G(\Theta) \rangle$. Let γ be the product induced by $\tilde{\gamma}$, and let $\eta : R \to F_{1,1}^{pre}(\Theta)$ be the unit given by $\eta(1_R) = \mathbf{1}$. Then $(F^{pre}(\Theta), \gamma, \eta)$ is the **free prematrad generated by** Θ.

There is the obvious projection

$$\varsigma\varsigma_{n,m}^{pre} : \tilde{F}_{n,m}^{pre}(\Theta) \to F_{n,m}^{pre}(\Theta). \tag{6.5}$$

In the operadic cases, we denote $\varsigma\varsigma_{1,n}^{pre}$ and $\varsigma\varsigma_{n,1}^{pre}$ by ς_n; which we mean will be clear from context.

Example 6.10. **The Bialgebra Prematrad**. Let $\Theta = \left\{\theta_1^1 = \mathbf{1},\ \theta_2^1,\ \theta_1^2\right\}$ and consider $\mathcal{H}^{pre} = F^{pre}(\Theta)/\sim$, where $A \sim B$ if $bideg\,(A) = bideg\,(B)$. Let $\bar{\gamma}$ be the product on $\mathcal{H}^{pre}$ induced by the projection $F^{pre}(\Theta) \to \mathcal{H}^{pre}$. The prematrad $(\mathcal{H}^{pre}, \bar{\gamma}, \eta)$ is the **bialgebra prematrad generated by** Θ if

(i) θ_2^1 is associative: $\left[\theta_2^1\right]\left[\ \theta_2^1 \quad \mathbf{1}\ \right] = \left[\theta_2^1\right]\left[\ \mathbf{1} \quad \theta_2^1\ \right]$,

(ii) θ_1^2 is coassociative: $\begin{bmatrix} \theta_1^2 \\ \mathbf{1} \end{bmatrix}\left[\theta_1^2\right] = \begin{bmatrix} \mathbf{1} \\ \theta_1^2 \end{bmatrix}\left[\theta_1^2\right]$, and

(iii) θ_1^2 and θ_2^1 are Hopf compatible: $\left[\theta_1^2\right]\left[\theta_2^1\right] = \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix}\left[\ \theta_1^2 \quad \theta_1^2\ \right]$.

The component $\mathcal{H}_{n,m}^{pre}$ of $\mathcal{H}^{pre}$ is generated by a class $c_{n,m}$, where

$$c_{1,1} = \{\mathbf{1}\}\,,\ c_{1,2} = \left\{\theta_2^1\right\},\ c_{2,1} = \left\{\theta_1^2\right\},\ c_{1,3} = \left\{\gamma(\theta_2^1;\theta_2^1,\mathbf{1}), \gamma(\theta_2^1;\mathbf{1},\theta_2^1)\right\},$$
$$c_{3,1} = \{\gamma(\theta_1^2,\mathbf{1};\theta_1^2), \gamma(\mathbf{1},\theta_1^2;\theta_1^2)\},\ c_{2,2} = \left\{\gamma(\theta_1^2;\theta_2^1), \gamma(\theta_2^1,\theta_2^1;\theta_1^2,\theta_1^2)\right\},$$

and so on. Note that $\mathcal{H}_{1,*}^{pre}$ is a non-sigma operad generated by $c_{1,2}$ subject to axiom (i) and $\mathcal{H}_{*,1}^{pre}$ is a non-sigma operad generated by $c_{2,1}$ subject to axiom (ii); both are isomorphic to the associativity operad $\underline{\mathcal{A}ss}$ [25]. Given a graded R-module H, a map of prematrads $\mathcal{H}^{\,pre} \to \mathcal{E}nd_{TH}$ defines a bialgebra structure on H. In this case, each representative of $c_{n,m}$ determines a directed piece-wise linear path from $(m,1)$ to $(1,n)$ in $\mathbb{N}^2$, and each such path represents the class $c_{n,m}$ (see Figure 6).

6.3 The Dimension of $\tilde{G}_{n,m}(\Theta)$

Given a bigraded set $Q = \{Q_{m,n}\}_{mn\geq 1}$, let $\tilde{G}(Q) = \{\tilde{G}_{m,n}(Q)\}$ be the set of free bigraded monomials generated by Q. There is a canonical map

$$F : \tilde{G}(Q_{m+1,n+1}) \to \mathfrak{m} \circledast \mathfrak{n}, \tag{6.6}$$

which is a bijection if $Q_{m,n}$ is a singleton for all m and n. The definition of F requires some preliminaries.

Definition 6.11. Let $B = (b_{ij})$ be a $q \times p$ GBSM. The *input and output leaf sequences of* B are defined and denoted by

$$ils\,(B) := (indeg\,(b_{11})\,, \ldots, indeg\,(b_{1p})) \quad \text{and} \quad ols\,(B) := (outdeg\,(b_{11})\,, \ldots, outdeg\,(b_{q1}))\,.$$

Let $\xi = B_1 \cdots B_k$ be a monomial in $\tilde{G}_{m,n}(Q)$.

(i) The **Input Leaf Decomposition of** ξ is the sequence

$$ild(\xi) := (ils(B_1), \ldots, ils(B_k))\,;$$

when $B_1 \cdots B_k$ is indecomposable we write $ILD(\xi)$. Given $ILD(\xi)$, the **Reduced Input Leaf Decomposition of** ξ, denoted by $RILD(\xi)$, is obtained from $ILD\,(\xi)$ by suppressing all unital components.

(ii) The **Output Leaf Decomposition of** ξ is the sequence

$$old(\xi) := (ols(B_1), \ldots, ols(B_k))\,;$$

when $B_1 \cdots B_k$ is indecomposable we write $OLD\,(\xi)$. Given $OLD(\xi)$, the **Reduced Output Leaf Decomposition of** ξ, denoted by $ROLD(\xi)$, is obtained from $OLD(\xi)$ by suppressing all unital components.

Given an indecomposable monomial $\xi = B_1 \cdots B_k \in \tilde{G}_{m,n}(Q)$ and its i/o leaf decompositions $(\alpha, \beta) := ((\mathbf{m}_1, \ldots, \mathbf{m}_k), (\mathbf{n}_1, \ldots, \mathbf{n}_k))$, consider the corresponding reduced i/o leaf decompositions $(R\alpha, R\beta)$ and form the bipartition $\bar{R}\beta/\bar{R}\alpha$, where $\bar{R}\alpha$ and $\bar{R}\beta$ are obtained from $\overset{\wedge}{\varepsilon}{}^{-1}(T_{R\alpha})$ and $\overset{\vee}{\varepsilon}{}^{-1}(T^{R\beta})$ by inserting empty blocks in positions corresponding to the unital components of α and β.

Now define the map F in (6.6) on a generator $Q_{m+1,n+1} \in Q$ by $F(Q_{m+1,n+1}) := \frac{\mathfrak{n}}{\mathfrak{m}}$ and on a monomial $\xi \in \tilde{G}_{n+1,m+1}(Q)$ as follows: For each monomial ζ within ξ (including $\zeta = \xi$), define $F(\zeta) := \bar{R}OLD(\zeta)/\bar{R}ILD(\zeta)$; then $F(\xi)$ is the top level component of an element in $\mathfrak{m} \circledast \mathfrak{n}$. Since the top level component of a framed element uniquely determines its canonical representative, we obtain the desired map F. Note that ξ is a Θ-monomial if and only if $h(F(\xi)) = 1$.

Example 6.12. Consider the Θ-monomial

$$\xi = B_1B_2B_3B_4 = \begin{bmatrix} \theta_2^1 \\ \theta_2^2 \\ \theta_2^2 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_2^2 \\ \mathbf{1} & \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_1^2 \; \theta_2^2 \; \theta_1^2 \end{bmatrix} \begin{bmatrix} \theta_2^1 \; \mathbf{1} \; \mathbf{1} \; \mathbf{1} \end{bmatrix} \in \tilde{G}_{5,5}\,(\Theta)\,.$$

The reduced leaf decompositions $RILD(\xi) = ((2),(12),(121),(2111))$ and $ROLD(\xi) = ((122),(21),(2))$ are the leaf decompositions of $T_{RILD(\xi)}$ and $T^{ROLD(\xi)}$ pictured in Figure 7 (reading from top-down). But $ILD(\xi) = RILD(\xi)$ and $OLD(\xi) = ((122),(21),(2),(1))$ imply $\overset{\wedge}{\varepsilon}{}^{-1}(T_{RILD(\xi)}) = 1|3|4|2$ and $\overset{\vee}{\varepsilon}{}^{-1}(T^{ROLD(\xi)}) = 24|1|3$. Hence

$$F(\xi) = \frac{\bar{R}OLD(\xi)}{\bar{R}ILD(\xi)} = \frac{24|1|3|0}{1|3|4|2} \in 4 \circledast 4.$$

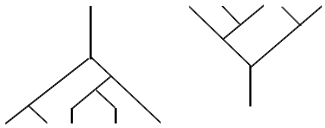


Figure 7. The PLTs $T_{RILD(\xi)}$ and $T^{ROLD(\xi)}$.

Definition 6.13. The **dimension** of a monomial $\xi \in \tilde{G}(\Theta)$ is defined and denoted by

$$|\xi| := |F(\xi)|.$$

Then $|\theta_s^t| = s+t-3$, $|(\theta^1_{x_1} \cdots \theta^1_{x_p})| = x_1 + \cdots + x_p - p - 1$ and $|(\theta_1^{y_1} \cdots \theta_1^{y_q})^T| = y_1 + \cdots + y_q - q - 1$.

Example 6.14. Consider the 2×2 matrix

$$B = \begin{bmatrix} \begin{bmatrix} \theta^1_2 \\ [\theta^1_2]\,[\mathbf{1}\ \ \mathbf{1}] \end{bmatrix} \begin{bmatrix} \theta^2_1 & \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} & [\theta^2_1] \end{bmatrix} & \begin{bmatrix} \theta^1_2 \\ [\mathbf{1}]\,[\theta^1_2] \end{bmatrix} \begin{bmatrix} \theta^2_1\ \theta^2_1 \end{bmatrix} \\ \\ \begin{bmatrix} \theta^1_2 \\ \theta^1_2 \end{bmatrix} \begin{bmatrix} \theta^2_1 & [\theta^2_1]\,[\mathbf{1}] \end{bmatrix} & \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} \theta^2_2 \end{bmatrix} [\mathbf{1}\ \ \mathbf{1}] \end{bmatrix},$$

whose first row and first column are decomposable (cf. Example 4.7). To evaluate $|\xi_{1,*}|$ for example, we decompose and obtain

$$|\xi_{1,*}| = \left| \begin{bmatrix} \theta^1_2 & \theta^1_2 \\ [\theta^1_2]\,[\mathbf{1}\ \ \mathbf{1}] & [\mathbf{1}]\,[\theta^1_2] \end{bmatrix} \right| + \left| \begin{bmatrix} \theta^2_1 & \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} [\theta^2_1] & \theta^2_1 & \theta^2_1 \end{bmatrix} \right| = 1 + 0 = 1.$$

Completing the dimension calculation we have

$$|B| = |B|^{row} + |B|^{col} + |B|^{ent} = 1 + 1 + 1 = 3.$$

7 Free Matrads

In this section we construct the free non-unital matrad $\tilde{F}(\Theta)$, which is an auxiliary object, then obtain the free matrad $F(\Theta)$ by defining the unit relation.

7.1 Free Non-unital Matrads

Definition 7.1. Given the map $\digamma$ defined in (6.6), let

$$\tilde{\mathcal{B}}_{n,m}(\Theta) := \digamma^{-1}(\mathfrak{m} \circledast_{pp} \mathfrak{n}) \subset \tilde{G}_{n,m}(\Theta).$$

The **free non-unital matrad generated by** Θ is the bigraded module

$$\tilde{F}(\Theta) := \langle \tilde{\mathcal{B}}(\Theta) \rangle.$$

Then by definition, $\tilde{\mathcal{B}}(\Theta)$ inherits the face operator $\tilde{\partial}$ from $(\mathfrak{m} \circledast_{pp} \mathfrak{n}, \tilde{\partial})$. For example,

$$\tilde{\partial}(\theta_2^2) = \left[\theta_1^2\right]\left[\theta_2^1\right] - \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_1^2 \end{bmatrix}. \tag{7.1}$$

Let $\tilde{F}(\Theta)_{n,m} := \langle \tilde{\mathcal{B}}_{n,m}(\Theta) \rangle$. By Proposition 5.17 we immediately obtain

Proposition 7.2. The face operator $\tilde{\partial}$ on $\mathfrak{m} \circledast_{pp} \mathfrak{n}$ induces a degree -1 differential $\tilde{\partial} : \tilde{F}(\Theta)_{n,m} \to \tilde{F}(\Theta)_{n,m}$ and a degree -1 differential on $\tilde{F}(\Theta)$.

Given $\xi \in \tilde{\mathcal{B}}(\Theta)$, the analogs of the operators η_1 and η_2 are denoted by the same symbols and given by $\eta_1(\xi) = \mathbf{1} \cdot \xi$ and $\eta_2(\xi) = \xi \cdot \mathbf{1}$.

Example 7.3. Let $B = \begin{bmatrix} \theta_2^2 & \mathbf{1} \cdot \theta_1^2 \\ \theta_2^2 & \theta_1^2 \cdot \mathbf{1} \end{bmatrix} \in \tilde{\mathbf{B}}$; then $\tilde{\partial}(B)$ has two components $\tilde{\partial}_1(B)$ and $\tilde{\partial}_2(B)$. The first is given by applying (7.1) to the $(1,1)$ entry:

$$\tilde{\partial}_1(B) = \begin{bmatrix} \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_1^2 \end{bmatrix} & \mathbf{1} \cdot \theta_1^2 \\ \theta_2^2 & \theta_1^2 \cdot \mathbf{1} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_1^2 \end{bmatrix} & \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} \theta_1^2 \end{bmatrix} \\ \begin{bmatrix} \theta_2^2 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{1} \end{bmatrix} & \theta_1^2 \cdot \mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \theta_2^1 & \mathbf{1} \\ \theta_2^1 & \mathbf{1} \\ \theta_2^2 & \theta_1^2 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_1^2 & \theta_1^2 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}.$$

The second arises by applying (7.1) to the entries in the first column, then coheretizing:

$$B \to \begin{bmatrix} \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_1^2 \end{bmatrix} & \mathbf{1} \cdot \theta_1^2 \\ \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_1^2 \end{bmatrix} & \theta_1^2 \cdot \mathbf{1} \end{bmatrix} \xrightarrow{coheretize}$$

$$\tilde{\partial}_2(B) = \begin{bmatrix} \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \mathbf{1} \cdot \theta_1^2 \end{bmatrix} & \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} \cdot \theta_1^2 \end{bmatrix} \\ \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_1^2 \cdot \mathbf{1} \end{bmatrix} & \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \begin{bmatrix} \theta_1^2 \cdot \mathbf{1} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \theta_2^1 & \mathbf{1} \\ \theta_2^1 & \mathbf{1} \\ \theta_2^1 & \mathbf{1} \\ \theta_2^1 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \theta_1^2 & \mathbf{1} \cdot \theta_1^2 & \mathbf{1} \cdot \theta_1^2 \\ \theta_1^2 & \theta_1^2 \cdot \mathbf{1} & \theta_1^2 \cdot \mathbf{1} \end{bmatrix}.$$

Example 7.4. Consider the 2-dimensional monomial $\xi = AB = [\theta_2^1\theta_2^1]^T[\theta_2^2\theta_2^2] \in \tilde{\mathcal{B}}_{2,4}$, which corresponds to the element ρ in Example 5.14; then $\tilde{\partial}(\xi) = \tilde{\partial}(A)B \cup A\tilde{\partial}(B) = A\tilde{\partial}(B)$. The 1-dimensional components in $\tilde{\partial}(B) := \{\tilde{\partial}_k(B)\}_{1\le k\le 7}$ are

$$\tilde{\partial}_1(B) = \left[\begin{bmatrix}\theta_2^1\\ \theta_2^1\end{bmatrix}[\theta_1^2\ \ \theta_1^2]\ \ \begin{bmatrix}\mathbf{1}\\ \mathbf{1}\end{bmatrix}[\theta_2^2]\right] = \begin{bmatrix}\theta_2^1 & \mathbf{1}\\ \theta_2^1 & \mathbf{1}\end{bmatrix}[\theta_1^2\ \ \theta_1^2\ \ \theta_2^2]\,;$$

$$\tilde{\partial}_2(B) = \left[\begin{bmatrix}\mathbf{1}\\ \mathbf{1}\end{bmatrix}[\theta_2^2]\ \ \begin{bmatrix}\theta_2^1\\ \theta_2^1\end{bmatrix}[\theta_1^2\ \ \theta_1^2]\right] = \begin{bmatrix}\mathbf{1} & \theta_2^1\\ \mathbf{1} & \theta_2^1\end{bmatrix}[\theta_2^2\ \ \theta_1^2\ \ \theta_1^2];$$

$$\tilde{\partial}_3(B) = \left[\begin{bmatrix}[\mathbf{1}][\theta_2^1]\\ \theta_2^1\end{bmatrix}[\theta_1^2\ \ \theta_1^2]\ \ \begin{bmatrix}[\theta_2^1][\mathbf{1}\ \ \mathbf{1}]\\ \theta_2^1\end{bmatrix}[\theta_1^2\ \ \theta_1^2]\right] =$$

$$\begin{bmatrix}[\mathbf{1}][\theta_2^1] & [\theta_2^1][\mathbf{1}\ \ \mathbf{1}]\\ \theta_2^1 & \theta_2^1\end{bmatrix}[\theta_1^2\ \ \theta_1^2\ \ \theta_1^2\ \ \theta_1^2]\,;$$

$$\tilde{\partial}_4(B) = \left[\begin{bmatrix}\theta_2^1\\ [\theta_2^1][\mathbf{1}\ \ \mathbf{1}]\end{bmatrix}[\theta_1^2\ \ \theta_1^2]\ \ \begin{bmatrix}\theta_2^1\\ [\mathbf{1}][\theta_2^1]\end{bmatrix}[\theta_1^2\ \ \theta_1^2]\right] =$$

$$\begin{bmatrix}\theta_2^1 & \theta_2^1\\ [\theta_2^1][\mathbf{1}\ \mathbf{1}] & [\mathbf{1}][\theta_2^1]\end{bmatrix}[\theta_1^2\ \ \theta_1^2\ \ \theta_1^2\ \ \theta_1^2];$$

$$\tilde{\partial}_5(B) = [\,[\theta_1^2][\theta_2^1]\ \ [\theta_2^2][\mathbf{11}]\,] = [\theta_1^2\ \ \theta_2^2][\theta_2^1\ \ \mathbf{1}\ \ \mathbf{1}];$$

$$\tilde{\partial}_6(B) = [\,[\theta_2^2][\mathbf{1}\ \ \mathbf{1}]\ \ [\theta_1^2][\theta_2^1]\,] = [\theta_2^2\ \ \theta_1^2][\mathbf{1}\ \ \mathbf{1}\ \ \theta_2^1];$$

$$\tilde{\partial}_7(B) = [\,[\theta_1^2][\theta_2^1]\ \ [\theta_1^2][\theta_2^1]\,] = [\theta_1^2\ \ \theta_1^2][\theta_2^1\ \ \theta_2^1].$$

7.2 Free Matrads Defined

Definition 7.5. For $m, n \ge 0$, define $\mathcal{B}_{n+1,m+1}(\Theta) := \tilde{\mathcal{B}}_{n,m}(\Theta)/\,(B \sim \mathbf{1}B \sim B\mathbf{1})$. Let $F(\Theta) := \langle \mathcal{B}(\Theta)\rangle$, let γ be the product induced by $\tilde{\gamma}$ on $\tilde{F}(\Theta)$, and let $\eta : R \to F_{1,1}(\Theta)$ be the unit given by $\eta(1_R) = \mathbf{1}$. Then $(F(\Theta), \gamma, \eta)$ is the **free matrad generated by** Θ.

The map

$$\varsigma\varsigma := \varsigma\varsigma^{pre}|_{\tilde{F}(\Theta)} : \tilde{F}_{m,n}(\Theta) \to F_{m+1,n+1}(\Theta) \tag{7.2}$$

is the canonical projection and the dimension of a balanced monomial $\xi \in \mathcal{B}(\Theta)$ is

$$|\xi| := \sum_{\theta_s^t \text{ in } \xi} |\theta_s^t|.$$

From (6.6) we immediately establish the following commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{B}}_{n,m}(\Theta) & \xrightarrow{\approx} & \mathfrak{m} \circledast_{pp} \mathfrak{n} \\ \varsigma\varsigma \downarrow & & \downarrow \vartheta\vartheta \\ \mathcal{B}_{n+1,m+1}(\Theta) & \xrightarrow{\approx} & \mathfrak{m} \circledast_{kk} \mathfrak{n}. \end{array}$$

We have (cf. Proposition 7.2)

Theorem 7.6. The differential $\tilde{\partial}$ induces a differential ∂ on $F(\Theta)$ such that

$$(F(\Theta)),\partial) \xrightarrow{\approx} (C_*(\mathfrak{m} \circledast_{kk} \mathfrak{n}),\partial)$$

is a canonical isomorphism.

