Linear groups related to Fibonacci polynomials

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Abstract

We examine the generators of linear groups associated with the roots of Fibonacci polynomials, matrix representations of the generators, some elements, and group structures. From the viewpoint of the roots of Fibonacci polynomials, we obtain the features of various classes of linear groups under certain circumstances. Furthermore, we analyze the relationships between the roots of Fibonacci polynomials and the new generating matrices. We get some amazing relationships between the roots of Fibonacci polynomials and the extended modular group, the extended Hecke groups, and the extended generalized Hecke groups with geometric viewpoints.

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1 Introduction

In the present day, numerous recurrence sequences have been investigated in the literature. The most famous sequence is Fibonacci. Fibonacci polynomials can be defined by Fibonacci-like recurrence relation. On the other hand, the sum of the coefficients of a Fibonacci polynomial is a Fibonacci number. Additionally, the ratio of two consecutive numbers or polynomials of the Fibonacci family converges to the golden ratio, which appears in many fields in the literature, such as nature, art, architecture, biology, physics, chemistry, cosmos, theology, finance, and so on (see [2], [14], [15], [21], [26], [29], [34], and [36]).

In [22], V. E. Hoggatt and M. Bicknell obtain the roots of large classes of polynomials Fibonacci and Lucas using hyperbolic trigonometric functions. Hence, the general root formulas for the polynomials have been achieved. This contribution is quite remarkable considering the Abel-Ruffini theorem. There are many papers on the issue from different aspects (see [10], [17], [20], [32], and [38]). On the other hand, in the literature, there are many generating matrices for the numbers and polynomials (see [7], [18], [26], and [27]). Furthermore, there are generating matrices associated with linear groups (see [8], [9], [23], [25], [33], and [37] for more details).

Many studies state the conditions under which these groups or semi-groups are free groups (semi-groups) or not free groups (semi-groups). The main ones can be given as [3], [5], [11], [12], [13], [16], [30], [31], [35], [40], [41], and [42]. In the studies, the provision of different conditions about the freeness of linear groups or semi-groups that have two or more generators is obtained with similar approaches. We briefly outline some of these studies: In [35], when $a, b, c, d, \alpha, \beta, \gamma, \delta \geq 0$; $d - a \geq 2$; $\delta - \alpha \geq 2$; the matrices

$$A = \begin{bmatrix} -a & b \\ -c & d \end{bmatrix} , B = \begin{bmatrix} -\alpha & -\beta \\ \gamma & \delta \end{bmatrix} \in SL(2, \mathbb{R})$$

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generate a free group. In [3], Bachmuth proved when $x, y, z \in \mathbb{C}$; $|x|, |y|, |z| \geq 4.45$; the following three matrices

$$A = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1-z & -z \\ z & 1-z \end{bmatrix}$$

generate a free group.

In [11], [12], [40], and [41], the matrix representations of the generators were defined by the following form of a linear group are in the same form and complementary studies were carried out for different conditions of this form in these articles. We briefly express these studies as follows.

They examine the generators of the linear group which

$$A_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \text{ and } B_b = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$

when a and $b \in \mathbb{C}$.

Sanov, Brenner, and Chang proved the group's freeness while these generators A_a and B_b have the following conditions

$a = b = 2$; $a = b$ and $|a| \geq 2$; $|ab| \geq 2, |ab - 2| \geq 2$ and $|ab + 2| \geq 2$, respectively.

Also, in [41], Słanina demonstrated the group is not free which generators mentioned above A_a and B_a for a that root of Fibonacci polynomials. Fibonacci polynomials $F_n(x)$ are defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \tag{1.1}$$

where $F_1(x) = 1$, $F_2(x) = x$ and $n \geq 3$ (see [26] for more details). It is expressed the following equalities in [41]:

$$F_{n+2}(x) = (1 + x^2)F_n(x) + xF_{n-1}(x). \tag{1.2}$$

$$(A_x B_x)^n = \begin{bmatrix} F_{2n+1}(x) & F_{2n}(x) \\ F_{2n}(x) & F_{2n-1}(x) \end{bmatrix}. \tag{1.3}$$

Słanina showed some features about the linear group as follows.

(i) a is a root of Fibonacci polynomial $F_{2n}(x)$ if and only if $(A_a B_a)^n = \pm \mathrm{I}$.

(ii) a is a root of Fibonacci polynomial $F_{2n+1}(x)$ if and only if $(A_a B_a)^n (B_a A_a)^n$ is a lower triangular matrix which commutes with B_a .