Example 7.7. Regarding Example 7.4, let $\zeta = \varsigma\varsigma(\xi)$ and $C = \varsigma\varsigma(B)$. Once again, $\zeta \in \mathcal{B}_{4,2}(\Theta)$ is 2-dimensional and has four 1-dimensional faces $\partial_1(C)$, $\partial_2(C)$, $\partial_5(C)$, and $\partial_6(C)$ respectively induced by $\tilde{\partial}_1(B)$, $\tilde{\partial}_2(B)$, $\tilde{\partial}_5(B)$, and $\tilde{\partial}_6(B)$, because $|\varsigma\varsigma(\tilde{\partial}_3(\xi))| = |\varsigma\varsigma(\tilde{\partial}_4(\xi))| = |\varsigma\varsigma(\tilde{\partial}_7(\xi))| = 0$ in $\mathcal{B}_{4,2}(\Theta)$.

We denote the chain complex $(F(\Theta)),\partial)$ by $(\mathcal{H}_\infty,\partial)$ and refer to it as the *A_∞-bialgebra matrad* (cf. Theorem 8.4 below). Thus an A_∞-bialgebra structure on a DG module A is defined by a prematrad map $\mathcal{H}_\infty \to \mathcal{E}nd_{TA}$ (cf. Example 6.3).

8 Constructions of PP and KK

In this section we construct the polytopes PP and KK as geometric realizations of the balanced and reduced balanced framed joins. We subsequently identify the free matrad $\mathcal{H}_\infty$ with the cellular chains of KK.

8.1 The Bipermutahedron $PP_{n,m}$

Let $w \in P_{m+n}$. By equality $P_m *_c P_n = P_{m+n}$, which follows from (2.10), the decomposition $w = \alpha \Cup \beta \in P_m *_c P_n$ uniquely determines a bipartition $\beta/\alpha \in P'_r(m) \times P'_r(n)$. Define the *combinatorial join*

$$\mathfrak{m} *_c \mathfrak{n} := \{\beta/\alpha \in P'_r(m) \times P'_r(n) : \alpha \Cup \beta \in P_m *_c P_n,\ 1 \le r \le m+n\};$$

then the bijection $g_1 : \mathfrak{m} *_c \mathfrak{n} \to P_{m+n}$ given by $\beta/\alpha \mapsto \alpha \Cup \beta$ identifies $\beta/\alpha \in \mathfrak{m} *_c \mathfrak{n}$ with the face $C_1|\cdots|C_r := \alpha \Cup \beta$ of P_{m+n}, and $\partial_{M^k,N^k}(\beta/\alpha)$ with the boundary component $\partial_{X^k}(C_1|\cdots|C_r)$, where $X^k := M^k \cup (N^k + m) \subseteq C_k$ (cf. (2.6)). Furthermore, g_1 extends to a bijection

$$g : \mathfrak{m} \circledast_{pp} \mathfrak{n} \xrightarrow{\approx} P_m *_{pp} P_n = PP_{m,n} \tag{8.1}$$

via $(c_1, c_2, c_3) \overset{g}{\mapsto} (g_1(c_1), c_2)$, where $(g_1(c_1), c_2)$ ranges over the faces of $PP_{n,m}$ to be constructed.

Given $m, n \ge 0$, construct the polytope $PP_{n,m}$ by subdividing P_{m+n} in the following way: A face $e \sqsubseteq P_{m+n}$ is indexed by a partition $C_1|\cdots|C_r \in P(\mathfrak{m} \Cup \mathfrak{n}) = P_{m+n}$, and in particular, $\mathfrak{m} \Cup \mathfrak{n}$ indexes the top dimensional cell. While the top dimensional cell of $PP_{n,m}$ is also indexed by $\mathfrak{m} \Cup \mathfrak{n} = g_1(\mathfrak{n}/\mathfrak{m})$ thought of as the elementary bipartition $\mathfrak{n}/\mathfrak{m}$, a proper face $e \sqsubset PP_{n,m}$ is indexed by a pair $(w, T(w))$, where $w = \alpha \Cup \beta$ is expressed as the bipartition $\beta/\alpha = \mathbf{C}_1 \cdots \mathbf{C}_r \in P'_r(\mathfrak{m}) \times P'_r(\mathfrak{n})$ and $T(w) = C_1 \cdots C_r := T^1(\mathbf{C}_1) \cdots T^1(\mathbf{C}_r)$ is a coherent formal product of bipartition matrices. When β/α is coherent (when $|w| = 0$ for example), we identify $(w, T(w))$ with w.

Express w as a product $w = E_1 \times \cdots \times E_r$ of permutahedra. A proper cell $e = (w, T(w)) \sqsubset PP_{n,m}$ is a subdivision cell of the form $e = e_1 \times \cdots \times e_r \sqsubseteq w$ such that $e_k \subseteq E_k$ and no proper face of w contains e. The dimension $|e| := |e_1| + \cdots + |e_r|$, where

$$|e_k| := |\overset{\wedge}{\mathbf{e}}(C_k)| + |\overset{\vee}{\mathbf{e}}(C_k)| - \hat{v}(C_k) - \overset{\vee}{v}(C_k) + 1.$$

Thus, $|e| = |w|$ if and only if C_k is TD coherent for all k.

The action of $\tilde{\partial}$ on the top dimensional cell $\mathfrak{m} \Cup \mathfrak{n} \sqsubset PP_{n,m}$ produces the set of all codimension 1 faces

$$\tilde{\partial}(\mathfrak{m} \Cup \mathfrak{n}) = \{(C_1|C_2, T(C_1|C_2)) : C_1|C_2 \in P_2(\mathfrak{m} \Cup \mathfrak{n})\},$$

and its action on a proper cell $e = (w, T(w))$ is defined by the set

$$\tilde{\partial}(e) := \bigcup_{k \in \mathfrak{r}} \left\{ \left(w, \partial^k_{\mathbf{M}_1,\mathbf{N}_1} T(w)\right), \left(\partial_{M^k}(w), \text{coheretizations of } \partial^k_{\mathbf{M}_2,\mathbf{N}_2} T(w)\right) \right\}$$

where $\partial^k_{\mathbf{M}_1,\mathbf{N}_1} T(w)$ is indecomposable and $\partial^k_{\mathbf{M}_2,\mathbf{N}_2} T(w)$ is decomposable with coheretizable matrix factors.

A cell of codimension 2 or more is given by the iterated action of $\tilde{\partial}$ on some cell of codimension 1. Furthermore, a careful analysis of Lemma 5.9 reveals that the face structure of e_k agrees with that of $\overset{\wedge}{\mathbf{e}}(C_k)$ or $\overset{\vee}{\mathbf{e}}(C_k)$ thought of as cells in the $(q_k - 1)^{st}$ or $(p_k - 1)^{st}$ subdivision complexes of $P_{\#\mathbf{is}(C_k)}$ or $P_{\#\mathbf{os}(C_k)}$, respectively (see Example 8.1 and Figure 8 below). It is straightforward to check that g is compatible with face operators.

Example 8.1. To obtain the subdivision cells of $P_{2,2}$ contained in $w = 1|234 \subset P_4$, construct the decomposition $1|234 = 1|2 \Cup 0|34$ and obtain the bipartition

$$\frac{0|34}{1|2} = \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{34}{0} & \frac{34}{2} \end{pmatrix}.$$

Then w subdivides as $a \cup b \cup c$, where a and b are squares and c is a heptagon. The squares are labeled by

$$a = \left(\frac{0|34}{1|2}, \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{34}{0} & \frac{4|3}{2|0} \end{pmatrix} \right) \quad \text{and} \quad b = \left(\frac{0|34}{1|2}, \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{34}{0} & \frac{4|3}{0|2} \end{pmatrix} \right);$$

the heptagon c is labeled by

$$c = \left(\frac{0|34}{1|2}, \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{3|4}{0|0} & \frac{34}{2} \end{pmatrix} \right).$$

One subdivision vertex with initial bipartition $\frac{0|34}{1|2}$ lies in the interior of w:

$$z_1 = \left(\frac{0|34}{1|2}, \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{3|4}{0|0} & \frac{4|0|3}{0|2|0} \end{pmatrix} \right).$$

Three subdivision vertices lie in the boundary of w :

$$z_2 = \left(\frac{0|34|0}{1|0|2}, \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{3|4}{0|0} & \frac{4|3}{0|0} \end{pmatrix} \begin{pmatrix} \frac{0}{0} & \frac{0}{2} \end{pmatrix} \right),$$

$$u_1 = \left(\frac{0|0|34}{1|2|0}, \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{0}{0} & \frac{0}{2} \\ \frac{0}{0} & \frac{0}{2} \\ \frac{0}{0} & \frac{0}{2} \end{pmatrix} \begin{pmatrix} \frac{3|4}{0|0} & \frac{3|4}{0|0} & \frac{4|3}{0|0} \end{pmatrix} \right),$$

and

$$u_2 = \left(\frac{0|0|34}{1|2|0}, \begin{pmatrix} \frac{0}{1} \\ \frac{0}{1} \\ \frac{0}{1} \end{pmatrix} \begin{pmatrix} \frac{0}{0} & \frac{0}{2} \\ \frac{0}{0} & \frac{0}{2} \\ \frac{0}{0} & \frac{0}{2} \end{pmatrix} \begin{pmatrix} \frac{3|4}{0|0} & \frac{4|3}{0|0} & \frac{4|3}{0|0} \end{pmatrix} \right).$$

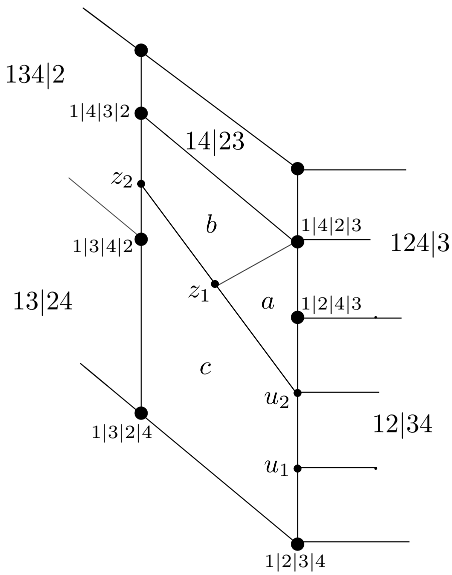


Figure 8. The subdivision cells of 1|234 in $PP_{2,2}$.

8.2 The Biassociahedron $KK_{n+1,m+1}$

The *reduced combinatorial framed join of* $\mathfrak{m}$ *and* $\mathfrak{n}$, denoted by $\mathfrak{m} *_{ck} \mathfrak{n}$, is the image of the subset $\mathfrak{m} *_c \mathfrak{n} \subseteq \mathfrak{m} \circledast \mathfrak{n}$ under the projection $\vartheta\vartheta$, i.e., $\mathfrak{m} *_{ck} \mathfrak{n} := \vartheta\vartheta(\mathfrak{m} *_c \mathfrak{n})$. In particular, the equivalence relation on $\mathfrak{m} \circledast \mathfrak{n}$ restricted to 0-dimensional bipartitions $\frac{\beta_1}{\alpha_1} = \mathbf{C}_{1,1} \cdots \mathbf{C}_{m+n,1}$ and $\frac{\beta_2}{\alpha_2} = \mathbf{C}_{1,2} \cdots \mathbf{C}_{m+n,2}$ in $\mathfrak{m} *_c \mathfrak{n}$ implies $\frac{\beta_1}{\alpha_1} \sim \frac{\beta_2}{\alpha_2}$ if and only if there exist associations $A_1 \cdots A_s = \mathbf{C}_{1,1} \cdots \mathbf{C}_{m+n,1}$, $s \leq m+n$, and $B_1 \cdots B_t = \mathbf{C}_{1,2} \cdots \mathbf{C}_{m+n,2}$, $t \leq m+n$, such that

- the number of semi-null matrix factors A_k with $\#\mathbf{is}(A_k) \neq 0$ equals the number of semi-null matrix factors B_k with $\#\mathbf{is}(B_k) \neq 0$,
- corresponding semi-null matrix factors A_k and B_k differ only in the empty biblocks in their corresponding entries,

- the number of semi-null matrix factors A_ℓ with $\#\mathbf{os}(A_\ell) \neq 0$ equals the number of semi-null matrix factors B_ℓ with $\#\mathbf{os}(B_\ell) \neq 0$, and
- corresponding semi-null matrix factors A_ℓ and B_ℓ differ only in the empty biblocks in their corresponding entries.

(See Example 5.24 and Remark 8.2.)

Thus we obtain the quotient polytopes $K_{n+1,m+1} := P_{m+n}/\sim$ of dimension $m+n-1$ determined by the set $\mathfrak{m} *_{ck} \mathfrak{n}$ together with the cellular projection

$$\vartheta_{n,m} : P_{m+n} \to K_{n+1,m+1}.$$

While $K_{n+1,1} = K_{1,n+1}$ is the associahedron K_{n+1} for all $n \geq 1$, $K_{n+1,m+1}$ is the permutahedron P_{m+n} for $1 \leq m, n \leq 2$ and $K_{n,2} = K_{2,n}$ is the multiplihedron J_n for all n (cf. Remark 8.2).

The canonical projection $\vartheta\vartheta$ defined in (5.5) and the bijection g defined in (8.1) induce the subdivision of $K_{n+1,m+1}$ that produces the biassociahedron $KK_{n+1,m+1}$ and commutes the following diagram:

$$\begin{array}{ccc} PP_{n,m} & \overset{\approx}{\longrightarrow} & P_{m+n} \\ \vartheta\vartheta \downarrow & & \downarrow \vartheta_{n,m} \\ KK_{n+1,m+1} & \underset{\approx}{\longrightarrow} & K_{n+1,m+1}, \end{array}$$

where the horizontal maps are non-cellular homeomorphisms induced by the subdivision process. We also refer to $KK_{m+1,n+1}$ as the *reduced balanced framed join of the associahedra* K_{m+1} and K_{n+1}, and denote $K_{m+1} *_{kk} K_{n+1} := KK_{m+1,n+1}$.

Remark 8.2. Regarding Example 5.24, the associahedron K_{n+1} is identified with $\mathfrak{n} \circledast_{kk} \mathfrak{o}$ but not with $K_{n,2} = \mathfrak{n} \smallsetminus \{n\} \circledast_{kk} \mathfrak{1}$. In the simplest case $n = 4$, the vertices $\frac{0|0|0|0}{2|3|4|1}$ and $\frac{0|0|0|0}{2|1|4|3}$ of P_4 are identified in K_5 but not when viewed as bipartitions $\frac{0|0|4|0}{2|3|0|1}$ and $\frac{0|0|4|0}{2|1|0|3}$ in $P_4 = P_3 *_c P_1$ in $K_{4,2}$. Thus, $K_{4,2} = J_4 \neq K_5$ while $|K_{4,2}| = |K_5| = 3$. Verification is an exercise in bipartition matrix factorizations and left to the reader.

Let $(C_*(PP_{n,m}), \tilde{\partial})$ and $(C_*(KK_{n+1,m+1}), \partial)$ be the cellular chain complexes of $PP_{n,m}$ and $KK_{n+1,m+1}$, respectively. There exist isomorphisms

$$(C_*(\mathfrak{m} \circledast_{pp} \mathfrak{n}), \tilde{\partial}) \overset{\approx}{\longrightarrow} (C_*(PP_{n,m}), \tilde{\partial})$$

and

$$(C_*(\mathfrak{m} \circledast_{kk} \mathfrak{n}), \partial) \overset{\approx}{\longrightarrow} (C_*(KK_{n+1,m+1}), \partial).$$

These facts, together with Proposition 7.2 and Theorem 7.6, immediately imply

Theorem 8.3. There is a canonical isomorphism of chain complexes

$$\tilde{\imath}_* : (\tilde{F}(\Theta)_{n,m}, \tilde{\partial}) \overset{\approx}{\longrightarrow} (C_*(PP_{n,m}), \tilde{\partial}) \tag{8.2}$$

extending the isomorphisms

$$\tilde{F}(\Theta)_{n+1,0} \overset{\approx}{\longrightarrow} C_*(PP_{n,0}) = C_*(P_n)$$

and

$$\tilde{F}(\Theta)_{0,m+1} \xrightarrow{\approx} C_*(PP_{0,m}) = C_*(P_m).$$

Theorem 8.4. There is a canonical isomorphism of chain complexes

$$\iota_* : (\mathcal{H}_\infty)_{n,m} \xrightarrow{\approx} C_*(KK_{n,m}) \tag{8.3}$$

extending the standard isomorphisms

$$\mathcal{A}_\infty(n) = (\mathcal{H}_\infty)_{n,1} \xrightarrow{\approx} C_*(KK_{n,1}) = C_*(K_n)$$

and

$$\mathcal{A}_\infty(m) = (\mathcal{H}_\infty)_{1,m} \xrightarrow{\approx} C_*(KK_{1,m}) = C_*(K_m).$$

The contractibility of $KK_{n,m}$ implies that the canonical projection $\varrho : (\mathcal{H}_\infty, \partial) \to (\mathcal{H}^{pre}, 0)$ is the minimal resolution mentioned in the introduction, and an A_∞-bialgebra structure on a DGM H is given by a morphism of matrads $\mathcal{H}_\infty \to U_H$; thus, an A_∞-bialgebra is an algebra over $\mathcal{H}_\infty$.

Pictures of $KK_{n,m}$ with codimension 1 faces labeled by bipartitions in $\mathfrak{m} \circledast_{kk} \mathfrak{n}$ for $2 \leq m+n \leq 6$ appear below. These pictures first appeared in [22] and subsequently in [31].

For $KK_{1,3}$:

$$1|2 \quad \leftrightarrow \quad \tfrac{0|0}{1|2} \quad \leftrightarrow \quad \gamma(\theta_2^1\,;\theta_1^1\theta_2^1)$$

$$2|1 \quad \leftrightarrow \quad \tfrac{0|0}{2|1} \quad \leftrightarrow \quad \gamma(\theta_2^1\,;\theta_2^1\theta_1^1)$$

For $KK_{3,1}$:

$$1|2 \quad \leftrightarrow \quad \tfrac{1|2}{0|0} \quad \leftrightarrow \quad \gamma(\theta_1^2\theta_1^1\,;\theta_1^2)$$

$$2|1 \quad \leftrightarrow \quad \tfrac{2|1}{0|0} \quad \leftrightarrow \quad \gamma(\theta_1^1\theta_1^2\,;\theta_1^2)$$

Figure 9. The biassociahedra $KK_{1,3} = K_3$ and $KK_{3,1} = K_3$ (intervals).

For $KK_{2,2}$:

$$1|2 \quad \leftrightarrow \quad \tfrac{0|1}{1|0} \quad \leftrightarrow \quad \gamma(\theta_2^1\theta_2^1\,;\theta_1^2\theta_1^2)$$

$$2|1 \quad \leftrightarrow \quad \tfrac{1|0}{0|1} \quad \leftrightarrow \quad \gamma(\theta_1^2\,;\theta_2^1)$$

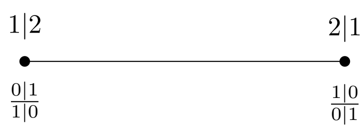


Figure 10. The biassociahedron $KK_{2,2}$ (an interval).

For $KK_{3,2}$:

$1|23 \leftrightarrow \frac{0|12}{1|0} \leftrightarrow \gamma(\theta_2^1\theta_2^1\theta_2^1\,;\gamma(\theta_1^2\theta_1^1;\theta_1^2)\theta_1^3+\theta_1^3\gamma(\theta_1^1\theta_1^2;\theta_1^2))$

$13|2 \leftrightarrow \frac{2|1}{1|0} \leftrightarrow \gamma(\theta_2^1\theta_2^2\,;\theta_1^2\theta_1^2)$

$3|12 \leftrightarrow \frac{2|1}{0|1} \leftrightarrow \gamma(\theta_1^1\theta_1^2\,;\theta_2^2)$

$12|3 \leftrightarrow \frac{1|2}{1|0} \leftrightarrow \gamma(\theta_2^2\theta_2^1\,;\theta_1^2\theta_1^2)$

$2|13 \leftrightarrow \frac{1|2}{0|1} \leftrightarrow \gamma(\theta_1^2\theta_1^1\,;\theta_2^2)$

$23|1 \leftrightarrow \frac{12|0}{0|1} \leftrightarrow \gamma(\theta_1^3\,;\theta_2^1)$

For $KK_{2,3}$:

$1|23 \leftrightarrow \frac{0|1}{1|2} \leftrightarrow \gamma(\theta_2^1\theta_2^1\,;\theta_1^2\theta_2^2)$

$13|2 \leftrightarrow \frac{1|0}{1|2} \leftrightarrow \gamma(\theta_2^2\,;\theta_1^1\theta_2^1)$

$3|12 \leftrightarrow \frac{1|0}{0|12} \leftrightarrow \gamma(\theta_1^2\,;\theta_3^1)$

$12|3 \leftrightarrow \frac{0|1}{12|0} \leftrightarrow \gamma(\gamma(\theta_2^1\,;\theta_1^1\theta_2^1)\theta_3^1+\theta_3^1\gamma(\theta_2^1\,;\theta_2^1\theta_1^1)\,;\theta_1^2\theta_1^2\theta_1^2)$

$2|13 \leftrightarrow \frac{0|1}{2|1} \leftrightarrow \gamma(\theta_2^1\theta_2^1\,;\theta_2^2\theta_1^2)$

$23|1 \leftrightarrow \frac{1|0}{2|1} \leftrightarrow \gamma(\theta_2^2\,;\theta_2^1\theta_1^1)$

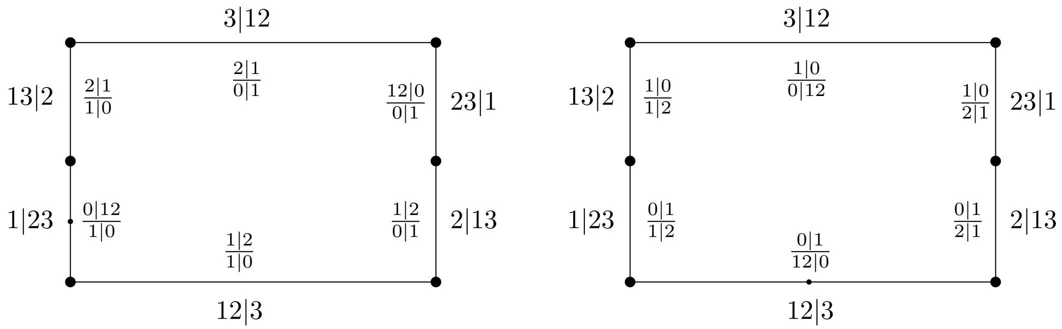


Figure 11. The biassociahedra $KK_{3,2}$ and $KK_{2,3}$ (heptagons).

For $KK_{3,3}$:

$1|234 \leftrightarrow \frac{0|12}{1|2} \leftrightarrow \gamma(\theta_2^1\theta_2^1\theta_2^1\,;\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_2^3+a+b)$, where
$a+b=\theta_1^3\gamma(\theta_2^1\theta_2^2\,;\theta_1^2\theta_1^2)+\theta_1^3\gamma(\theta_1^1\theta_1^2\,;\theta_2^2)$

$123|4 \leftrightarrow \frac{1|2}{12|0} \leftrightarrow \gamma(c+d+\theta_3^2\gamma(\theta_2^1\,;\theta_2^1\theta_1^1)\,;\theta_1^2\theta_1^2\theta_1^2)$, where
$c+d=\gamma(\theta_2^1\theta_2^1\,;\theta_1^2\theta_2^2)\theta_3^1+\gamma(\theta_2^2\,;\theta_1^1\theta_2^1)\theta_3^1$

$2|134 \leftrightarrow \frac{0|12}{2|1} \leftrightarrow \gamma(\theta_2^1\theta_2^1\theta_2^1\,;\theta_2^3\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)+e+f)$, where
$e+f=\gamma(\theta_2^2\theta_2^1\,;\theta_1^2\theta_1^2)\theta_1^3+\gamma(\theta_1^2\theta_1^1\,;\theta_2^2)\theta_1^3$

$$124|3 \quad \leftrightarrow \quad \frac{2|1}{12|0} \quad \leftrightarrow \quad \gamma(g+h+\gamma(\theta_2^1\,;\theta_1^1\theta_2^1)\theta_3^2\,;\theta_1^2\theta_1^2\theta_1^2),\ \text{where}$$
$$g+h=\theta_3^1\gamma(\theta_2^1\theta_2^1\,;\theta_2^2\theta_1^2)+\theta_3^1\gamma(\theta_2^2\,;\theta_2^1\theta_1^1)$$

$$134|2 \quad \leftrightarrow \quad \frac{12|0}{1|2} \quad \leftrightarrow \quad \gamma(\theta_2^3\,;\theta_1^1\theta_2^1)$$

$$234|1 \quad \leftrightarrow \quad \frac{12|0}{2|1} \quad \leftrightarrow \quad \gamma(\theta_2^3\,;\theta_2^1\theta_1^1)$$

$$3|124 \quad \leftrightarrow \quad \frac{1|2}{0|12} \quad \leftrightarrow \quad \gamma(\theta_1^2\theta_1^1\,;\theta_3^2)$$

$$4|123 \quad \leftrightarrow \quad \frac{2|1}{0|12} \quad \leftrightarrow \quad \gamma(\theta_1^1\theta_1^2\,;\theta_3^2)$$

$$23|14 \quad \leftrightarrow \quad \frac{1|2}{2|1} \quad \leftrightarrow \quad \gamma(\theta_2^2\theta_2^1\,;\theta_2^2\theta_1^2)$$

$$14|23 \quad \leftrightarrow \quad \frac{2|1}{1|2} \quad \leftrightarrow \quad \gamma(\theta_2^1\theta_2^2\,;\theta_1^2\theta_2^2)$$

$$24|13 \quad \leftrightarrow \quad \frac{2|1}{2|1} \quad \leftrightarrow \quad \gamma(\theta_2^1\theta_2^2\,;\theta_2^2\theta_1^2)$$

$$13|24 \quad \leftrightarrow \quad \frac{1|2}{1|2} \quad \leftrightarrow \quad \gamma(\theta_2^2\theta_2^1\,;\theta_1^2\theta_2^2)$$

$$34|12 \quad \leftrightarrow \quad \frac{12|0}{0|12} \quad \leftrightarrow \quad \gamma(\theta_1^3\,;\theta_3^1)$$

$$12|34 \quad \leftrightarrow \quad \frac{0|12}{12|0} \quad \leftrightarrow \quad \gamma[\theta_3^1\gamma(\theta_2^1\,;\theta_2^1\theta_1^1)\gamma(\theta_2^1\,;\theta_2^1\theta_1^1)+\gamma(\theta_2^1\,;\theta_1^1\theta_2^1)\theta_3^1\gamma(\theta_2^1\,;\theta_2^1\theta_1^1)$$
$$+\gamma(\theta_2^1\,;\theta_2^1\theta_1^1)\gamma(\theta_2^1\,;\theta_2^1\theta_1^1)\theta_3^1\,;$$
$$\theta_1^3\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)+\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_1^3\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)+$$
$$+\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_1^3]$$

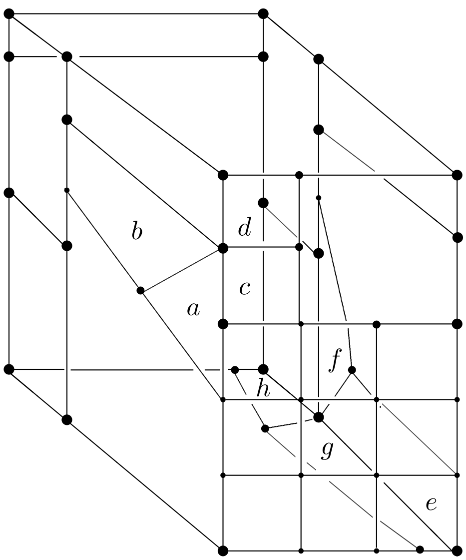


Figure 12. The biassociahedron $KK_{3,3}$ as a subdivision of P_4.

For $KK_{4,2}$:

$$1|234 \leftrightarrow \tfrac{0|123}{1|0} \leftrightarrow \gamma(\theta_2^1\theta_2^1\theta_2^1\theta_2^1\,;a+b-c+d+e+f),\ \text{where}$$

$$a=\theta_1^4\gamma(\theta_1^1\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)\,;\theta_1^2)$$
$$b=\gamma(\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_1^1\,;\theta_1^2)\theta_1^4$$
$$c=\gamma(\theta_1^2\theta_1^1\theta_1^1\,;\theta_1^3)\,\gamma(\theta_1^1\theta_1^1\theta_1^2\,;\theta_1^3)$$
$$d=\gamma(\theta_1^3\theta_1^1\,;\theta_1^2)\,\gamma(\theta_1^1\theta_1^2\theta_1^1\,;\theta_1^3)$$
$$e=\gamma(\theta_1^3\theta_1^1\,;\theta_1^2)\,\gamma(\theta_1^1\theta_1^3\,;\theta_1^2)$$
$$f=\gamma(\theta_1^1\theta_1^2\theta_1^1\,;\theta_1^3)\,\gamma(\theta_1^1\theta_1^3\,;\theta_1^2)$$

$$123|4 \leftrightarrow \tfrac{12|3}{1|0} \leftrightarrow \gamma(\theta_2^3\theta_2^1\,;\theta_1^2\theta_1^2)$$
$$2|134 \leftrightarrow \tfrac{1|23}{0|1} \leftrightarrow \gamma(\theta_1^2\theta_1^1\theta_1^1\,;\theta_2^3)$$
$$124|3 \leftrightarrow \tfrac{13|2}{1|0} \leftrightarrow \gamma(\theta_2^2\theta_2^2\,;\theta_1^2\theta_1^2)$$
$$134|2 \leftrightarrow \tfrac{23|1}{1|0} \leftrightarrow \gamma(\theta_2^1\theta_2^3\,;\theta_1^2\theta_1^2)$$
$$234|1 \leftrightarrow \tfrac{123|0}{0|1} \leftrightarrow \gamma(\theta_1^4\,;\theta_2^1)$$
$$3|124 \leftrightarrow \tfrac{2|13}{0|1} \leftrightarrow \gamma(\theta_1^1\theta_1^2\theta_1^1\,;\theta_2^3)$$
$$4|123 \leftrightarrow \tfrac{3|12}{0|1} \leftrightarrow \gamma(\theta_1^1\theta_1^1\theta_1^2\,;\theta_2^3)$$
$$13|24 \leftrightarrow \tfrac{2|13}{1|0} \leftrightarrow \gamma[\theta_2^1\theta_2^2\theta_2^1\,;\theta_1^3\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)+\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_1^3]$$
$$14|23 \leftrightarrow \tfrac{3|12}{1|0} \leftrightarrow \gamma[\theta_2^1\theta_2^1\theta_2^2\,;\theta_1^3\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)+\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_1^3]$$
$$23|14 \leftrightarrow \tfrac{12|3}{0|1} \leftrightarrow \gamma(\theta_1^3\theta_1^1\,;\theta_2^2)$$
$$34|12 \leftrightarrow \tfrac{23|1}{0|1} = \gamma(\theta_1^1\theta_1^3\,;\theta_2^2)$$
$$12|34 \leftrightarrow \tfrac{1|23}{1|0} \leftrightarrow \gamma[\theta_2^2\theta_2^1\theta_2^1\,;\theta_1^3\gamma(\theta_1^1\theta_1^2\,;\theta_1^2)+\gamma(\theta_1^2\theta_1^1\,;\theta_1^2)\theta_1^3]$$

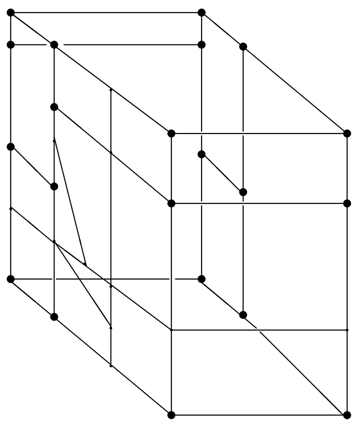


Figure 13. The biassociahedron $KK_{4,2}$ as a subdivision of $J_4=K_{4,2}=\vartheta_{3,1}(P_4)$.

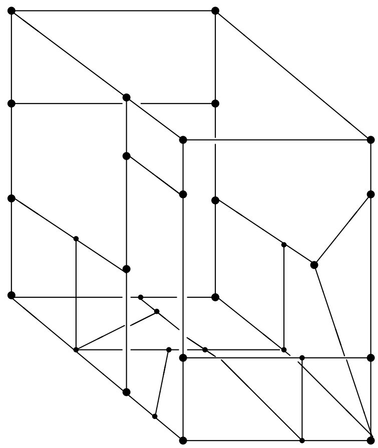


Figure 14. The biassociahedron $KK_{2,4}$ as a subdivision of $J_4 = K_{2,4} = \vartheta_{1,3}(P_4)$.

9 The A_∞-Bialgebra Morphism Matrad

9.1 The Combinatorial Cylinder $\mathcal{ZJ}_{n+1,m+1}$

Let $m, n \geq 0$. In this subsection we define the combinatorial cylinder $\mathcal{ZJ}_{n+1,m+1}$ on the reduced balanced framed join $\mathfrak{m} \circledast_{kk} \mathfrak{n}$ as a quotient of the combinatorial cylinder $\mathcal{ZP}_{n,m}$ on the balanced framed join $\mathfrak{m} \circledast_{pp} \mathfrak{n}$.

Given an integer $z > \max\{m, n\}$, define $\mathfrak{m}_z := \mathfrak{m} \cup \{z\}$ and $\mathfrak{n}_z := \mathfrak{n} \cup \{z\}$, and consider the disjoint union $(\mathfrak{m}_z \circledast \mathfrak{n}) \sqcup (\mathfrak{m} \circledast \mathfrak{n}_z)$. A framed element $c \in (\mathfrak{m}_z \circledast \mathfrak{n}) \sqcup (\mathfrak{m} \circledast \mathfrak{n}_z)$ has the form $c = C_1 \cdots C_{k-1} \cdot F \cdot C_{k+1} \cdots C_r$, where F is the matrix factor containing z and $z \in \mathbf{is}(F)$ if and only if $c \in \mathfrak{m}_z \circledast \mathfrak{n}$. A *structure bipartition within* c is a bipartition within c of the form $\frac{B_1|\cdots|B_r}{A_1|\cdots|A_r} = \mathbf{C}_1 \cdots \mathbf{C}_{k-1} \cdot \mathbf{F} \cdot \mathbf{C}_{k+1} \cdots \mathbf{C}_r$.

Let $ZF_{n,m}$ denote the subset of $(\mathfrak{m}_z \circledast \mathfrak{n}) \sqcup (\mathfrak{m} \circledast \mathfrak{n}_z)$ with the following property: If $c \in ZF_{n,m}$ and $\frac{B_1|\cdots|B_r}{A_1|\cdots|A_r}$ is a structure bipartition within c, then z lies in A_i or B_i but not in A_r or B_1 when $r > 1$. Given $c \in ZF_{n,m}$, let $c_{m,n}$ denote the framed element obtained from c by discarding $z \in A_k \cup B_k$ when $r = 1$ or $1 < k < r$.

Definition 9.1. Given $m, n \geq 0$, choose an integer $z > \max\{m, n\}$. The **combinatorial cylinder on the framed join** $\mathfrak{m} \circledast \mathfrak{n}$ is the set $\widetilde{\mathcal{ZF}}_{n,m} :=$

- $\{\frac{0}{z} \sim \frac{z}{0}\}$ when $m = n = 0$,
- $\mathfrak{m} *_c \{z\}$ when $m > 0$ and $n = 0$,
- $\{z\} *_c \mathfrak{n}$ when $m = 0$ and $n > 0$, and
- $ZF_{n,m}/\sim$ when $mn > 0$, where $c^1 \sim c^2$ if and only if $c^1_{m,n} = c^2_{m,n}$.

The **combinatorial cylinder on the prebalanced framed join** $\mathfrak{m} \,\widetilde{\circledast}_{pp}\, \mathfrak{n}$ is the set $\widetilde{\mathcal{ZP}}_{n,m} :=$

- $\widetilde{\mathcal{ZF}}_{n,0}$ for all $n \geq 0$,
- $\widetilde{\mathcal{ZF}}_{0,m}$ for all $m \geq 0$, and
- $\{c \in \widetilde{\mathcal{ZF}}_{n,m} : c \in \left(m_z \widetilde{\circledast}_{pp} \mathfrak{n}\right) \sqcup \left(\mathfrak{m} \widetilde{\circledast}_{pp} n_z\right)\}$ for all $mn > 0$.

The **combinatorial cylinder on the reduced prebalanced framed join** $\mathfrak{m} \widetilde{\circledast}_{pp} \mathfrak{n}$ is the set

$$\widetilde{\mathcal{ZJ}}_{n+1,m+1} := \widetilde{\mathcal{ZP}}_{n,m}/\sim,$$

where $c' \sim c \in \widetilde{\mathcal{ZP}}_{n,m}$ if and only if c and c' have the same number of indecomposable matrix factors and corresponding indecomposable matrix factors differ only in the number of empty blocks in their entries. As in the definitions of $\mathfrak{m} \circledast_{pp} \mathfrak{n} \subseteq \mathfrak{m} \widetilde{\circledast}_{pp} \mathfrak{n}$ and $\mathfrak{m} \circledast_{kk} \mathfrak{n} = \mathfrak{m} \circledast_{pp} \mathfrak{n}/\sim$, the **combinatorial cylinder on the reduced balanced framed join** $\mathfrak{m} \circledast_{pp} \mathfrak{n}$ is the set

$$\mathcal{ZJ}_{n+1,m+1} := \mathcal{ZP}_{n,m}/\sim .$$

Then in particular, $\mathcal{ZJ}_{n+1,m+1} = \mathcal{ZP}_{n,m}$ when $0 \leq m+n \leq 1$.

As in the absolute case, there is the face operator $\tilde{\partial}_{n,m}$ on $\widetilde{\mathcal{ZP}}_{n,m}$, the projection of chain complexes $C_*(\widetilde{\mathcal{ZP}}_{n,m}) \to C_*(\widetilde{\mathcal{ZJ}}_{n+1,m+1})$, the induced face operator $\partial_{n+1,m+1}$ on $\widetilde{\mathcal{ZJ}}_{n+1,m+1}$, and the restricted projection

$$C_*(\mathcal{ZP}_{n,m}) \to C_*(\mathcal{ZJ}_{n+1,m+1}).$$

Thus, denoting

$$\mathfrak{g}_{m+1}^{n+1} := \begin{cases} \left[\frac{z}{0}\right] = \left[\frac{0}{z}\right], & m = n = 0 \\ \left[\frac{\mathfrak{n}}{z}\right], & m = 0,\ n > 0 \\ \left[\frac{z}{\mathfrak{m}}\right], & m > 0,\ n = 0 \\ \left[\frac{n_z}{\mathfrak{m}}\right] = \left[\frac{\mathfrak{n}}{m_z}\right], & mn > 0 \end{cases}$$

we have

$$\tilde{\partial}_{n,m}\left(\mathfrak{g}_{m+1}^{n+1}\right) := \begin{cases} \varnothing, & m = n = 0 \\ \{F_1C_2, C_1F_2\}, & m+n > 0. \end{cases} \tag{9.1}$$

9.2 Relative Prematrads

Let (M, γ_M, η_M) and (N, γ_N, η_N) be R-prematrads, where $\gamma_M = \{\gamma_{\mathbf{x}}^{\mathbf{y}} : \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{M}_{\mathbf{x}}^q \to \mathbf{M}_{\|\mathbf{x}\|}^{\|\mathbf{y}\|}\}$ and $\gamma_N = \{\gamma_{\mathbf{x}}^{\mathbf{y}} : \mathbf{N}_p^{\mathbf{y}} \otimes \mathbf{N}_{\mathbf{x}}^q \to \mathbf{N}_{\|\mathbf{x}\|}^{\|\mathbf{y}\|}\}$, let $E = \{E_{n,m}\}_{mn\geq 1}$ be a bigraded R-module, and let $\lambda = \{\lambda_{\mathbf{x}}^{\mathbf{y}} : \mathbf{M}_p^{\mathbf{y}} \otimes \mathbf{E}_{\mathbf{x}}^q \to \mathbf{E}_{\|\mathbf{x}\|}^{\|\mathbf{y}\|}\}$ and $\rho = \{\rho_{\mathbf{x}}^{\mathbf{y}} : \mathbf{E}_p^{\mathbf{y}} \otimes \mathbf{N}_{\mathbf{x}}^q \to \mathbf{E}_{\|\mathbf{x}\|}^{\|\mathbf{y}\|}\}$ be left and right R-module actions ($\gamma_M, \gamma_N, \lambda,$ and ρ are subject to $\mathbf{x} \times \mathbf{y} \in \mathbb{N}^{1\times p} \times \mathbb{N}^{q\times 1}$ and $pq \geq 1$). Then λ and ρ induce left and right global products $\Upsilon_\lambda : \mathbf{M} \otimes \mathbf{E} \to \mathbf{E}$ and $\Upsilon_\rho : \mathbf{E} \otimes \mathbf{N} \to \mathbf{E}$ in the same way that γ_M induces the associative global product $\Upsilon_M : \mathbf{M} \otimes \mathbf{M} \to \mathbf{M}$.

Definition 9.2. A tuple (M, E, N, λ, ρ) is a **relative prematrad** if

(i) associativity holds, i.e.,

(a) $\Upsilon_\rho(\Upsilon_\lambda \otimes \mathbf{1}) = \Upsilon_\lambda(\mathbf{1} \otimes \Upsilon_\rho)$,

(b) $\Upsilon_\lambda(\Upsilon_M \otimes \mathbf{1}) = \Upsilon_\lambda(\mathbf{1} \otimes \Upsilon_\lambda)$,

(c) $\Upsilon_\rho(\Upsilon_\rho \otimes \mathbf{1}) = \Upsilon_\rho(\mathbf{1} \otimes \Upsilon_N)$, and

(ii) for all $a, b \in \mathbb{N}$, the units η_M and η_N induce the canonical isomorphisms

$$R^{\otimes b} \otimes \mathbf{E}_a^b \overset{\eta_M^{\otimes b} \otimes \mathbf{1}}{\longrightarrow} \mathbf{M}_1^{\mathbf{1}^{b\times 1}} \otimes \mathbf{E}_a^b \overset{\lambda_a^{\mathbf{1}^{b\times 1}}}{\longrightarrow} \mathbf{E}_a^b \text{ and } \mathbf{E}_a^b \otimes R^{\otimes a} \overset{\mathbf{1} \otimes \eta_N^{\otimes a}}{\longrightarrow} \mathbf{E}_a^b \otimes \mathbf{N}_{\mathbf{1}^{1\times a}}^1 \overset{\rho^b_{\mathbf{1}^{1\times a}}}{\longrightarrow} \mathbf{E}_a^b.$$

Under conditions (i) and (ii), E is as an M-N-**bimodule** and an M-**bimodule** when $M = N$.