In this paper, we examine the group in detail and determine its characteristics in the context of combinatorial group theory. We express the properties of elements of the group and some new generator matrices regarding the widespread problem.

Fibonacci polynomials hold the following properties:

$$F_{n+1}(x) + F_{n-1}(x) = xF_n(x) + 2F_{n-1}(x) \tag{1.4}$$

and

$$F_{n+1}(x)F_{n-1}(x) - F_n^2(x) = (-1)^n \tag{1.5}$$

In [22], the roots of Fibonacci polynomials $F_n(x)$ are given as follows:

$$x = 2i \cos \frac{k\pi}{n},\ k = 1, 2, \ldots, n-1. \tag{1.6}$$

In [6], when a is a root of Fibonacci polynomial $F_{2n+1}(a)$ satify the following properties:

$$F_{2n}(a) = \pm i \text{ and } F_{2n-1}(a) = \mp ai \tag{1.7}$$

In [6], when a is a root of Fibonacci polynomial $F_{2n-1}(a)$ satify the following properties:

$$F_{2n}(a) = \pm i \;\text{ and } F_{2n+1}(a) = \pm ai \tag{1.8}$$

In addition to the groups mentioned above, there are different linear groups in the literature. In [39], extended Hecke groups $\overline{H}(\lambda_q)$ defined analogously with the extended modular group. The groups generated by three linear fractional transformations

$$T(z) = -\tfrac{1}{z} \;\;, \;\; U(z) = z + \lambda \;\text{ and }\; R(z) = -\tfrac{1}{\overline{z}}$$

where $\lambda_q = 2\cos\frac{\pi}{q}, q \in \mathbb{N}$. Let $S = TU$ i.e.

$$S(z) = -\tfrac{1}{z+\lambda}.$$

Notice that T, S and R have matrix representation

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} , S = \begin{bmatrix} 0 & -1 \\ 1 & \lambda \end{bmatrix} \text{ and } R = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

respectively.

The extended Hecke group $\overline{H}_q = \overline{H}(\lambda_q)$ has the presentation

$$\overline{H}(\lambda_q) = \langle T, S, R \mid T^2 = S^q = R^2 = (TR)^2 = (SR)^2 = I \rangle \cong D_2 *_{\mathbb{Z}_2} D_q$$

The first several of these groups are $\overline{H}(\lambda_3) = \overline{\Gamma} = PGL(2,\mathbb{Z})$ (the extended modular group), $\overline{H}(\sqrt{2})$, $\overline{H}(\frac{1+\sqrt{5}}{2})$, and $\overline{H}(\sqrt{3})$.

In [24], a more general class $\overline{H}_{p,q}$ of extended Hecke groups $\overline{H}(\lambda_q)$ introduced by taking

$$X(z) = \tfrac{-1}{z-\lambda_p} \;\;, \;\; V(z) = z + \lambda_p + \lambda_q \;\text{ and }\; R(z) = -\tfrac{1}{\overline{z}}$$

where $2 \leq p \leq q,\; p + q > 4$. Here, if we take $Y = XV = \frac{-1}{z+\lambda_q}$ then the group presentation

$$\overline{H}_{p,q} = \langle X, Y, R \mid X^p = Y^q = R^2 = (XR)^2 = (YR)^2 = I \rangle \cong D_p *_{\mathbb{Z}_2} D_q$$

The group $\overline{H}_{p,q}$ named as extended generalized Hecke groups. Also, known as $\overline{H}_{2,q} = \overline{H}_q$. Moreover, all extended Hecke groups $\overline{H}_q$ are included in extended generalized Hecke groups $\overline{H}_{2,q}$. The extended modular group, extended Hecke groups, and extended generalized Hecke groups have been extensively studied from many points of view in the literature. (See for more details [8], [9], [23], [24], [25], [39], and [43].) Furthermore, there are numerous notable studies on $2\cos\frac{\pi}{q}$ and $\cos\frac{2\pi}{q}$ in the literature. Finding the minimal polynomial of $\cos\frac{2\pi}{q}$ is an old problem due to its connection to the cyclotomic polynomials. The algebraic numbers are investigated in many papers related to Chebyshev polynomials, Gaussian periods, Dickson polynomials, Ramanujan sums, and Möbius inversion (see for more details [1], [4], [19], [28]).

2 Relationships between the roots of Fibonacci polynomials and extended modular group & extended Hecke groups & extended generalized Hecke groups

In this section, we examine the complex numbers as vectors in the complex plane. All the roots of Fibonacci polynomials are pure imaginary complex numbers. Also, each norm of the roots of a Fibonacci polynomial is smaller than two. We describe the roots in the complex plane as related to the parameter of the extended modular group $\overline{\Gamma}$, the extended Hecke groups $\overline{H}(\lambda_q)$, the extended generalized Hecke groups $\overline{H}_{p,q}$.