Definition 9.3. A **morphism** $f : (M, E, N, \lambda, \rho) \to (M', E', N', \lambda', \rho')$ of relative prematrads is a triple $(f_M : M \to M',\ f_E : E \to E',\ f_N : N \to N')$ such that

(i) f_M and f_N are maps of prematrads, and

(ii) f_E commutes with left and right actions, i.e., for all $\mathbf{x} \times \mathbf{y} \in \mathbb{N}^{p\times 1} \times \mathbb{N}^{1\times q}$ and $pq \geq 1$ we have $f_E\, \lambda_{\mathbf{x}}^{\mathbf{y}} = \lambda'^{\mathbf{y}}_{\mathbf{x}} (f_M^{\otimes q} \otimes f_E^{\otimes p})$ and $f_E\, \rho_{\mathbf{x}}^{\mathbf{y}} = \rho'^{\mathbf{y}}_{\mathbf{x}} (f_E^{\otimes q} \otimes f_N^{\otimes p})$.

Tree representations of $\lambda_1^{\mathbf{x}}$ and $\rho_1^{\mathbf{x}}$ are related to those of $\lambda_{\mathbf{x}}^1$ and $\rho_{\mathbf{x}}^1$ by a reflection in some horizontal axis. Although $\rho_{\mathbf{x}}^1$ agrees with Markl, Shnider, and Stasheff's right module action over an operad [25], $\lambda_{\mathbf{x}}^1$ differs fundamentally from their left module action, and our definition of an "operadic bimodule" is consistent with their definition of an operadic ideal.

Given graded R-modules A and B, let

$$\begin{array}{lcllcl} U_A & := & \mathcal{E}nd_{TA} & = & \{N_{s,p} = Hom\,(A^{\otimes p}, A^{\otimes s})\}_{p,s\geq 1}\,, \\ U_{A,B} & := & Hom\,(TA, TB) & = & \{E_{t,q} = Hom\,(A^{\otimes q}, B^{\otimes t})\}_{q,t\geq 1}\,, \\ U_B & := & \mathcal{E}nd_{TB} & = & \{M_{u,r} = Hom\,(B^{\otimes r}, B^{\otimes u})\}_{r,u\geq 1}\,, \end{array}$$

and define left and right actions λ and ρ in terms of the horizontal and vertical operations $\times$ and $\circ$ analogous to those in the prematrad structures on U_A and U_B (see Section 2 above and [31]). Then the relative PROP $(U_A, U_{A,B}, U_B, \lambda, \rho)$ is the universal example of a relative prematrad.

Definition 9.4. Given bigraded sets $F = \{F_{n,m}\}_{mn\geq 1}$ and $Q = \{Q_{n,m}\}_{mn\geq 1}$ with based element $\mathbf{1} \in Q_{1,1}$, construct the bigraded set $\tilde{E}\,(F;Q) = \{\tilde{E}_{n,m}\,(F;Q)\}_{mn\geq 1}$ of **relative free monomials generated by** F **and** Q inductively as follows: Define $\tilde{E}_{1,1}\,(F;Q) := F_{1,1}$. If $m + n \geq 3$ and $\tilde{E}_{j,i}\,(F;Q)$ has been constructed for all $(i,j) \leq (m,n)$ such that $i + j < m + n$, define

$$\tilde{E}_{n,m}(F;Q) := F_{n,m} \cup \{C_1 \cdots C_{r+s+1} = A_1 \cdots A_r \cdot F \cdot B_1 \cdots B_s : r + s \geq 1\}\,,$$

where

(i) F is a GBSM over $\{\tilde{E}_{j,i}\,(F;Q) : (i,j) \leq (m,n)\,,\ i + j < m + n\}$,

(ii) F is a column matrix with $outdeg(F) = n$ when $r = 0$,

(iii) F is a row matrix with $indeg(F) = m$ when $s = 0$, *(iv)* A_k and B_ℓ are GBSMs over $\{\tilde{G}_{j,i}(Q) : (i,j) \leq (m,n), i+j < m+n\}$,

(v) A_1 is a column matrix with $outdeg(A_1) = n$ when $r > 0$,

(vi) B_s is a row matrix with $indeg(B_s) = m$ when $s > 0$,

(vii) if $B = (\xi_{tu})$ is a $q \times p$ GBSM over $\tilde{G}_{*,*}(Q)$ within C_i and $\xi_{tu} = B_1 \cdots B_{k-1} \cdot \mathbf{1} \cdot B_{k+1} \cdots B_r$ for some k, define $\xi_{tu} := B_1 \cdots B_{k-1} B_{k+1} \cdots B_r$ when either $pq = 1$ and $B = C_i$ or $pq > 1$ and B_{t*} and B_{*u} are indecomposable,

(viii) $\mathfrak{f}_s^t = [\mathbf{1} \cdots \mathbf{1}]^T [\mathfrak{f}_s^t] = [\mathfrak{f}_s^t][\mathbf{1} \cdots \mathbf{1}]$,

(ix) $C_i \times C_{i+1}$ is a BTP for all i,

(x) $C_i C_{i+1}$ is the formal product, and

(xi) $(C_i C_{i+1}) C_{i+2} = C_i (C_{i+1} C_{i+2})$.

An (F,Q)-**monomial** is an elementary monomial in $\tilde{E}(F;Q)$.

Note that in view of item (vii), the sets $\tilde{E}_{n,1}(F;Q)$ and $\tilde{E}_{1,m}(F;Q)$ consist of (F,Q)-monomials exclusively.

Let $\mathfrak{F} = \{\mathfrak{f}_m^n\}_{mn \geq 1}$ and $\Theta = \left\{\theta_m^n : \theta_1^1 = \mathbf{1}\right\}_{mn \geq 1}$ be bigraded sets with at most one element of bidegree (m,n).

Definition 9.5. Let $\tilde{F}^{pre}(\Theta;\mathfrak{F};\Theta) = \left\langle \tilde{E}(\mathfrak{F};\Theta) \right\rangle$ and define left and right actions $\tilde{\lambda}^{pre} : \tilde{F}^{pre}(\Theta) \otimes \tilde{F}^{pre}(\Theta,\mathfrak{F},\Theta) \to \tilde{F}^{pre}(\Theta,\mathfrak{F},\Theta)$ and $\tilde{\rho}^{pre} : \tilde{F}^{pre}(\Theta,\mathfrak{F},\Theta) \otimes \tilde{F}^{pre}(\Theta) \to \tilde{F}^{pre}(\Theta,\mathfrak{F},\Theta)$ by juxtaposition. Then

$$(\tilde{F}^{pre}(\Theta), \tilde{F}^{pre}(\Theta,\mathfrak{F},\Theta), \tilde{F}^{pre}(\Theta), \tilde{\lambda}^{pre}, \tilde{\rho}^{pre})$$

is **relative free non-unital prematrad generated by $\mathfrak{F}$ and Θ**.

Define $E_{n,m}(\mathfrak{F};\Theta) := \widetilde{E}_{n,m}(\mathfrak{F};\Theta)/(A \sim \mathbf{1}A \sim A\mathbf{1})$ and let

$$F^{pre}(\Theta,\mathfrak{F},\Theta) = \langle E(\mathfrak{F};\Theta) \rangle.$$

The **relative free prematrad generated by $\mathfrak{F}$ and Θ** is the relative prematrad

$$(F^{pre}(\Theta), F^{pre}(\Theta,\mathfrak{F},\Theta), F^{pre}(\Theta), \lambda^{pre}, \rho^{pre}).$$

Example 9.6. The bimodule $F_{2,2}^{pre}(\Theta,\mathfrak{F},\Theta)$ contains 25 generators, namely, the indecomposable $\mathfrak{f}_2^2$ and the following twenty-four $(\lambda^{pre}, \rho^{pre})$-decomposables:

two of the form AFB:

$$\left[\theta_1^2\right]\left[\mathfrak{f}_1^1\right]\left[\theta_2^1\right] \text{ and } \begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix} \begin{bmatrix} \mathfrak{f}_1^1 & \mathfrak{f}_1^1 \\ \mathfrak{f}_1^1 & \mathfrak{f}_1^1 \end{bmatrix} \begin{bmatrix} \theta_1^2 & \theta_1^2 \end{bmatrix}; \tag{9.2}$$

eleven of the form $FB_1 \cdots B_s$:

$$\left[\mathfrak{f}_1^2\right]\left[\theta_2^1\right], \begin{bmatrix} \mathfrak{f}_1^1 \\ \mathfrak{f}_1^1 \end{bmatrix}\left[\theta_2^2\right], \begin{bmatrix} \mathfrak{f}_1^1 \\ \mathfrak{f}_1^1 \end{bmatrix}\left[\theta_1^2\right]\left[\theta_2^1\right], \begin{bmatrix} \mathfrak{f}_1^1 \\ \mathfrak{f}_1^1 \end{bmatrix}\begin{bmatrix} \theta_2^1 \\ \theta_2^1 \end{bmatrix}\left[\begin{matrix} \theta_1^2 & \theta_1^2 \end{matrix}\right],$$

$$\begin{bmatrix} \mathfrak{f}_2^1 \\ \mathfrak{f}_2^1 \end{bmatrix}\left[\begin{matrix} \theta_1^2 & \theta_1^2 \end{matrix}\right], \begin{bmatrix} \mathfrak{f}_2^1 \\ \left[\mathfrak{f}_1^1\right]\left[\theta_2^1\right] \end{bmatrix}\left[\begin{matrix} \theta_1^2 & \theta_1^2 \end{matrix}\right], \begin{bmatrix} \left[\mathfrak{f}_1^1\right]\left[\theta_2^1\right] \\ \mathfrak{f}_2^1 \end{bmatrix}\left[\begin{matrix} \theta_1^2 & \theta_1^2 \end{matrix}\right],$$

$$\begin{bmatrix} \left[\theta_2^1\right]\left[\begin{matrix} \mathfrak{f}_1^1 & \mathfrak{f}_1^1 \end{matrix}\right] \\ \left[\mathfrak{f}_1^1\right]\left[\theta_2^1\right] \end{bmatrix}\left[\begin{matrix} \theta_1^2 & \theta_1^2 \end{matrix}\right], \begin{bmatrix} \left[\mathfrak{f}_1^1\right]\left[\theta_2^1\right] \\ \left[\theta_2^1\right]\left[\begin{matrix} \mathfrak{f}_1^1 & \mathfrak{f}_1^1 \end{matrix}\right] \end{bmatrix}\left[\begin{matrix} \theta_1^2 & \theta_1^2 \end{matrix}\right],$$

$$\begin{bmatrix} \left[\theta_2^1\right]\left[\begin{matrix} \mathfrak{f}_1^1 & \mathfrak{f}_1^1 \end{matrix}\right] \\ \mathfrak{f}_2^1 \end{bmatrix}\left[\begin{matrix} \theta_1^2 & \theta_1^2 \end{matrix}\right], \begin{bmatrix} \mathfrak{f}_2^1 \\ \left[\theta_2^1\right]\left[\begin{matrix} \mathfrak{f}_1^1 & \mathfrak{f}_1^1 \end{matrix}\right] \end{bmatrix}\left[\begin{matrix} \theta_1^2 & \theta_1^2 \end{matrix}\right];$$

and their eleven respective duals of the form $A_1 \cdots A_r F$.

Example 9.7. Recall that the bialgebra prematrad $\mathcal{H}^{pre}$ has two prematrad generators $c_{1,2}$ and $c_{2,1}$, and a single module generator $c_{n,m}$ of bidegree (m,n) for $m+n \geq 4$. Consequently, the $\mathcal{H}^{pre}$-bimodule $\mathcal{J}\mathcal{J}^{pre}$ has a single bimodule generator $\mathfrak{f}$ of bidegree $(1,1)$, and a single module generator $c_{n,m}$ of bidegree (m,n) satisfying the structure relation

$$\lambda(c_{n,m}; \underbrace{\mathfrak{f}, \ldots, \mathfrak{f}}_{m}) = \rho(\underbrace{\mathfrak{f}, \ldots, \mathfrak{f}}_{n}; c_{n,m}).$$

More precisely, if $\Theta = \left\langle \theta_1^1 = \mathbf{1}, \theta_2^1, \theta_1^2 \right\rangle$ and $\mathfrak{F} = \left\langle \mathfrak{f} = \mathfrak{f}_1^1 \right\rangle$, then

$$\mathcal{J}\mathcal{J}^{pre} = F^{pre}(\Theta, \mathfrak{F}, \Theta) / \sim,$$

where $u \sim u'$ if and only if $bideg(u) = bideg(u')$. A bialgebra morphism $f : A \to B$ is the image of $\mathfrak{f}$ under a map $\mathcal{J}\mathcal{J}^{pre} \to U_{A,B}$ of relative prematrads. In this case, we omit the superscript and simply write $\mathcal{J}\mathcal{J}$.

Example 9.8. Whereas $F_{1,*}^{pre}(\Theta)$ with bimodule generators $\{\theta_m^1\}_{m\geq 1}$ and $F_{*,1}^{pre}(\Theta)$ with bimodule generators $\{\theta_1^n\}_{n\geq 1}$ can be identified with the A_∞-operad $\mathcal{A}_\infty$, $F_{1,*}(\Theta, \mathfrak{F}, \Theta)$ and $F_{*,1}(\Theta, \mathfrak{F}, \Theta)$ can be identified with the $\mathcal{A}_\infty$-bimodule $\mathcal{J}_\infty$. Thus an A_∞-(co)algebra morphism $f : A \to B$ is the image of the A_∞-bimodule generators under a map $\mathcal{J}_\infty \to Hom\,(TA, B)$ (or $\mathcal{J}_\infty \to Hom\,(A, TB)$) of relative prematrads.

When $\Theta = \left\langle \theta_m^n : \theta_1^1 = \mathbf{1} \right\rangle_{mn\geq 1}$ and $\mathfrak{F} = \langle \mathfrak{f}_m^n \rangle_{mn\geq 1}$, the canonical projections $\varrho_\Theta^{pre} : F^{pre}(\Theta) \to \mathcal{H}^{pre}$ and $\varrho^{pre} : F^{pre}(\Theta, \mathfrak{F}, \Theta) \to \mathcal{J}\mathcal{J}^{pre}$ define a map $(\varrho_\Theta^{pre}, \varrho^{pre}, \ \varrho_\Theta^{pre})$ of relative prematrads. If

∂^{pre} is a differential on $F^{pre}(\Theta, \mathfrak{F}, \Theta)$ such that ϱ^{pre} is a free resolution in the category of relative prematrads, the induced isomorphism $\varrho^{pre} : H_*\left(F^{pre}(\Theta, \mathfrak{F}, \Theta), \partial^{pre}\right) \approx \mathcal{J}\mathcal{J}^{pre}$ implies

$$\partial^{pre}(\mathfrak{f}_1^1) = 0$$

$$\partial^{pre}(\mathfrak{f}_2^1) = \rho(\mathfrak{f}_1^1; \theta_2^1) - \lambda(\theta_2^1; \mathfrak{f}_1^1, \mathfrak{f}_1^1)$$

$$\partial^{pre}(\mathfrak{f}_1^2) = \rho(\mathfrak{f}_1^1, \mathfrak{f}_1^1; \theta_1^2) - \lambda(\theta_1^2; \mathfrak{f}_1^1).$$

This gives rise to the standard isomorphisms

$$\begin{array}{ccccccc} F_{1,2}^{pre}(\Theta, \mathfrak{F}, \Theta) & = & \left\langle \left[\theta_2^1\right]\left[\ \mathfrak{f}_1^1 \quad \mathfrak{f}_1^1\ \right] \right. & , & \left[\mathfrak{f}_2^1\right] & , & \left. \left[\mathfrak{f}_1^1\right]\left[\theta_2^1\right] \right\rangle \\ \approx \ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ C_*(J_2) & = & \langle 1|2 & , & 12 & , & 2|1\rangle \\ \approx \ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ F_{2,1}^{pre}(\Theta, \mathfrak{F}, \Theta) & = & \left\langle \left[\theta_1^2\right]\left[\mathfrak{f}_1^1\right] \right. & , & \left[\mathfrak{f}_1^2\right] & , & \left. \begin{bmatrix} \mathfrak{f}_1^1 \\ \mathfrak{f}_1^1 \end{bmatrix} \left[\theta_1^2\right] \right\rangle \end{array}$$

(see Figure 17). A similar application of ∂^{pre} to $\mathfrak{f}_1^n$ and $\mathfrak{f}_m^1$ gives the isomorphisms

$$F_{n,1}^{pre}(\Theta, \mathfrak{F}, \Theta) \xrightarrow{\approx} \mathcal{J}_\infty(n) = C_*(J_n) \tag{9.3}$$

and

$$F_{1,m}^{pre}(\Theta, \mathfrak{F}, \Theta) \xrightarrow{\approx} \mathcal{J}_\infty(m) = C_*(J_m) \tag{9.4}$$

(see [33], [34], [29]). Combinatorially, J_{n+1} is the cylinder $K_{n+1} \times I$ and the projection

$$\pi_{n+1} : P_{n+1} \to J_{n+1} \tag{9.5}$$

in (10.2) below is given by $\pi_{n+1} := \pi_{0,n}$ (cf. [29]).

9.3 Relative Matrads.

Let $f := A_1 \cdots A_{k-1} \cdot F \cdot A_{k+1} \cdots A_r \in \tilde{E}_{n+1,m+1}(\mathfrak{F};\Theta)$. As in the absolute case, the *dimension* of f is the sum of the dimensions of its matrix factors, i.e.,

$$|f| = |F| + \sum_{1 \leq k \leq r} |A_k|\,,$$

and in particular, $|\mathfrak{f}_m^n| = m + n - 2$.

Given a positive integer n and integer sequences $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_k)$, define $\mathbf{x} \smile n := (x_1, \ldots, x_k, n)$ and $\mathbf{x} + \mathbf{y} := (x_1 + y_1, \ldots, x_k + y_k)$.

Definition 9.9. Let $f = A_1 \cdots A_{k-1} \cdot F \cdot A_{k+1} \cdots A_r \in \tilde{E}_{n+1,m+1}(\mathfrak{F};\Theta)$.

(i) The **Input Leaf Decomposition of** f is the sequence

$$ILD(f) := \{ils(A_1), \ldots, ils(A_{k-1}), ils(F), ils(A_{k+1}), \ldots, ils(A_r)\};$$

the **Augmented Input Leaf Decomposition of** f is the sequence

$$AILD(f) := \{ils(A_1), \ldots, ils(A_{k-1}), ils(F) + (0, \ldots, 0, 1), ils(A_{k+1} \smile 1), \ldots, ils(A_r) \smile 1\}.$$

(ii) The **Output Leaf Decomposition of** f is the sequence

$$OLD\,(f) := \{ols(A_1), \ldots, ols(A_{k-1}), ols(F), ols(A_{k+1}), ..., ols(A_r)\};$$

the **Augmented Output Leaf Decomposition of** f is the sequence

$$AOLD\,(f) := \{ols(A_1) \smile 1, \ldots, ols(A_{k-1}) \smile 1, ols(F) + (0, \ldots, 0, 1), ols(A_{k+1}), \ldots, ols(A_r)\}.$$

Now define the map

$$\digamma_z : \tilde{E}_{n+1,m+1}(\mathfrak{F};\Theta) \to \widetilde{\mathcal{ZF}}_{n,m} \tag{9.6}$$

in a manner similar to the definition of $\digamma$ in (6.6), but this time define $\digamma_z$ on *three* types of monomials: Let $f = A_1 \cdots A_{k-1} \cdot F \cdot A_{k+1} \cdots A_r \in \tilde{E}_{n+1,m+1}(\mathfrak{F};\Theta)$, let $[c] := \digamma_z(f)$, and let $c = (c^1, c^2, \ldots, c^h)$ be the canonical representative. If $r = 1$, then $f = \mathfrak{f}^{n+1}_{m+1}$ and $\digamma_z\,(f) := \mathfrak{g}^{n+1}_{m+1}$. Otherwise,

1. if $m = 0$, then $\digamma_z(f) = \frac{\bar{R}OLD(f)}{\bar{R}AILD(f)} = c^1 = c$.
2. if $n = 0$, then $\digamma_z(f) = \frac{\bar{R}AOLD(f)}{\bar{R}ILD(f)} = c^1 = c$.
3. if $mn > 0$ and g is a monomial within f (the case $g = f$ included)
   (a) $1 < k = r$, then $\digamma_z(g) = \frac{\bar{R}AOLD(f)}{\bar{R}ILD(f)}$.
   (b) $1 = k < r$, then $\digamma_z(g) = \frac{\bar{R}OLD(f)}{\bar{R}AILD(f)}$.
   (c) $1 < k < r$, then $\digamma_z(g)$ is given by either $3(a)$ or $3(b)$.

Then, for example,

$$\digamma_z\left(\left[\theta_1^{n+1}\right]\left[\mathfrak{f}_1^1\right]\right) = \frac{\mathfrak{n}|0}{0|z}, \quad \digamma_z\left(\begin{bmatrix} \mathfrak{f}_1^1 \\ \vdots \\ \mathfrak{f}_1^1 \end{bmatrix}\left[\theta_1^{n+1}\right]\right) = \frac{0|\mathfrak{n}}{z|0},$$

$$\digamma_z\left(\left[\theta_{m+1}^1\right]\left[\mathfrak{f}_1^1 \cdots \mathfrak{f}_1^1\right]\right) = \frac{0|z}{\mathfrak{m}|0}, \quad \text{and} \quad \digamma_z\left(\left[\mathfrak{f}_1^1\right]\left[\theta_{m+1}^1\right]\right) = \frac{z|0}{0|\mathfrak{m}}.$$

Note that f is an $(\mathfrak{F},\Theta)$-monomial if and only if $h(\digamma_z(f)) = 1$.

Remark 9.10. If g is a monomial of bidegree $(t,1)$ or $(1,t)$, $t>1$, its value $F_z(g)$ given by (1) or (2) may differ from its value as a monomial within but distinct from f in (3) (see Example 9.11).

Example 9.11. Consider the monomial

$$f=\begin{bmatrix} \mathfrak{f}_2^1 \\ [\mathfrak{f}_1^1][\theta_2^1] \end{bmatrix}[\theta_1^2\ \theta_1^2]\in \tilde{E}_{2,2}(\mathfrak{F};\Theta).$$

Apply item 3(b) in the definition of F_z and obtain

$$F_z(f)=(c^1,c^2)=\left(\frac{0|1}{1z|0}=\begin{pmatrix}\frac{0}{1z}\\ \frac{0}{1z}\end{pmatrix}\begin{pmatrix}\frac{1}{0}\ \frac{1}{0}\ \frac{1}{0}\end{pmatrix},\begin{pmatrix}\frac{0}{1z}\\ \frac{0|0}{z|1}\end{pmatrix}\begin{pmatrix}\frac{1}{0}\ \frac{1}{0}\ \frac{1}{0}\end{pmatrix}\right)\in\widetilde{\mathcal{ZF}}_{1,1}.$$

Note that the incoherent left-hand factor of c^2 implies $F_z(f)\notin\widetilde{\mathcal{ZP}}_{1,1}$. Also note that $F_z(\mathfrak{f}_2^1)=\frac{z}{1}\neq\frac{0}{1z}$, which is the value of $\mathfrak{f}_2^1$ as an $(\mathfrak{F},\Theta)$-monomial within f.

Apply (9.6) to define the subset $\tilde{\mathcal{B}}_{n,m}(\Theta,\mathfrak{F},\Theta):=F_z^{-1}(\mathcal{ZP}_{n,m})\subset\tilde{E}_{n,m}(\mathfrak{F};\mathfrak{f})$; then define

$$\mathcal{B}_{n,m}(\Theta,\mathfrak{F},\Theta):=\tilde{\mathcal{B}}_{n,m}(\Theta,\mathfrak{F},\Theta)/\left(A\sim\mathbf{1}A\sim A\mathbf{1}\right).$$

Definition 9.12. The **relative free non-unital balanced matrad generated by** $\mathfrak{F}$ and Θ is the bigraded module

$$\tilde{F}(\Theta,\mathfrak{F},\Theta):=\langle\tilde{\mathcal{B}}(\Theta,\mathfrak{F},\Theta)\rangle,$$

and the **relative free balanced matrad generated by** $\mathfrak{F}$ and Θ is the bigraded module

$$F(\Theta,\mathfrak{F},\Theta):=\langle\mathcal{B}(\Theta,\mathfrak{F},\Theta)\rangle.$$

Example 9.13. Consider the map $\pi_{n+1}:P_{n+1}\to J_{n+1}$ for $n=3$ (cf. (9.5) and Remark 8.2). Then bijection (9.4) is represented on the vertices of J_4 in $\pi_4(2|134\cup 24|13)$ as follows:

$$\begin{array}{ccc}
[\theta_2^1][\theta_2^1\ \mathbf{1}][\mathbf{1}\ \mathbf{1}\ \theta_2^1][\mathfrak{f}_1^1\ \mathfrak{f}_1^1\ \mathfrak{f}_1^1\ \mathfrak{f}_1^1] & \sim & [\theta_2^1][\mathbf{1}\ \theta_2^1][\theta_2^1\ \mathbf{1}\ \mathbf{1}][\mathfrak{f}_1^1\ \mathfrak{f}_1^1\ \mathfrak{f}_1^1\ \mathfrak{f}_1^1]\\
\updownarrow & & \updownarrow\\
\frac{0|0|0|z}{2|1|3|0} & \sim & \frac{0|0|0|z}{2|3|1|0}\\
\updownarrow & & \updownarrow\\
2|1|3|4 & \overset{\pi_4}{\sim} & 2|3|1|4
\end{array}$$

$$\begin{array}{ccc}
[\theta_2^1][\theta_2^1\ \mathbf{1}][\mathfrak{f}_1^1\ \mathfrak{f}_1^1\ \mathfrak{f}_1^1][\mathbf{1}\ \mathbf{1}\ \theta_2^1] & & [\theta_2^1][\mathbf{1}\ \theta_2^1][\mathfrak{f}_1^1\ \mathfrak{f}_1^1\ \mathfrak{f}_1^1][\theta_2^1\ \mathbf{1}\ \mathbf{1}]\\
\updownarrow & & \updownarrow\\
\frac{0|0|z|0}{2|1|0|3} & & \frac{0|0|z|0}{2|3|0|1}\\
\updownarrow & & \updownarrow\\
2|1|4|3 & & 2|3|4|1
\end{array}$$

$$\begin{array}{ccc}
[\theta_2^1][\mathfrak{f}_1^1\ \mathfrak{f}_1^1][\theta_2^1\ \mathbf{1}][\mathbf{1}\ \mathbf{1}\ \theta_2^1] & \sim & [\theta_2^1][\mathfrak{f}_1^1\ \mathfrak{f}_1^1][\mathbf{1}\ \theta_2^1][\theta_2^1\ \mathbf{1}\ \mathbf{1}] \\
\updownarrow & & \updownarrow \\
\frac{0|z|0|0}{2|0|1|3} & \sim & \frac{0|z|0|0}{2|0|3|1} \\
\updownarrow & & \updownarrow \\
2|4|1|3 & \overset{\pi_4}{\sim} & 2|4|3|1
\end{array}$$

$$\begin{array}{ccc}
[\mathfrak{f}_1^1][\theta_2^1][\theta_2^1\ \mathbf{1}][\mathbf{1}\ \mathbf{1}\ \theta_2^1] & \sim & [\mathfrak{f}_1^1][\theta_2^1][\mathbf{1}\ \theta_2^1][\theta_2^1\ \mathbf{1}\ \mathbf{1}] \\
\updownarrow & & \updownarrow \\
\frac{z|0|0|0}{0|2|1|3} & \sim & \frac{z|0|0|0}{0|2|3|1} \\
\updownarrow & & \updownarrow \\
4|2|1|3 & \overset{\pi_4}{\sim} & 4|2|3|1.
\end{array}$$

Note that the 2-cell 24|13 and the edge 2|13|4 of P_4 degenerate under π_4 above, while the 2-cell 24|13 and the edge 1|24|3 of P_4 degenerate under the map π_4 defined in [29].

Proposition 9.14. The map $\tilde{\partial}_{n,m}$ on the cellular chains $C_*(\mathcal{ZP}_{n,m})$ induces a map

$$\tilde{\partial}_{n,m} : \tilde{F}_{n,m}(\Theta, \mathfrak{F}, \Theta) \to \tilde{F}_{n,m}(\Theta, \mathfrak{F}, \Theta)$$

of degree -1 and a differential $\tilde{\partial} = \sum_{mn \geq 1} \tilde{\partial}_{n,m}$ on $\tilde{F}(\Theta, \mathfrak{F}, \Theta)$. Consequently, there is the identification of chain complexes

$$F(\Theta, \mathfrak{F}, \Theta) = C_*(\mathcal{ZJ}_{n,m}).$$

Example 9.15. The set $F_{2,2}(\Theta, \mathfrak{F}, \Theta)$ consists of the following coherent dimension 1 generators and their duals in Example 9.6 (cf. Figure 18):

$$[\mathfrak{f}_1^2][\theta_2^1]\,,\ \begin{bmatrix} \mathfrak{f}_1^1 \\ \mathfrak{f}_1^1 \end{bmatrix}[\theta_2^2]\,,\ \begin{bmatrix} [\mathfrak{f}_1^1][\theta_2^1] \\ \mathfrak{f}_2^1 \end{bmatrix}[\ \theta_1^2 \quad \theta_1^2\ ]\,,\ \begin{bmatrix} \mathfrak{f}_2^1 \\ [\theta_2^1][\ \mathfrak{f}_1^1 \quad \mathfrak{f}_1^1\ ] \end{bmatrix}[\ \theta_1^2 \quad \theta_1^2\ ]\,.$$

Definition 9.16. The $\mathcal{A}_\infty$- **bialgebra morphism matrad is the** DG $\mathcal{H}_\infty$-bimodule

$$r\mathcal{H}_\infty := \left(F\left(\Theta, \mathfrak{F}, \Theta\right), \partial\right).$$

Thus a map $A \Rightarrow B$ of A_∞-bialgebras is defined by a relative prematrad map $r\mathcal{H}_\infty \to U_{A,B}$ (cf. Definitions 9.3 and 11.1).

10 The Bimultiplihedron $JJ_{n,m}$

Let $m, n \geq 1$. First, we construct the polytopes $JP_{n,m}$ and a face-preserving bijection

$$\mathcal{ZP}_{n,m} \longrightarrow \{\text{cells of } \ JP_{n,m}\}, \tag{10.1}$$

which induces the structural bijection

$$\mathcal{ZJ}_{n+1,m+1} \longrightarrow \{\text{cells of } \ JJ_{n+1,m+1}\}$$

and the commutative diagram

$$\begin{array}{ccc} \mathcal{ZP}_{n,m} & \overset{\approx}{\longrightarrow} & \{\text{cells of } \ JP_{n,m}\} \\ \downarrow & & \downarrow \pi_{n,m} \\ \mathcal{ZJ}_{n+1,m+1} & \overset{\approx}{\longrightarrow} & \{\text{cells of } \ JJ_{n+1,m+1}\}. \end{array} \tag{10.2}$$

We refer to the polytopes $JJ = \{JJ_{n+1,m+1}\}$ as *bimultiplihedra*. As in the absolute case, apply the bijections $m_z *_c \mathfrak{n} \to P_{m+n+1}$ and $\mathfrak{m} *_c n_z \to P_{m+n+1}$ with $z = m+n+1$ and fix the bijection (10.1). Then $JP_{n,m}$ forms a subdivision of the cylinder $PP_{n,m} \times I$ in which codimension 1 cells of the form $C_1F_2 := \left(\frac{B_1|(B_2\cup z)}{A_1|A_2}, \dots\right)$ and $F_1C_2 := \left(\frac{B_1|B_2}{(A_1\cup z)|A_2}, \dots\right)$ correspond to the subdivision cells of the codimension 1 cells $A|(B \cup m+n+1)$ and $(A \cup m+n+1)|B$ for $A|B = A_1|A_2 \Cup B_1|B_2 \subset P_{m+n+1}$, respectively. Then the bimultiplihedron $JJ_{n+1,m+1}$ is a subdivision of the cylinder $KK_{n+1,m+1} \times I$; in particular, the octagon $JJ_{2,2}$ is a subdivision of a hexagon (see Figure 18) and $JJ_{n+1,1} = JJ_{1,n+1}$ can be identified with multiplihedra J_{n+1} (cf. Example 9.13).

Thus, there is an isomorphism of chain complexes

$$C_*(\mathcal{ZJ}_{n+1,m+1}) \overset{\approx}{\longrightarrow} C_*(JJ_{n+1,m+1}), \tag{10.3}$$

and $C_*(JJ)$ realizes the $\mathcal{A}_\infty$-bialgebra morphism matrad $r\mathcal{H}_\infty$. Furthermore, the standard isomorphisms (9.3) and (9.4) extend to isomorphisms

$$(r\mathcal{H}_\infty)_{n+1,m+1} \overset{\approx}{\longrightarrow} C_*(JJ_{n+1,m+1}), \tag{10.4}$$

and one recovers $\mathcal{J}_\infty$ by restricting the differential ∂ to $(r\mathcal{H}_\infty)_{1,*}$ or $(r\mathcal{H}_\infty)_{*,1}$.

Since $\rho \in \mathcal{ZJ}_{n,m}$ can be expressed as a matrix product $\rho = A_1 \cdots A_r \cdot F \cdot B_1 \cdots B_s$ in which only F contains z, the element ρ can be represented by a piecewise linear path in $\mathbb{N}^3$ with $r+s+1$ directed components from $(m+1, 1, 0) \in \mathbb{N}^2 \times 0$ to $(1, n+1, 1) \in \mathbb{N}^2 \times 1$, where $B_1 \cdots B_s$ is represented by a path in $\mathbb{N}^2 \times 0$ with directed components $B_s, B_{s-1}, \dots, B_1$, F is represented by a directed component $\vec{F}$ from $\mathbb{N}^2 \times 0$ to $\mathbb{N}^2 \times 1$, and $A_1 \cdots A_r$ is represented by a path in $\mathbb{N}^2 \times 1$ with directed components $A_r, A_{r-1}, \dots, A_1$ (see Figure 15).

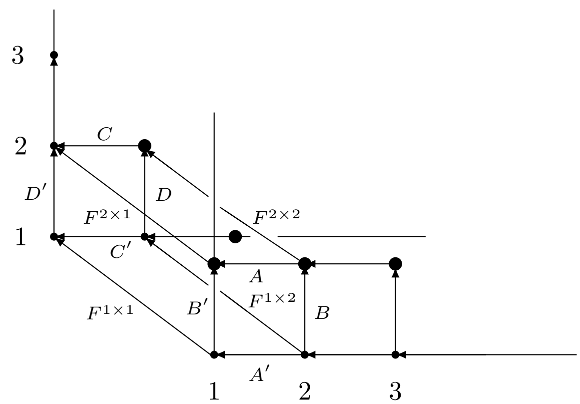


Figure 15. The paths in $\mathbb{N}^3$ with directed 3 components corresponding to common vertices of $JJ_{2,2}$ and P_3.

Here are some pictures of $JJ_{n,m}$ for small values of m and n :

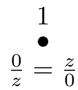


Figure 16. The point $JJ_{1,1}$.

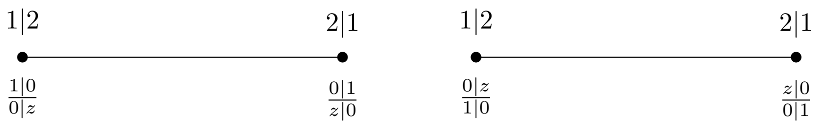


Figure 17. The intervals $JJ_{2,1}$ and $JJ_{1,2}$.

<u>For $JJ_{2,2}$:</u>

$$1|23 \leftrightarrow \frac{0|2z}{1|0} \leftrightarrow \begin{bmatrix}\theta_2^1\\ \theta_2^1\end{bmatrix}\left(\left[\,[\theta_1^2][\mathfrak{f}_1^1] \quad \mathfrak{f}_1^2\,\right] + \left[\,\mathfrak{f}_1^2 \quad \begin{bmatrix}\mathfrak{f}_1^1\\ \mathfrak{f}_1^1\end{bmatrix}[\theta_1^2]\,\right]\right)$$

$$13|2 \leftrightarrow \frac{0|2}{1z|0} \leftrightarrow \left(\begin{bmatrix}\mathfrak{f}_2^1\\ [\theta_2^1][\mathfrak{f}_1^1\ \mathfrak{f}_1^1]\end{bmatrix} + \begin{bmatrix}[\mathfrak{f}_1^1][\theta_2^1]\\ \mathfrak{f}_2^1\end{bmatrix}\right)\left[\,\theta_1^2 \quad \theta_1^2\,\right]$$

$$3|12 \leftrightarrow \frac{0|2}{z|1} \leftrightarrow \begin{bmatrix}\mathfrak{f}_1^1\\ \mathfrak{f}_1^1\end{bmatrix}[\theta_2^2]$$

$$12|3 \leftrightarrow \frac{2|z}{1|0} \leftrightarrow [\theta_2^2][\mathfrak{f}_1^1\ \mathfrak{f}_1^1]$$

$$2|13 \leftrightarrow \frac{2|z}{0|1} \leftrightarrow [\theta_1^2][\mathfrak{f}_2^1]$$

$$23|1 \leftrightarrow \frac{2|0}{z|1} \leftrightarrow [\mathfrak{f}_1^2][\theta_2^1]$$

$$1|2|3 \leftrightarrow \frac{0|2|z}{1|0|0} \leftrightarrow \begin{bmatrix}\theta_2^1\\ \theta_2^1\end{bmatrix}\left[\,\theta_1^2 \quad \theta_1^2\,\right]\left[\,\mathfrak{f}_1^1 \quad \mathfrak{f}_1^1\,\right]$$

$$1|3|2 \leftrightarrow \frac{0|z|2}{1|0|0} \leftrightarrow \begin{bmatrix}\theta_2^1\\ \theta_2^1\end{bmatrix}\begin{bmatrix}\mathfrak{f}_1^1 & \mathfrak{f}_1^1\\ \mathfrak{f}_1^1 & \mathfrak{f}_1^1\end{bmatrix}\left[\,\theta_1^2 \quad \theta_1^2\,\right]$$

$$3|1|2 \leftrightarrow \frac{0|0|2}{z|1|0} \leftrightarrow \begin{bmatrix}\mathfrak{f}_1^1\\ \mathfrak{f}_1^1\end{bmatrix}\begin{bmatrix}\theta_2^1\\ \theta_2^1\end{bmatrix}\left[\,\theta_1^2 \quad \theta_1^2\,\right]$$

$$2|1|3 \leftrightarrow \frac{2|0|z}{0|1|0} \leftrightarrow [\theta_1^2][\theta_2^1]\left[\,\mathfrak{f}_1^1 \quad \mathfrak{f}_1^1\,\right]$$

$$2|3|1 \leftrightarrow \frac{2|z|0}{0|0|1} \leftrightarrow [\theta_1^2][\mathfrak{f}_1^1][\theta_2^1]$$

$$3|2|1 \leftrightarrow \frac{z|2|0}{0|0|1} \leftrightarrow \begin{bmatrix}\mathfrak{f}_1^1\\ \mathfrak{f}_1^1\end{bmatrix}[\theta_1^2][\theta_2^1]$$

$$v_1 \quad \leftrightarrow \quad \left(\frac{0|2z}{1|0}, \begin{pmatrix}\frac{0}{1}\\ \frac{0}{1}\\ \frac{0}{1}\end{pmatrix}\left(\frac{2|z}{0|0}\ \frac{z|2}{0|0}\right)\right) \quad \leftrightarrow \quad \begin{bmatrix}\theta_2^1\\ \theta_2^1\end{bmatrix}\left[[\theta_1^2][\mathfrak{f}_1^1] \quad \begin{bmatrix}\mathfrak{f}_1^1\\ \mathfrak{f}_1^1\end{bmatrix}[\theta_1^2]\right]$$

$$v_2 \quad \leftrightarrow \quad \left(\frac{0|2}{1z|0}, \begin{pmatrix}\frac{0|0}{z|1}\\ \frac{0|0}{1|z}\end{pmatrix}\left(\frac{2}{0}\ \frac{2}{0}\ \frac{2}{0}\right)\right) \quad \leftrightarrow \quad \begin{bmatrix} [\mathfrak{f}_1^1][\theta_2^1] \\ [\theta_2^1][\ \mathfrak{f}_1^1 \quad \mathfrak{f}_1^1\ ]\end{bmatrix}[\,\theta_1^2 \quad \theta_1^2\,],$$

where v_1 and v_2 are the respective midpoints of the edges $1|23$ and $13|2$ in P_3.

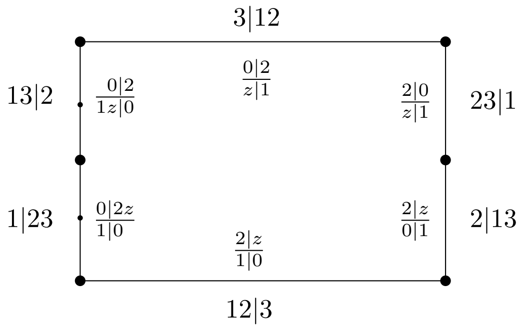


Figure 18. The octagon $JJ_{2,2}$ as a subdivision of P_3.

For $JJ_{2,3}$:

$123\|4$	$\leftrightarrow$	$\frac{3\|z}{12\|0}$	$\leftrightarrow$	$\theta_3^2/\mathfrak{f}_1^1\mathfrak{f}_1^1\mathfrak{f}_1^1$
$4\|123$	$\leftrightarrow$	$\frac{0\|3}{z\|12}$	$\leftrightarrow$	$\mathfrak{f}_1^1\mathfrak{f}_1^1/\theta_3^2$
$13\|24$	$\leftrightarrow$	$\frac{3\|z}{1\|2}$	$\leftrightarrow$	$\theta_2^2/\mathfrak{f}_1^1\mathfrak{f}_2^1$
$134\|2$	$\leftrightarrow$	$\frac{3\|0}{1z\|2}$	$\leftrightarrow$	$\mathfrak{f}_2^2/\theta_1^1\theta_2^1$
$3\|124$	$\leftrightarrow$	$\frac{3\|z}{0\|12}$	$\leftrightarrow$	$\theta_1^2/\mathfrak{f}_3^1$
$34\|12$	$\leftrightarrow$	$\frac{3\|0}{z\|12}$	$\leftrightarrow$	$\mathfrak{f}_1^2/\theta_3^1$
$1\|234$	$\leftrightarrow$	$\frac{0\|3z}{1\|2}$	$\leftrightarrow$	$\theta_2^1\theta_2^1/\left(\theta_1^2\mathfrak{f}_1^1\right)\mathfrak{f}_2^2$
$14\|23$	$\leftrightarrow$	$\frac{0\|3}{1z\|2}$	$\leftrightarrow$	$[(\mathfrak{f}_1^1/\theta_2^1)\mathfrak{f}_2^1+\mathfrak{f}_2^1\left(\theta_2^1/\mathfrak{f}_1^1\mathfrak{f}_1^1\right)]/\theta_1^2\theta_2^2$
$2\|134$	$\leftrightarrow$	$\frac{0\|3z}{2\|1}$	$\leftrightarrow$	$\theta_2^1\theta_2^1/\mathfrak{f}_2^2(\mathfrak{f}_1^1\mathfrak{f}_1^1/\theta_1^2)$
$24\|13$	$\leftrightarrow$	$\frac{0\|3}{2z\|1}$	$\leftrightarrow$	$[(\mathfrak{f}_1^1/\theta_2^1)\mathfrak{f}_2^1+\mathfrak{f}_2^1\left(\theta_2^1/\mathfrak{f}_1^1\mathfrak{f}_1^1\right)]/\theta_2^2\theta_1^2$
$23\|14$	$\leftrightarrow$	$\frac{3\|z}{2\|1}$	$\leftrightarrow$	$\theta_2^2/\mathfrak{f}_2^1\mathfrak{f}_1^1$
$234\|1$	$\leftrightarrow$	$\frac{3\|0}{2z\|1}$	$\leftrightarrow$	$\mathfrak{f}_2^2/\theta_2^1\theta_1^1$

$$\begin{array}{lcccl} 12|34 & \leftrightarrow & \frac{0|3z}{12|0} & \leftrightarrow & [\left(\theta_2^1/\theta_2^1\theta_1^1\right)\theta_3^1 + \theta_3^1\left(\theta_2^1/\theta_1^1\theta_2^1\right)]/ \\ &&&& [(\theta_1^2/\mathfrak{f}_1^1)(\theta_1^2/\mathfrak{f}_1^1)\mathfrak{f}_1^2 + (\theta_1^2/\mathfrak{f}_1^1)\mathfrak{f}_1^2(\mathfrak{f}_1^1\mathfrak{f}_1^1/\theta_1^2) \\ &&&& +\mathfrak{f}_1^2(\mathfrak{f}_1^1\mathfrak{f}_1^1/\theta_1^2)(\mathfrak{f}_1^1\mathfrak{f}_1^1/\theta_1^2)] \\ 124|3 & \leftrightarrow & \frac{0|3}{12z|0} & \leftrightarrow & [(\theta_2^1/\theta_1^1\theta_2^1/\mathfrak{f}_1^1\mathfrak{f}_1^1\mathfrak{f}_1^1)\mathfrak{f}_3^1 + \mathfrak{f}_3^1(\mathfrak{f}_1^1/\theta_2^1/\theta_2^1\theta_1^1) \\ &&&& +(\theta_3^1/\mathfrak{f}_1^1\mathfrak{f}_1^1\mathfrak{f}_1^1)(\theta_2^1/\mathfrak{f}_2^1\mathfrak{f}_1^1) + (\theta_3^1/\mathfrak{f}_1^1\mathfrak{f}_1^1\mathfrak{f}_1^1)(\mathfrak{f}_2^1/\theta_2^1\theta_1^1) \\ &&&& +(\theta_2^1/\mathfrak{f}_2^1\mathfrak{f}_1^1)(\mathfrak{f}_2^1/\theta_2^1\theta_1^1) - (\theta_2^1/\mathfrak{f}_1^1\mathfrak{f}_2^1)(\mathfrak{f}_2^1/\theta_1^1\theta_2^1) \\ &&&& -(\theta_2^1/\mathfrak{f}_1^1\mathfrak{f}_2^1)(\mathfrak{f}_1^1/\theta_3^1) - (\mathfrak{f}_2^1/\theta_1^1\theta_2^1)(\mathfrak{f}_1^1/\theta_3^1)]/\,\theta_1^2\theta_1^2\theta_1^2 \end{array}$$

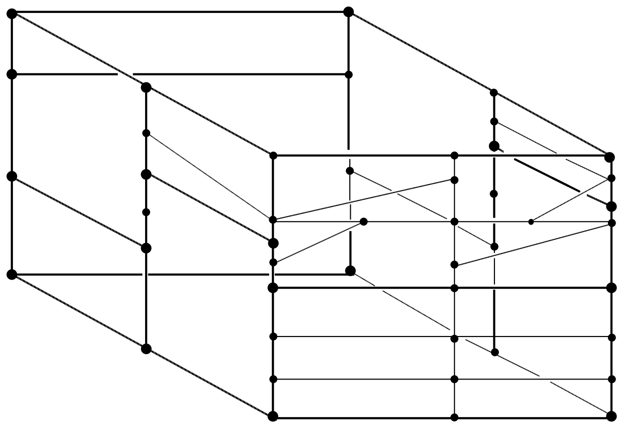


Figure 19. The bimultiplihedra $JJ_{2,3}$ as a subdivision of P_4.

11 Morphisms and the Transfer of A_∞-Structure

In this section we define the morphisms of A_∞-bialgebras. But before we begin, we mention three settings in which A_∞-bialgebras naturally appear:

(1) Let X be a space. The bar construction $\Omega S_*(X)$ on the simplicial singular chain complex of X is a DG bialgebra with coassociative coproduct [1], [5], [18], but whether or not $\Omega^2 S_*(X)$ admits a coassociative coproduct is unknown. However, there is an A_∞-coalgebra structure on $\Omega^2 S_*(X)$, which is compatible with the product, and $\Omega^2 S_*(X)$ is an A_∞-bialgebra.

(2) Let R be a commutative ring with unity, let H be a graded R-bialgebra with nontrivial product and coproduct, and let $\rho : RH \longrightarrow H$ be a (bigraded) free resolution as an R-algebra. Since RH cannot be simultaneously free and cofree, it is difficult to introduce a coassociative coproduct on RH so that ρ is a bialgebra map. However, there is always an A_∞-bialgebra structure on RH such that ρ is a morphism of A_∞-bialgebras.

(3) If A is an A_∞-bialgebra over a field, and $g : H(A) \to A$ is a cycle-selecting homomorphism, there is an A_∞-bialgebra structure on $H(A)$, which is unique up to isomorphism, and a morphism

$G : H(A) \Rightarrow A$ of A_∞-bialgebras extending g (see Theorem 11.5).

Definition 11.1. Let (A, ω_A) and (B, ω_B) be A_∞-bialgebras. An element

$$G = \{g_m^n \in Hom^{m+n-2}(A^{\otimes m}, B^{\otimes n})\}_{m,n \geq 1} \in U_{A,B}$$

is an **A_∞-bialgebra morphism from A to B** if the map $\mathfrak{f}_m^n \mapsto g_m^n$ extends to a map $r\mathcal{H}_\infty \to U_{A,B}$ of relative prematrads. When G is an A_∞-bialgebra morphism from A **to** B we write $G : A \Rightarrow B$. An A_∞-bialgebra morphism $\Phi = \{\varphi_m^n\}_{m,n\geq 1} : A \Rightarrow B$ is an **isomorphism** if $\varphi_1^1 : A \to B$ is an isomorphism of underlying modules.

If A is a free DGM, B is an A_∞-coalgebra, and $g : A \to B$ is a homology isomorphism (weak equivalence) with a right-homotopy inverse, the Coalgebra Perturbation Lemma (CPL) transfers the A_∞-coalgebra structure from B to A (see [15], [21]). When B is an A_∞-bialgebra, Theorem 11.3 generalizes the CPL in two directions:

1. The A_∞-bialgebra structure transfers from B to A.
2. Neither freeness nor a existence of a right-homotopy inverse is required.

Note that (2) formulates the transfer of A_∞-algebra structure in maximal generality (see Remark 11.7).

Proposition 11.2. Let A and B be DGMs. If $g : A \to B$ is a chain map and $u \in Hom\,(A^{\otimes m}, A^{\otimes n})$, the induced map $\tilde{g} : U_A \to U_{A,B}$ defined by $\tilde{g}\,(u) = g^{\otimes n}u$ is a cochain map. If in addition, g is a homology isomorphism, $\tilde{g}$ is a cohomology isomorphism if either

(i) A is free as an R-module or

(ii) for each $n \geq 1$, there is a DGM $X\,(n)$ and a splitting $B^{\otimes n} = A^{\otimes n} \oplus X(n)$ as a chain complex such that $H^* Hom\left(A^{\otimes k}, X\,(n)\right) = 0$ for all $k \geq 1$.

The proof is left to the reader.

Theorem 11.3. Let (A, d_A) be a DGM, let (B, d_B, ω_B) be an A_∞-bialgebra, and let $g : A \to B$ be a chain map/homology isomorphism. If $\tilde{g}$ is a cohomology isomorphism, then

(i) (Existence) g induces an A_∞-bialgebra structure $\omega_A = \{\omega_A^{n,m}\}$ on A, and extends to a map $G = \{g_m^n \mid g_1^1 = g\} : A \Rightarrow B$ of A_∞-bialgebras.

(ii) (Uniqueness) (ω_A, G) is unique up to isomorphism, i.e., if (ω_A, G) and $\left(\bar{\omega}_A, \bar{G}\right)$ are induced by chain homotopic maps g and $\bar{g}$, there is an isomorphism of A_∞-bialgebras $\Phi : (A, \bar{\omega}_A) \Rightarrow (A, \omega_A)$ and a chain homotopy $T : \bar{G} \simeq G \circ \Phi$.

Proof. **(The Transfer Algorithm)** We obtain the desired structures by simultaneously constructing a map of matrads $\alpha_A : C_*\,(KK) \to (U_A, \nabla)$ and a map of relative matrads $\beta : C_*\,(JJ) \to (U_{A,B}, \nabla)$. Thinking of $JJ_{n,m}$ as a subdivision of the cylinder $KK_{n,m} \times I$, identify the top dimensional cells of $KK_{n,m}$ and $JJ_{n,m}$ with θ_m^n and $\mathfrak{f}_m^n$, and the faces $KK_{n,m} \times 0$ and $KK_{n,m} \times 1$

of $JJ_{n,m}$ with $\theta_m^n\left(\mathfrak{f}_1^1\right)^{\otimes m}$ and $\left(\mathfrak{f}_1^1\right)^{\otimes n}\theta_m^n$, respectively. By hypothesis, there is a map of matrads $\alpha_B : C_*(KK) \to (U_B, \nabla)$ such that $\alpha_B(\theta_m^n) = \omega_B^{n,m}$.

To initialize the induction, define $\beta : C_*\left(JJ_{1,1}\right) \to Hom^0\left(A,B\right)$ by $\beta\left(\mathfrak{f}_1^1\right) = g_1^1 = g$, and extend β to $C_*\left(JJ_{1,2}\right) \to Hom^1\left(A^{\otimes 2}, B\right)$ and $C_*\left(JJ_{2,1}\right) \to Hom^1\left(A, B^{\otimes 2}\right)$ in the following way: On the vertices $\theta_2^1\left(\mathfrak{f}_1^1 \otimes \mathfrak{f}_1^1\right) \in JJ_{1,2}$ and $\theta_1^2\mathfrak{f}_1^1 \in JJ_{2,1}$, define $\beta\left(\theta_2^1\left(\mathfrak{f}_1^1 \otimes \mathfrak{f}_1^1\right)\right) = \omega_B^{1,2}\left(g \otimes g\right)$ and $\beta\left(\theta_1^2\mathfrak{f}_1^1\right) = \omega_B^{2,1}g$. Since $\omega_B^{1,2}\left(g \otimes g\right)$ and $\omega_B^{2,1}g$ are ∇-cocycles, and $\tilde{g}_*$ is an isomorphism, there exist cocycles $\omega_A^{1,2}$ and $\omega_A^{2,1}$ in U_A such that

$$\tilde{g}_*[\omega_A^{1,2}] = [\omega_B^{1,2}\left(g \otimes g\right)] \text{ and } \tilde{g}_*[\omega_A^{2,1}] = [\omega_B^{2,1}g].$$

Thus $\left[\omega_B^{1,2}\left(g \otimes g\right) - g\omega_A^{1,2}\right] = \left[\omega_B^{2,1}g - \left(g \otimes g\right)\omega_A^{2,1}\right] = 0$, and there exist cochains g_2^1 and g_1^2 in $U_{A,B}$ such that

$$\nabla g_2^1 = \omega_B^{1,2}\left(g \otimes g\right) - g\omega_A^{1,2} \text{ and } \nabla g_1^2 = \omega_B^{2,1}g - \left(g \otimes g\right)\omega_A^{2,1}.$$

For $m = 1, 2$ and $n = 3 - m$, define $\alpha_A : C_*\left(KK_{n,m}\right) \to Hom\left(A^{\otimes m}, A^{\otimes n}\right)$ by $\alpha_A(\theta_m^n) = \omega_A^{n,m}$, and define $\beta : C_*\left(JJ_{n,m}\right) \to Hom\left(A^{\otimes m}, B^{\otimes n}\right)$ by

$$\begin{aligned} \beta(\mathfrak{f}_m^n) &= g_m^n \\ \beta(\mathfrak{f}_1^1\,\theta_2^1) &= g\,\omega_A^{1,2} && (m=2) \\ \beta((\mathfrak{f}_1^1 \otimes \mathfrak{f}_1^1)\,\theta_1^2) &= (g \otimes g)\,\omega_A^{2,1} && (m=1). \end{aligned}$$

Inductively, given (m,n), $m + n \geq 4$, assume that for $i + j < m + n$ there exists a map of matrads $\alpha_A : C_*\left(KK_{j,i}\right) \to Hom\left(A^{\otimes i}, A^{\otimes j}\right)$ and a map of relative matrads $\beta : C_*\left(JJ_{j,i}\right) \to Hom\left(A^{\otimes i}, B^{\otimes j}\right)$ such that $\alpha_A(\theta_i^j) = \omega_A^{j,i}$ and $\beta(\mathfrak{f}_i^j) = g_i^j$. Thus we are given chain maps $\alpha_A : C_*\left(\partial KK_{n,m}\right) \to Hom\left(A^{\otimes m}, A^{\otimes n}\right)$ and $\beta : C_*\left(\partial JJ_{n,m} \smallsetminus \operatorname{int} KK_{n,m} \times 1\right) \to Hom\left(A^{\otimes m}, B^{\otimes n}\right)$; we wish to extend α_A to the top cell θ_m^n of $KK_{n,m}$, and β to the codimension 1 cell $\left(\mathfrak{f}_1^1\right)^{\otimes n}\theta_m^n$ and the top cell $\mathfrak{f}_m^n$ of $JJ_{n,m}$. Since α_A is a map of matrads, the components of the cocycle

$$z = \alpha_A\left(C_*(\partial KK_{n,m})\right) \in Hom^{m+n-4}\left(A^{\otimes m}, A^{\otimes n}\right)$$

are expressed in terms of $\omega_A^{j,i}$ with $i + j < m + n$; similarly, since β is a map of relative matrads, the components of the cochain

$$\varphi = \beta\left(C_*(\partial JJ_{n,m} \smallsetminus \operatorname{int} KK_{n,m} \times 1)\right) \in Hom^{m+n-3}\left(A^{\otimes m}, B^{\otimes n}\right)$$

are expressed in terms of ω_B, $\omega_A^{j,i}$ and g_i^j with $i + j < m + n$. Clearly $\tilde{g}\left(z\right) = \nabla\varphi$; and $[z] = [0]$ since $\tilde{g}$ is a homology isomorphism. Now choose a cochain $b \in Hom^{m+n-3}\left(A^{\otimes m}, A^{\otimes n}\right)$ such that $\nabla b = z$. Then

$$\nabla\left(\tilde{g}\left(b\right) - \varphi\right) = \nabla\tilde{g}\left(b\right) - \tilde{g}\left(z\right) = 0.$$

Choose a class representative $u \in \tilde{g}_*^{-1}\left[\tilde{g}\left(b\right) - \varphi\right]$, set $\omega_A^{n,m} = b - u$, and define $\alpha_A\left(\theta_m^n\right) = \omega_A^{n,m}$. Then $\left[\tilde{g}\left(\omega_A^{n,m}\right) - \varphi\right] = \left[\tilde{g}\left(b - u\right) - \varphi\right] = \left[\tilde{g}\left(b\right) - \varphi\right] - \left[\tilde{g}\left(u\right)\right] = [0]$. Choose a cochain $g_m^n \in Hom^{m+n-2}$ $\left(A^{\otimes m}, B^{\otimes n}\right)$ such that

$$\nabla g_m^n = g^{\otimes n}\omega_A^{n,m} - \varphi,$$

and define $\beta\left(\mathfrak{f}_m^n\right) = g_m^n$. To extend β as a map of relative matrads, define $\beta(\left(\mathfrak{f}_1^1\right)^{\otimes n}\theta_m^n) = g^{\otimes n}\omega_A^{n,m}$. Passing to the limit we obtain the desired maps α_A and β.

Furthermore, if chain maps $\bar{\alpha}_A$ and $\bar{\beta}$ are defined in terms of different choices, beginning with a chain map $\bar{g}$ chain homotopic to g, let $\bar{\omega}_A = \operatorname{Im}\bar{\alpha}_A$ and $\bar{G} = \operatorname{Im}\bar{\beta}$. There is an inductively defined isomorphism $\Phi = \sum \varphi_m^n : (A, \bar{\omega}_A) \Rightarrow (A, \omega_A)$ with $\varphi_1^1 = \mathbf{1}$, and a chain homotopy $T : \tilde{G} \simeq G \circ \Phi$. To initialize the induction, set $\varphi_1^1 = \mathbf{1}$, and note that

$$\nabla g_2^1 = g\omega_A^{1,2} - \omega_B^{1,2}(g \otimes g) \text{ and } \nabla \bar{g}_2^1 = \bar{g}\bar{\omega}_A^{1,2} - \omega_B^{1,2}(\bar{g} \otimes \bar{g}).$$

Let $s : \bar{g} \simeq g$; then $c_2^1 = \omega_B^{1,2}\left(s \otimes g + \bar{g} \otimes s\right)$ satisfies

$$\nabla c_2^1 = \omega_B^{1,2}(g \otimes g) - \omega_B^{1,2}(\bar{g} \otimes \bar{g}).$$

Hence

$$\nabla(g_2^1 - \bar{g}_2^1 + c_2^1) = g\omega_A^{1,2} - \bar{g}\bar{\omega}_A^{1,2}$$

and

$$\bar{g}(\omega_A^{1,2} - \bar{\omega}_A^{1,2}) = \nabla(g_2^1 - \bar{g}_2^1 + c_2^1 - s\,\omega_A^{1,2}).$$

Consequently, there is $\varphi_2^1 : A^{\otimes 2} \to A$ such that $\nabla\varphi_2^1 = \omega_A^{1,2} - \bar{\omega}_A^{1,2}$; and, as above, φ_2^1 may be chosen so that $\bar{g}\varphi_2^1 - (g_2^1 - \bar{g}_2^1 + c_2^1 - s\,\omega_A^{1,2})$ is cohomologous to zero. Thus there is a component t_2^1 of T such that

$$\nabla(t_2^1) = \bar{g}\varphi_2^1 - (g_2^1 - g_2^1 + c_2^1 + s\,\omega_A^{1,2}).$$

Q.E.D.

Since $\tilde{g}$ is a homology isomorphism whenever A is free (cf. Proposition 11.2) we have:

Corollary 11.4. Let (A, d_A) be a free DGM, let (B, d_B, ω_B) be an A_∞-bialgebra, and let $g : A \to B$ be a chain map/homology isomorphism. Then

(i) (Existence) g induces an A_∞-bialgebra structure ω_A on A, and extends to a map $G : A \Rightarrow B$ of A_∞-bialgebras.

(ii) (Uniqueness) (ω_A, G) is unique up to isomorphism.

Given a chain complex B of (not necessarily free) R-modules, there is always a chain complex of free R-modules (A, d_A), and a homology isomorphism $g : A \to B$. To see this, let $(RH : \cdots \to R_1H \to R_0H \overset{\rho}{\to} H, d)$ be a free R-module resolution of $H = H_*\left(B\right)$. Since R_0H is projective, there is a cycle-selecting homomorphism $g_0' : R_0H \to Z\left(B\right)$ lifting ρ through the projection $Z\left(B\right) \to H$, and extending to a chain map $g_0 : (RH, 0) \to \left(B, d_B\right)$. If $RH : 0 \to R_1H \to R_0H \to H$ is a short R-module resolution of H, then g_0 extends to a homology isomorphism $g : (RH, d + h) \to (B, d_B)$ with $(A, d_A) = (RH, d)$. Otherwise, there is a perturbation h of d such that $g : (RH, d + h) \to (B, d_B)$ is a homology isomorphism with $(A, d_A) = (RH, d + h)$ (see [3], [27]). Thus an A_∞-structure on B always transfers to an A_∞-structure on $(RH, d + h)$ via Corollary 11.4, and we obtain our main result concerning the transfer of A_∞-structure to homology:

Theorem 11.5. *Let B be an A_∞-bialgebra with homology $H = H_*\left(B\right)$, let (RH, d) be a free R-module resolution of H, and let h be a perturbation of d such that $g : (RH, d + h) \to (B, d_B)$ is a homology isomorphism. Then* *(i)* (Existence) g induces an A_∞-bialgebra structure ω_{RH} on RH, and extends to a map $G : RH \Rightarrow B$ of A_∞-bialgebras.

(ii) (Uniqueness) (ω_{RH}, G) is unique up to isomorphism.

Remark 11.6. Note that A_∞-bialgebra structures induced by the Transfer Algorithm are isomorphic for all choices of the map $g : (RH, d + h) \to (B, d_B)$. Thus we obtain an isomorphism class of A_∞-bialgebra structures on RH.

Remark 11.7. When $H = H_*(B)$ is a free module, we recover the classical results of Kadeishvili [17], Markl [21], and others, which transfer a DG (co)algebra structure to an A_∞-(co)algebra structure on homology, by setting $RH = H$. Furthermore, any pair of A_∞-(co)algebra structures $\{\omega_H^{n,1}\}_{n\geq 1}$ and $\{\omega_H^{1,m}\}_{m\geq 1}$ on H induced by the same cycle-selecting map $g : H \to B$ extend to an A_∞-bialgebra structure $(H, \omega_H^{n,m})$, by the proof of Theorem 11.5. For an example of a DGA B whose cohomology $H(B)$ is not free, and whose DGA structure transfers to an A_∞-algebra structure on $H(B)$ via Theorem 11.5 along a map $g : H(B) \to B$ having no right-homotopy inverse, see [37].

12 Applications and Examples

The applications and examples in this section apply the Transfer Algorithm given by the proof of Theorem 11.3. Three kinds of specialized A_∞-bialgebras $(A, \{\omega_m^n\})$ are relevant here:

1. $\omega_m^1 = 0$ for $m \geq 3$ (the A_∞-algebra substructure is trivial).
2. $\omega_m^n = 0$ for $m, n \geq 2$ (all higher order structure is concentrated in the A_∞-algebra and A_∞-coalgebra substructures).
3. Conditions (1) and (2) hold simultaneously.

Of these, A_∞-bialgebras of the first and third kind appear in the applications.

Structure relations defining A_∞-bialgebras of the second and third kind are expressed in terms of the S-U diagonal Δ_K on associahedra (see Subsection 2.4) and have especially nice form. Structure relations of the second kind were derived in [36]. Structure relations in an A_∞-bialgebra (A, ω) of the third kind with $\omega_1^1 = 0$, $\mu = \omega_2^1$ and $\psi^n = \omega_1^n$ are a special case of those derived in [36], and are given by the formula

$$\left\{\psi^n \mu = \mu^{\otimes n} \Psi^n\right\}_{n\geq 2}, \tag{12.1}$$

where the n-ary A_∞-coalgebra operation

$$\Psi^n = (\sigma_{n,2})_* \, \iota \, (\xi \otimes \xi) \, \Delta_K \left(e^{n-2}\right) : A \otimes A \to (A \otimes A)^{\otimes n}$$

is defined in terms of

- a map $\xi : C_*(K) \to Hom(A, TA)$ of operads that sends $e^{n-2} \subseteq K_n$ to ψ^n,
- the canonical isomorphism $\iota : Hom\,(A, A^{\otimes n})^{\otimes 2} \to Hom\left(A^{\otimes 2}, (A^{\otimes n})^{\otimes 2}\right)$,
- and the induced isomorphism $(\sigma_{n,2})_* : Hom\left(A^{\otimes 2}, (A^{\otimes n})^{\otimes 2}\right) \to Hom\left(A^{\otimes 2}, (A^{\otimes 2})^{\otimes n}\right)$.

Structure relations defining a morphism $G = \{g^n\} : (A, \omega_A) \Rightarrow (B, \omega_B)$ between A_∞-bialgebras of the third kind are expressed in terms of the S-U diagonal Δ_J on multiplihedra [29] by the formula

$$\left\{g^n \mu_A = \mu_B^{\otimes n} \mathbf{g}^n\right\}_{n\geq 1}, \tag{12.2}$$

where

$$\mathbf{g}^n = (\sigma_{n,2})_* \iota(\upsilon \otimes \upsilon)\Delta_J(e^{n-1}) : A \otimes A \to (B \otimes B)^{\otimes n},$$

and $\upsilon : C_*(J) \to Hom(A, TB)$ is a map of relative prematrads sending the top dimensional cell $e^{n-1} \subseteq J_n$ to g^n (the maps $\{\mathbf{g}^n\}$ define the tensor product morphism $G \otimes G : (A \otimes A, \Psi_{A\otimes A}) \Rightarrow (B \otimes B, \Psi_{B\otimes B})$).

Given a simply connected topological space X, consider the Moore loop space ΩX and the simplicial singular cochain complex $S^*(\Omega X; R)$. Under the hypotheses of the Transfer Algorithm, the DG bialgebra structure of $S^*(\Omega X; R)$ transfers to an A_∞-bialgebra structure on $H^*(\Omega X; R)$. Our next two theorems apply this principle, and identify some important A_∞-bialgebras of the third kind on loop space (co)homology.

Theorem 12.1. If X is simply connected, $H^*(\Omega X; \mathbb{Q})$ admits an induced A_∞-bialgebra structure of the third kind.

Proof. Let $\mathcal{A}_X$ be a free DG commutative algebra cochain model for X over $\mathbb{Q}$ (e.g., Sullivan's minimal or Halperin-Stasheff's filtered model); then $H^*(\mathcal{A}_X) \approx H^*(X; \mathbb{Q})$. The bar construction $(B = B\mathcal{A}_X, d_B, \Delta_B)$ with shuffle product is a cofree DG commutative Hopf algebra cochain model for ΩX, and $H = H^*(B, d_B)$ is a Hopf algebra with induced coproduct $\psi^2 = \omega_1^2$ and free graded commutative product $\mu = \omega_2^1$ (by a theorem of Hopf). Since H is a free commutative algebra, there is a multiplicative cocycle-selecting map $g_1^1 : H \to B$. Consequently, we may set $\omega_n^1 = 0$ for all $n \geq 3$ and $g_n^1 = 0$ for all $n \geq 2$, and obtain a trivial A_∞-algebra structure (H, μ) induced by g_1^1. There is an induced A_∞-coalgebra structure $(H, \psi^n)_{n\geq 2}$, and an A_∞-coalgebra map $G = \left\{g^n \mid g^1 = g_1^1\right\}_{n\geq 1} : H \Rightarrow B$ constructed as follows: For $n \geq 2$, assume ψ^n and g^{n-1} have been constructed, and apply the Transfer Algorithm to obtain candidates ω_1^{n+1} and g_1^n. Restrict ω_1^{n+1} to generators, and let ψ^{n+1} be the free extension of ω_1^{n+1} to all of H using Formula 12.1. Similarly, restrict g_1^n to generators, and let g^n be the free extension of g_1^n to all of H using Formula 12.2.

To complete the proof, we show that all other A_∞-bialgebra operations may be trivially chosen. Refer to the Transfer Algorithm, and note that the Hopf relation $\psi^2\mu = (\mu \otimes \mu)\, \sigma_{2,2} \left(\psi^2 \otimes \psi^2\right)$ implies $\beta(\partial \mathfrak{f}_2^2)|_{C_1(JJ_{2,2}\setminus \mathrm{int}(KK_{2,2}\times 1))} = 0$. Thus we may choose $\omega_2^2 = g_2^2 = 0$ so that $\beta(\partial \mathfrak{f}_2^2) = \beta(\mathfrak{f}_2^2) = 0$. Inductively, assume that $\omega_2^{n-1} = g_2^{n-1} = 0$ for $n \geq 3$. Then $\beta(\partial \mathfrak{f}_2^n)|_{C_{n-1}(JJ_{n,2}\setminus \mathrm{int}(KK_{n,2}\times 1))} = 0$, and we may choose $\omega_2^n = 0$ and $g_2^n = 0$ so that $\beta(\partial \mathfrak{f}_2^n) = \beta(\mathfrak{f}_2^n) = 0$. Finally, for $m \geq 3$ set $\omega_m^n = 0$ and $g_m^n = 0$. Then $(H, \mu, \psi^n)_{n\geq 2}$ as an A_∞-bialgebra of the third kind with structure relations given by Formula 12.1, and G is a map of A_∞-bialgebras satisfying Formula 12.2. Q.E.D.

Note that the components of the A_∞-bialgebra map G in the proof of Theorem 12.1 are exactly the components of a map of underlying A_∞-coalgebras given by the Transfer Algorithm.

Let R be a PID and let X be a connected space such that $H_*(X; R)$ is torsion free. Then the Bott-Samelson Theorem [4] asserts that $H_*(\Omega\Sigma X; R)$ is isomorphic as an algebra to the tensor algebra $T^a\tilde{H}_*(X; R)$ generated by the reduced homology of X, and the adjoint $i : X \to \Omega\Sigma X$ of the identity $\mathbf{1} : \Sigma X \to \Sigma X$ induces the canonical inclusion $i_* : \tilde{H}_*(X; R) \hookrightarrow T^a\tilde{H}_*(X; R) \approx$

$H_*(\Omega\Sigma X;R)$. Thus if $\{\psi^n\}_{n\geq 2}$ is an A_∞-coalgebra structure on $H_*(X;R)$, the tensor algebra $T^a\tilde{H}_*(X;R)$ admits a canonical A_∞-bialgebra structure of the third kind with respect to the free extension of each ψ^n via Formula 12.1.

Furthermore, the canonical inclusion $t : X \hookrightarrow \Omega\Sigma X$ induces a DG coalgebra map of simplicial singular chains $t_\# : S_*(X;R) \to S_*(\Omega\Sigma X;R)$, which extends to a homology isomorphism $t_\# : T^a\tilde{S}_*(X;R) \approx S_*(\Omega\Sigma X;R)$ of DG Hopf algebras. Thus the induced *Bott-Samelson Isomorphism* $t_* : T^a\tilde{H}_*(X;R) \approx H_*(\Omega\Sigma X;R)$ is an isomorphism of Hopf algebras ([16], [18]), and $T^a\tilde{S}_*(X;R)$ is a free DG Hopf algebra chain model for $\Omega\Sigma X$.

Theorem 12.2. Let R be a PID, and let X be a connected space such that $H_*(X;R)$ is torsion free.

(i) Then $T^a\tilde{H}_*(X;R)$ admits an A_∞ -bialgebra structure of the third kind, which is trivial if and only if the A_∞-coalgebra structure of $H_*(X;R)$ is trivial.

(ii) The Bott-Samelson Isomorphism $t_* : T^a\tilde{H}_*(X;R) \approx H_*(\Omega\Sigma X;R)$ extends to an isomorphism of A_∞-bialgebras.

Proof. Since $H_*(X;R)$ is free as an R-module, we may choose a cycle-selecting map $\bar{g} = \bar{g}_1^1 : H_*(X,R) \to S_*(X;R)$ and apply the Transfer Algorithm to obtain an induced A_∞-coalgebra structure $\bar{\omega} = \{\bar{\omega}_1^n\}_{n\geq 2}$ on $H_*(X,R)$ and a corresponding map of A_∞-coalgebras $\bar{G} = \{\bar{g}_1^n\}_{n\geq 1} : H_*(X;R) \Rightarrow S_*(X;R)$. Let $H = T^a\tilde{H}_*(X;R)$, let $B = T^a\tilde{S}_*(X;R)$, and consider the free (multiplicative) extension $g = T(\bar{g}) : H \to B$. As in the proof of Theorem 12.1, use formulas 12.1 and 12.2 to freely extend $\bar{\omega}$ and $\bar{G}$ to families $\omega = \{\omega_1^n\}$ and $G = \{g_1^n \mid g_1^1 = g\}_{n\geq 1}$ defined on H, and choose all other A_∞-bialgebra operations to be zero. Then $\bar{\omega}$ lifts to an A_∞-bialgebra structure (H,ω,μ) of the third kind with free product μ, and $\bar{G}$ lifts to a map $G : H \Rightarrow B$ of A_∞-bialgebras. Furthermore, restricting ω to the multiplicative generators $H_*(X;R)$ recovers the A_∞-coalgebra operations on $H_*(X;R)$. Thus A_∞ -bialgebra structure of H is trivial if and only if the A_∞-coalgebra structure of $H_*(X;R)$ is trivial. Finally, since B is a free DG Hopf algebra chain model for $\Omega\Sigma X$, the Bott-Samelson Isomorphism t_* extends to an isomorphism of A_∞-bialgebras, and identifies the A_∞-bialgebra structure of $H_*(\Omega\Sigma X;R)$ with (H,ω,μ). Q.E.D.

It is important to note that prior to this work, all known *rational* homology invariants of $\Omega\Sigma X$ are trivial for any space X. However, we now have the following:

Corollary 12.3. A nontrivial A_∞-coalgebra structure on $H_*(X;\mathbb{Q})$ induces a nontrivial A_∞-bialgebra structure on $H_*(\Omega\Sigma X;\mathbb{Q})$. Thus the A_∞-bialgebra structure of $H_*(\Omega\Sigma X;\mathbb{Q})$ is a nontrivial rational homology invariant.

Proof. First, $H = H_*(\Omega\Sigma X;\mathbb{Q})$ admits an induced A_∞-bialgebra structure of the third kind by Theorem 12.2, which is trivial if and only if the A_∞-coalgebra structure of $H_*(X;\mathbb{Q})$ is trivial. Second, the dual version of Theorem 12.1 imposes an induced A_∞-bialgebra structure on H whose A_∞-coalgebra substructure is trivial, and whose A_∞-algebra substructure is trivial if and only if the A_∞-coalgebra structure of $H_*(X;\mathbb{Q})$ is trivial. Q.E.D.

The two A_∞-bialgebras identified in the proof of Corollary 12.3–one with trivial A_∞-coalgebra substructure and the other with trivial A_∞-algebra substructure–are in fact isomorphic, and represent the same isomorphism class of A_∞-bialgebra structures on $H_*(\Omega\Sigma X;\mathbb{Q})$ (cf. Remark 11.6).

Indeed, choose a pair of isomorphisms for the two underlying A_∞-(co)algebra substructures—the first as an isomorphism of A_∞-algebras and the second as an isomorphism of A_∞-coalgebras (their component in bidegree $(1,1)$ is the identity $\mathbf{1}: H \to H$). Since $\omega_i^j = 0$ for $i, j \geq 2$, these isomorphisms clearly determine an isomorphism of A_∞-bialgebras.

Our next example exhibits an A_∞-bialgebra of the first but not the second kind. Given a 1-connected DGA (A, d_A), the bar construction of A, denoted by BA, is the cofree DGC $T^c\left(\downarrow \overline{A}\right)$ whose differential d and coproduct Δ are defined as follows: Let $\lfloor x_1|\cdots|x_n\rfloor$ denote the element $\downarrow x_1 \mid \cdots \mid\downarrow x_n \in BA$, and let e denote the unit $\lfloor\ \rfloor$. Then

$$d\lfloor x_1|\cdots|x_n\rfloor = \sum_{i=1}^{n} \pm \lfloor x_1|\cdots|d_A x_i|\cdots|x_n\rfloor + \sum_{i=1}^{n-1} \pm \lfloor x_1|\cdots|x_i x_{i+1}|\cdots|x_n\rfloor;$$

$$\Delta\lfloor x_1|\cdots|x_n\rfloor = e \otimes \lfloor x_1|\cdots|x_n\rfloor + \lfloor x_1|\cdots|x_n\rfloor \otimes e + \sum_{i=1}^{n-1} \lfloor x_1|\cdots|x_i\rfloor \otimes \lfloor x_i|\cdots|x_n\rfloor.$$

Given an A_∞-coalgebra $(C, \Delta^n)_{n\geq 1}$, the tilde-cobar construction of C, denoted by $\tilde{\Omega}C$, is the free DGA $T^a\left(\uparrow \overline{C}\right)$ with differential $d_{\tilde{\Omega}A}$ given by extending $\sum_{i\geq 1}\Delta^n$ as a derivation. Let $\lceil x_1|\cdots|x_n\rceil$ denote $\uparrow x_1 \mid \cdots \mid\uparrow x_n \in \tilde{\Omega}H$.

Example 12.4. Consider the DGA $A = \mathbb{Z}_2[a,b]/\left(a^4, ab\right)$ with $|a| = 3$, $|b| = 5$ and trivial differential. Define a homotopy Gerstenhaber algebra (HGA) structure $\{E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A\}_{p,q\geq 0;\ p+q>0}$ with $E_{p,q}$ acting trivially except $E_{1,0} = E_{0,1} = \mathbf{1}$ and $E_{1,1}(b;b) = a^3$ (cf. [10], [18]). Form the tensor coalgebra $BA \otimes BA$ with coproduct $\psi = \sigma_{2,2}(\Delta \otimes \Delta)$, and consider the induced map

$$\varphi = E'_{1,0} + E'_{0,1} + E'_{1,1} : BA \otimes BA \to A$$

of degree $+1$, which acts trivially except for $E'_{1,0}(\lfloor x\rfloor \otimes e) = E'_{0,1}(e \otimes \lfloor x\rfloor) = x$ for all $x \in A$, and $E'_{1,1}(\lfloor b\rfloor \otimes \lfloor b\rfloor) = a^3$. Since $E_{p,q}$ is an HGA structure, φ is a twisting cochain, which lifts to a chain map of DG coalgebras $\mu : BA \otimes BA \to BA$ defined by

$$\mu = \sum_{k=0}^{\infty} \downarrow^{\otimes k+1} \varphi^{\otimes k+1} \bar{\psi}^{(k)},$$

where $\bar{\psi}^{(0)} = \mathbf{1}$, $\bar{\psi}^{(k)} = \left(\bar{\psi} \otimes \mathbf{1}^{\otimes k-1}\right)\cdots\left(\bar{\psi} \otimes \mathbf{1}\right)\bar{\psi}$ for $k > 0$, and $\bar{\psi}$ is the reduced coproduct on $BA \otimes BA$. Then, for example, $\mu(\lfloor b\rfloor \otimes \lfloor b\rfloor) = \lfloor a^3\rfloor$ and $\mu(\lfloor b\rfloor \otimes \lfloor a|b\rfloor) = \lfloor a|a^3\rfloor + \lfloor b|a|b\rfloor$. It follows that (BA, d, Δ, μ) is a DG Hopf algebra. Let μ_H and Δ_H be the product and coproduct on $H = H^*(BA)$ induced by μ and Δ; then (H, Δ_H, μ_H) is a graded bialgebra. Let $\alpha = \text{cls}\lfloor a\rfloor$ and $z = \text{cls}\lfloor a|a^3\rfloor$ in H, and note that $\lfloor a^3\rfloor = d\lfloor a|a^2\rfloor$. Let $g : H \to BA$ be a cycle-selecting map such that $g(\text{cls}\lfloor x_1|\cdots|x_n\rfloor) = \lfloor x_1|\cdots|x_n\rfloor$. Then

$$\bar{\Delta}_H(z) = \text{cls}\,\bar{\Delta}\lfloor a|a^3\rfloor = \text{cls}\left\{\lfloor a\rfloor \otimes \lfloor a^3\rfloor\right\} = 0$$

so that

$$\{\Delta g + (g \otimes g)\Delta_H\}(z) = \lfloor a\rfloor \otimes \lfloor a^3\rfloor.$$

By the Transfer Algorithm, we may choose a map $g^2 : H \to BA \otimes BA$ such that $g^2(z) = \lfloor a \rfloor \otimes \lfloor a|a^2 \rfloor$; then

$$\nabla g^2(z) = \{\Delta g + (g \otimes g)\Delta_H\}(z).$$

Furthermore, note that

$$\left\{\left(g^2 \otimes g + g \otimes g^2\right)\Delta_H + (\Delta \otimes \mathbf{1} + \mathbf{1} \otimes \Delta) g^2\right\}(z) = \lfloor a \rfloor \otimes \lfloor a \rfloor \otimes \lfloor a^2 \rfloor.$$

Since $\lfloor a^2 \rfloor = d\lfloor a|a \rfloor$, there is an A_∞-coalgebra operation $\Delta_H^3 : H \to H^{\otimes 3}$, and a map $g^3 : H \to (BA)^{\otimes 3}$ satisfying the general relation on J_3 such that $\Delta_H^3(z) = 0$ and $g^3(z) = \lfloor a \rfloor \otimes \lfloor a \rfloor \otimes \lfloor a|a \rfloor$. In fact, we may choose Δ_H^3 to be identically zero on H so that

$$\nabla g^3 = (\Delta \otimes \mathbf{1} + \mathbf{1} \otimes \Delta) g^2 + \left(g^2 \otimes g + g \otimes g^2\right)\Delta_H.$$

Now the potentially non-vanishing terms in the image of J_4 are

$$\left(g^3 \otimes g + g^2 \otimes g^2 + g \otimes g^3\right)\Delta_H + (\Delta \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \Delta \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \Delta) g^3,$$

and evaluating at z gives $\lfloor a \rfloor \otimes \lfloor a \rfloor \otimes \lfloor a \rfloor \otimes \lfloor a \rfloor$. Thus there is an A_∞-coalgebra operation Δ_H^4 and a map $g^4 : H \to (BA)^{\otimes 4}$ satisfying the general relation on J_4 such that $\Delta_H^4(z) = \alpha \otimes \alpha \otimes \alpha \otimes \alpha$ and $g^4(z) = 0$. Now recall that the induced A_∞-coalgebra structure on $H \otimes H$ is given by

$$\begin{aligned}
\Delta_{H\otimes H} &= \sigma_{2,2}(\Delta_H \otimes \Delta_H)\\
\Delta_{H\otimes H}^4 &= \sigma_{4,2}[\Delta_H^4 \otimes (\mathbf{1}^{\otimes 2} \otimes \Delta_H)(\mathbf{1} \otimes \Delta_H)\Delta_H + (\Delta_H \otimes \mathbf{1}^{\otimes 2})(\Delta_H \otimes \mathbf{1})\Delta_H \otimes \Delta_H^4]\\
&\;\;\vdots
\end{aligned}$$

Let $\beta = \operatorname{cls}\lfloor b \rfloor$, $u = \operatorname{cls}\lfloor a|b \rfloor$, $v = \operatorname{cls}\lfloor b|a \rfloor$, and $w = \operatorname{cls}\lfloor b|a|b \rfloor$ in H, and consider the induced map of tilde-cobar constructions

$$\widetilde{\mu_H} = \sum_{n \geq 1} (\uparrow \mu_H \downarrow)^{\otimes n} : \tilde{\Omega}(H \otimes H) \to \tilde{\Omega}H.$$

Then

$$\widetilde{\mu_H}\lceil \beta \otimes u \rceil = \lceil \mu_H(\beta \otimes u) \rceil = \lceil \operatorname{cls}\mu(\lfloor b \rfloor \otimes \lfloor a|b \rfloor) \rceil = \lceil z + w \rceil$$

so that

$$d_{\tilde{\Omega}H}\widetilde{\mu_H}\lceil \beta \otimes u \rceil = d_{\tilde{\Omega}H}\lceil z + w \rceil = \lceil \alpha|\alpha|\alpha|\alpha \rceil + \lceil \beta|u \rceil + \lceil v|\beta \rceil.$$

But on the other hand,

$$\begin{aligned}
d_{\tilde{\Omega}(H\otimes H)}\lceil \beta \otimes u \rceil &= \lceil \Delta_{H\otimes H}(\beta \otimes u) \rceil\\
&= \lceil e \otimes u \mid \beta \otimes e \rceil + \lceil \beta \otimes e \mid e \otimes u \rceil\\
&\quad + \lceil \beta \otimes \alpha \mid e \otimes \beta \rceil + \lceil e \otimes \alpha \mid \beta \otimes \beta \rceil
\end{aligned}$$

so that

$$\widetilde{\mu_H} d_{\tilde{\Omega}(H\otimes H)}\lceil \beta \otimes u \rceil = \lceil \beta|u \rceil + \lceil v|\beta \rceil.$$

Although $\widetilde{\mu_H}$ fails to be a chain map, the Transfer Algorithm implies there is a chain map $\widetilde{\mu_H}^2 : \tilde{\Omega}(H \otimes H) \to \tilde{\Omega}H$ such that $\widetilde{\mu_H}^2 \lceil e \otimes \alpha \mid \beta \otimes \beta \rceil = \lceil \alpha|\alpha|\alpha|\alpha \rceil$, which can be realized by defining

$$\widetilde{\mu_H}^2 = \sum_{n \geq 1} \left(\uparrow \mu_H \downarrow + \uparrow^{\otimes 3} \omega_2^3 \downarrow \right)^{\otimes n},$$

where $\omega_2^3 (\beta \otimes \beta) = \alpha \otimes \alpha \otimes \alpha$. Indeed, to see that the required equality holds, note that $\mu_H (\beta \otimes \beta) = 0$ since $\lfloor a^3 \rfloor = d \lfloor a|a^2 \rfloor$. Thus there is a map $g_2 : H \otimes H \to BA$ such that $g_2 (\beta \otimes \beta) = \lfloor a|a^2 \rfloor$, and $\nabla g_2 = g\mu_H + \mu (g \otimes g)$. Furthermore, there is the following general relation on $JJ_{2,2}$:

$$\begin{aligned} \nabla g_2^2 = {} & \omega_{BA}^{2,2} (g \otimes g) + (\mu \otimes \mu) \sigma_{2,2} \left(\Delta g \otimes g^2 + g^2 \otimes (g \otimes g) \Delta_H \right) + g^2 \mu_H \\ & + (\mu (g \otimes g) \otimes g_2 + g_2 \otimes g\mu_H) \sigma_{2,2} (\Delta_H \otimes \Delta_H) + \Delta g_2 + (g \otimes g) \omega_2^2. \end{aligned} \tag{12.3}$$

The first expression on the right-hand side vanishes since BA has trivial higher order structure, and the next two expressions vanish since $\mu_H (\beta \otimes \beta) = 0$ and $g^2(\beta) = 0$ (β is primitive). However,

$$\{ (\mu (g \otimes g) \otimes g_2 + g_2 \otimes g\mu_H) \sigma_{2,2} (\Delta_H \otimes \Delta_H) + \Delta g_2 \} (\beta \otimes \beta) = \bar{\Delta}\, g_2(\beta \otimes \beta) = \lfloor a \rfloor \otimes \lfloor a^2 \rfloor .$$

Since $d \lfloor a|a \rfloor = \lfloor a^2 \rfloor$, there an operation $\omega_2^2 : H^{\otimes 2} \to H^{\otimes 2}$, and a map $g_2^2 : H^{\otimes 2} \to (BA)^{\otimes 2}$ satisfying relation (12.3) such that $\omega_2^2 (\beta \otimes \beta) = 0$ and $g_2^2 (\beta \otimes \beta) = \lfloor a \rfloor \otimes \lfloor a|a \rfloor$. Similarly, there is an operation $\omega_2^3 : H^{\otimes 2} \to H^{\otimes 3}$, and a map $g_2^3 : H^{\otimes 2} \to (BA)^{\otimes 3}$ satisfying the general relation on $JJ_{3,2}$ such that $\omega_2^3 (\beta \otimes \beta) = \alpha \otimes \alpha \otimes \alpha$ and $g_2^3 (\beta \otimes \beta) = 0$. Thus $\left(H, \mu_H, \Delta_H, \Delta_H^4, \omega_2^3, ... \right)$ is an A_∞-bialgebra of the first kind.

One can think of the algebra A in Example 12.4 as the singular $\mathbb{Z}_2$-cohomology algebra of a space X with the Steenrod algebra $\mathcal{A}_2$ acting nontrivially via $Sq_1 b = a^3$ (recall that $Sq_1 : H^n (X; \mathbb{Z}_2) \to H^{2n-1} (X; \mathbb{Z}_2)$ is a homomorphism defined by $Sq_1 [x] = [x \smile_1 x]$). Recall that a space X is $\mathbb{Z}_2$-*formal* if there exists a DGA B and cohomology isomorphisms $C^*(X; \mathbb{Z}_2) \leftarrow B \to H^* (X; \mathbb{Z}_2)$. Thus, when X is $\mathbb{Z}_2$-formal, $H^* (BA) \approx H^*(\Omega X; \mathbb{Z}_2)$ as graded coalgebras. Now consider a $\mathbb{Z}_2$-formal space X whose cohomology $H^* (X; \mathbb{Z}_2)$ is generated multiplicatively by $\{a_1, \ldots, a_{n+1}, b\}$, $n \geq 2$. Then Example 12.4 suggests the following conditions on X, which if satisfied, give rise to a nontrivial operation ω_2^n with $n \geq 2$, on the loop cohomology $H^*(\Omega X; \mathbb{Z}_2)$:

1. $a_1 b = 0$;
2. $a_1 \cdots a_{n+1} = 0$;
3. $a_{i_1} \cdots a_{i_k} \neq 0$ whenever $k \leq n$ and $i_p \neq i_q$ for all $p \neq q$;
4. $Sq_1(b) = a_2 \cdots a_{n+1}$.

To see this, consider the non-zero classes $\alpha_i = \operatorname{cls} \lfloor a_i \rfloor$, $\beta = \operatorname{cls} \lfloor b \rfloor$, $u = \operatorname{cls} \lfloor a_1|b \rfloor$, $w = \operatorname{cls} \lfloor b|a_1|b \rfloor$, and $z = \operatorname{cls} \lfloor a_1|a_2 \cdots a_{n+1} \rfloor$ in $H = H^* (BA)$. Conditions (2) and (3) give rise to an induced A_∞-coalgebra structure $\left\{ \Delta_H^k : H \to H^{\otimes k} \right\}$ such that $\Delta_H^k(z) = 0$ for $3 \leq k \leq n$, and $\Delta_H^{n+1}(z) = \alpha_1 \otimes \cdots \otimes \alpha_{n+1}$ with $g^k(z) = \lfloor a_1 \rfloor \otimes \cdots \otimes \lfloor a_{k-1} \rfloor \otimes \lfloor a_k | a_{k+1} \cdots a_{n+1} \rfloor$ for $2 \leq k \leq n$ and $g^{n+1}(z) = 0$. Next, condition (4) implies $\beta \smile u = w + z$, and we can define $\omega_2^k(\beta \otimes \beta) = 0$ for $2 \leq k < n$ and $\omega_2^n(\beta \otimes \beta) = \alpha_2 \otimes \cdots \otimes \alpha_{n+1}$ with $g_2^1(\beta \otimes \beta) = \lfloor a_2 | a_3 \cdots a_{n+1} \rfloor$, $g_2^k(\beta \otimes \beta) = \lfloor a_2 \rfloor \otimes \cdots \otimes \lfloor a_k \rfloor \otimes$

$\lfloor a_{k+1} \ |a_{k+2} \cdots a_{n+1} \rfloor$ for $2 \leq k \leq n-1$ and $g_2^n(\beta \otimes \beta) = 0$. Indeed, the Transfer Algorithm implies the existence of an A_∞-bialgebra structure in which ω_2^n satisfies the required structure relation on $JJ_{n,2}$. Note that the $\mathbb{Z}_2$-formality assumption is in fact superfluous here, as it is sufficient for α_i, β, and u to be non-zero.

Spaces X with $\mathbb{Z}_2$-cohomology satisfying conditions (1)-(4) abound.

Example 12.5. Given an integer $n \geq 2$, choose positive integers $r_1, \ldots, r_{n+1}$ and $m \geq 2$ such that $r_2 + \cdots + r_{n+1} = 4m - 3$. Consider the "thick bouquet" of spheres $S^{r_1} \veebar \cdots \veebar S^{r_{n+1}}$, i.e., $S^{r_1} \times \cdots \times S^{r_{n+1}}$ with top dimensional cell removed, and generators $\bar{a}_i \in H^{r_i}(S^{r_i}; \mathbb{Z}_2)$. Also consider the suspension of complex projective space $\Sigma\mathbb{C}P^{2m-2}$ with generators $\bar{b} \in H^{2m-1}(\Sigma\mathbb{C}P^{2m-2}; \mathbb{Z}_2)$ and $Sq_1\bar{b} \in H^{4m-3}(\Sigma\mathbb{C}P^{2m-2}; \mathbb{Z}_2)$. Let $Y_n = S^{r_1} \veebar \cdots \veebar S^{r_{n+1}} \vee \Sigma\mathbb{C}P^{2m-2}$, and choose a map $f : Y_n \to K(\mathbb{Z}_2, 4m-3)$ such that $f^*(\iota_{4m-3}) = \bar{a}_2 \cdots \bar{a}_{n+1} + Sq_1\bar{b}$. Finally, consider the pullback $p : X_n \to Y_n$ of the following path fibration:

$$\begin{array}{ccccc} K(\mathbb{Z}_2, 4m-4) & \longrightarrow & X_n & \longrightarrow & \mathcal{L}K(\mathbb{Z}_2, 4m-3) \\ & & p\downarrow & & \downarrow \\ & & Y_n & \xrightarrow{f} & K(\mathbb{Z}_2, 4m-3) \\ & & \bar{a}_2 \cdots \bar{a}_{n+1} + Sq_1\bar{b} & \xleftarrow[f^*]{} & \iota_{4m-3} \end{array}$$

Let $a_i = p^*(\bar{a}_i)$ and $b = p^*(\bar{b})$; then $a_1, \ldots, a_{n+1}, b$ are multiplicative generators of $H^*(X_n; \mathbb{Z}_2)$ satisfying conditions (1) - (4) above. We remark that one can also obtain a space X_2' with a non-trivial ω_2^2 on its loop cohomology by setting $Y_2' = (S^2 \times S^3) \vee \Sigma\mathbb{C}P^2$ in the construction above (see [37] for details).

Finally, we note that the cohomology of Eilenberg-MacLane spaces and Lie groups fail to satisfy all of (1)-(4), and it would not be surprising to find that the operations ω_2^n vanish in their loop cohomologies for all $n \geq 2$. In the case of Eilenberg-MacLane spaces, results due to Berciano and the second author [2] seems to support this conjecture. Indeed, each tensor factor $A = E(v, 2n+1) \otimes \Gamma(w, 2np+2) \subset H_*(K(\mathbb{Z}, n); \mathbb{Z}_p)$, where $n \geq 3$ and p an odd prime, is an A_∞-bialgebra of the third kind of the form $(A, \Delta^2, \Delta^p, \mu)$.

HGAs with nontrivial actions of the Steenrod algebra $\mathcal{A}_2$ were first considered by the first author in [28]. In general, the Steenrod $\smile_1$-cochain operation together with other higher cochain operations induce a nontrivial HGA structure on $S^*(X)$, but the failure of the differential to be a $\smile_1$-derivation prevents an immediate lifting of this HGA structure to cohomology (for some remarks on the history of lifting of $\smile_1$-operation on the homology level see [18] and [28]).

Example 12.6. Let $g : S^{2n-2} \to S^n$ be a map of spheres, and let $Y_{m,n} = S^m \times (e^{2n-1} \cup_g S^n)$. Let $*$ be the wedge point of $S^m \vee S^n \subset Y_{m,n}$, let $f : S^{2m-1} \to S^m \times *$, and let $X_{m,n} = e^{2m} \cup_f Y_{m,n}$. Then $X_{m,n}$ is $\mathbb{Z}_2$-formal for each m and n (by a dimensional argument), and we may consider $A = H^*(X_{m,n}; \mathbb{Z}_2)$ and $H = H^*(BA) \approx H^*(\Omega X_{m,n}; \mathbb{Z}_2)$. Below we prove that:

(i) The A_∞-coalgebra structure of H is nontrivial if and only if the Hopf invariant $h(f) = 1$, in which case $m = 2, 4, 8$.

(ii) If $h(f) = 1$, the A_∞-coalgebra structure on H extends to a nontrivial A_∞-bialgebra structure on H. Furthermore, let $a \in A^n$ and $c \in A^{2n-1}$ be multiplicative generators; then there is a

perturbed multiplication φ on H if and only if $Sq_1 a = c$, in which case $n = 3, 5, 9$; otherwise φ is induced by the shuffle product on BA.

Proof. Suppose $h(f) = 1$. Then A is generated multiplicatively by $a \in A^n$, $b \in A^m$, and $c \in A^{2n-1}$ subject to the relations $a^2 = c^2 = ac = ab^2 = b^2 c = b^3 = 0$. Let $\alpha = \operatorname{cls}\lfloor a \rfloor$, $\beta = \operatorname{cls}\lfloor b \rfloor$, $\gamma = \operatorname{cls}\lfloor c \rfloor$, and $z = \operatorname{cls}\lfloor b^2 \rfloor \in H = H^*(BA)$. Given $x_i = \operatorname{cls}\lfloor u_i \rfloor \in H$ with $u_i u_{i+1} = 0$, let $x_1|\cdots|x_n = \operatorname{cls}\lfloor u_1|\cdots|u_n \rfloor$. Note that $x = \alpha|z = z|\alpha$ and $y = \gamma|z = z|\gamma$. Let Δ_H denote the coproduct in H induced by the cofree coproduct Δ in BA. Then x and y are primitive, and $\Delta_H(\alpha|z|\alpha) = e \otimes \alpha|z|\alpha + x \otimes \alpha + \alpha \otimes x + \alpha|z|\alpha \otimes e$. Define $g(x) = \lfloor a|b^2 \rfloor$ and $g^2(x) = \lfloor a \rfloor \otimes \lfloor b|b \rfloor$; define $g(\alpha|z|\alpha) = \lfloor a|b^2|a \rfloor$ and $g^2(\alpha|z|\alpha) = \lfloor a \rfloor \otimes (\lfloor a|b|b \rfloor + \lfloor b|a|b \rfloor + \lfloor b|b|a \rfloor)$. There is an induced A_∞-coalgebra operation $\Delta_H^3 : H \to H^{\otimes 3}$, which vanishes except on elements of the form $\cdots|z|\cdots$, and may be defined on the elements x, y, and $\alpha|z|\alpha$ by $\Delta_H^3(x) = \alpha \otimes \beta \otimes \beta$, $\Delta_H^3(y) = \gamma \otimes \beta \otimes \beta$, and $\Delta_H^3(\alpha|z|\alpha) = \alpha \otimes (\alpha|\beta + \beta|\alpha) \otimes \beta + \alpha \otimes \beta \otimes (\alpha|\beta + \beta|\alpha)$. Then $\{\Delta_H, \Delta_H^3\}$ defines an A_∞-coalgebra structure on H. Furthermore, if $Sq_1 a = c$, which can only occur when $n = 3, 5, 9$, the induced HGA structure on A is determined by Sq_1, and induces a perturbation of the shuffle product $\mu : BA \otimes BA \to BA$ with $\mu(\lfloor a \rfloor \otimes \lfloor a \rfloor) = \lfloor c \rfloor$. The product μ lifts to a perturbed product φ on H such that $\varphi(\alpha \otimes \alpha|z) = \alpha|z|\alpha + \gamma|z$, and the A_∞-coalgebra structure $(H, \Delta_H, \Delta_H^3)$ extends to an A_∞-bialgebra structure $(H, \Delta_H, \Delta_H^3, \varphi)$ as in Example 12.4. On the other hand, if $Sq_1 a = 0$, then is induced by the shuffle product on BA and $\varphi(\alpha \otimes \alpha|z) = \alpha|z|\alpha$. Conversely, if $h(f) = 0$, then $b^2 = 0$ so that $\Delta_H^k = 0$, for all $k \geq 3$. Q.E.D.

We conclude with an investigation of the A_∞-bialgebra structure on the double cobar construction. To this end, we first prove a more general fact, which follows our next definition:

Definition 12.7. Let (A, d, ψ, φ) be a free DG bialgebra, i.e., free as a DGA. An **acyclic cover of** A is a collection of acyclic DG submodules $\mathcal{C}(A) := \{C^a \subseteq A : a \text{ is a monomial of } A\}$ such that $\psi(C^a) \subseteq C^a \otimes C^a$ *and* $\varphi(C^a \otimes C^b) \subseteq C^{ab}$.

Proposition 12.8. Let (A, d, ψ, φ) be a free DG bialgebra with acyclic cover $\mathcal{C}(A)$.

(i) Then φ and ψ extend to an A_∞-bialgebra structure of the third kind.

(ii) Let $(A', d', \psi', \varphi')$ be a free DG bialgebra with acyclic cover $\mathcal{C}(A')$, and let $f : A \to A'$ be a DGA map such that $f(C^a) \subseteq C^{f(a)}$ for all $C^a \in \mathcal{C}(A)$. Then f extends to a morphism of A_∞-bialgebras.

Proof. Define an A_∞-coalgebra structure as follows: Let $\psi^2 = \psi$; arbitrarily define ψ^3 on multiplicative generators, and extend ψ^3 to decomposables via $\psi^3 \mu_A = \mu_A^{\otimes 3} \sigma_{3,2} (\psi^3 \otimes \psi^3)$. Inductively, if $\{\psi^i\}_{i<n}$ have been constructed, arbitrarily define ψ^n on multiplicative generators, and use (12.1) to extend ψ^n to decomposables. Since each ψ^n preserves $\mathcal{C}(A)$ by hypothesis, $\{\varphi, \psi^2, \psi^3, \ldots\}$ is an A_∞-bialgebra structure of the third kind, as desired. The proof of part (ii) is similar. Q.E.D.

Given a space X, choose a base point $y \in X$. Let $\operatorname{Sing}^2 X$ denote the Eilenberg 2-subcomplex of $\operatorname{Sing} X$, and let $C_*(X) = C_*(\operatorname{Sing}^2 X)/C_{>0}(\operatorname{Sing} y)$. In [29] we constructed an explicit (non-coassociative) coproduct on the double cobar construction $\Omega^2 C_*(X)$, which imposes a DG bialgebra structure. Let $\mathbf{\Omega}^2$ denote the functor from the category of (2-reduced) simplicial sets to the category of permutahedral sets ([29], [19]) such that $\Omega^2 C_*(X) = C_*^\diamondsuit(\mathbf{\Omega}^2 \operatorname{Sing}^2 X)$, where $C_*^\diamondsuit(Y) =$

$C_*\big(\mathrm{Sing}^M Y\big)/\langle\text{degeneracies}\rangle$ and $\mathrm{Sing}^M Y$ is the multipermutahedral singular complex of Y (see Definition 15 in [29]; cf. [1],[5]). Now consider the monoidal permutahedral set $\mathbf{\Omega}^2\,\mathrm{Sing}^2 X$, and let V_* be its monoidal (non-degenerate) generators. For each $a \in V_n$, let $C^a = R\,\langle r\text{-faces of } a : 0 \leq r \leq n\rangle$. Then $\{C^a\}$ is an acyclic cover, and by Proposition 12.8 we have

Theorem 12.9. The DG bialgebra structure on the double cobar construction $\Omega^2 C_*(X)$ extends to an A_∞-bialgebra of the third kind.

Conjecture 12.10. Given a 2-connected space X, the chain complex $C_*^{\diamondsuit}(\Omega^2 X)$ admits an A_∞-bialgebra structure extending the DG bialgebra structure constructed in [29]. Moreover, there exists a morphism $G = \{g_m^n\} : \Omega^2 C_*(X) \Rightarrow C_*^{\diamondsuit}(\Omega^2 X)$ of A_∞-bialgebras such that g_1^1 is a homology isomorphism.

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