Observation 2.1. The parameter of the extended modular group is $\lambda_3 = 2\cos\frac{\pi}{3}$. All of the roots of Fibonacci polynomial $F_3(x)$ are known as $2i\cos\frac{\pi}{3}$ and $2i\cos\frac{2\pi}{3}$ from Equation 1.6 for $k = 1, 2$. If the first root $2i\cos\frac{\pi}{3}$ of the Fibonacci polynomial $F_3(x)$ is rotated 270 degrees counterclockwise around the origin in the complex plane, the parameter of the extended modular group is obtained. Consequently, we can state that the Fibonacci polynomial $F_3(x)$ generates a parameter for the extended modular group as a geometric explanation.

Observation 2.2. The parameter of the extended Hecke group is $\lambda_q = 2\cos\frac{\pi}{q}$ and all of the roots of Fibonacci polynomial $F_q(x)$ are known as $2i\cos\frac{\pi}{q}$, $2i\cos\frac{2\pi}{q}$, ..., $2i\cos\frac{(q-1)\pi}{q}$ from Equation 1.6 for $k = 1, 2, ...q-1$. If the first root $2i\cos\frac{\pi}{q}$ of the Fibonacci polynomial $F_q(x)$ is rotated 270 degrees counterclockwise around the origin in the complex plane, the parameter of the extended Hecke group is obtained. As a result, we can state geometrically that the Fibonacci polynomial $F_q(x)$ generates a parameter for the extended Hecke group.

Observation 2.3. The parameters of the extended generalized Hecke groups are $\lambda_p = 2\cos\frac{\pi}{p}$ and $\lambda_q = 2\cos\frac{\pi}{q}$. Also, all the roots of Fibonacci polynomial $F_p(x)$ known as $2i\cos\frac{\pi}{p}$, $2i\cos\frac{2\pi}{p}$, ... , $2i\cos\frac{(p-1)\pi}{p}$ from Equation 1.6 for $k = 1, 2, ...p-1$. All the roots of Fibonacci polynomial $F_q(x)$ known as $2i\cos\frac{\pi}{q}$, $2i\cos\frac{2\pi}{q}$, ... , $2i\cos\frac{(q-1)\pi}{q}$ from Equation 1.6 for $k = 1, 2, ...q-1$. If the first roots $2i\cos\frac{\pi}{p}$ of the Fibonacci polynomial $F_p(x)$ and $2i\cos\frac{\pi}{q}$ of the Fibonacci polynomial $F_q(x)$ are rotated 270 degrees counterclockwise around the origin in the complex plane, the parameters of the extended generalized Hecke groups are obtained. Consequently, we can state geometrically that the Fibonacci polynomial $F_p(x)$ and $F_q(x)$ generate parameters for the extended generalized Hecke groups.

Remark 2.4. The parameter of the extended Hecke group can not be derived from a unique Fibonacci polynomial. For instance, the parameter of the extended Hecke group $\overline{H}_5 = \overline{H}(\frac{1+\sqrt{5}}{2})$ is obtained from the Fibonacci polynomial $F_5(x)$. Also, the parameter is obtained from another Fibonacci polynomial $F_{10}(x)$ using Equation 1.6 for $k = 2$. Therefore, we can state the parameter of the extended Hecke group $\overline{H}(\frac{1+\sqrt{5}}{2})$ related to the Fibonacci polynomials $F_{5k}(x)$ when k is a whole number. More generally, the parameter of the extended Hecke group H_r can be derived from the Fibonacci polynomial $F_{nr}(x)$ when n is a whole number and r is an integer greater than two.

Remark 2.5. Each Fibonacci polynomial $F_n(x)$ for $n \geq 3$ generates at least one parameter for the extended Hecke group. For example, the Fibonacci polynomial $F_{10}(x)$ generates two parameters as $2\cos\frac{\pi}{10}$ and $2\cos\frac{\pi}{5}$ via the roots $2i\cos\frac{\pi}{10}$ and $2i\cos\frac{2\pi}{10}$ rotated 270 degrees counterclockwise around the origin in the complex plane. The Fibonacci polynomial $F_{11}(x)$ generates one parameter as $2\cos\frac{\pi}{11}$ via the root $2i\cos\frac{\pi}{11}$ rotated 270 degrees counterclockwise around the origin in the complex plane. Finally, using the same manner, the Fibonacci polynomial $F_{12}(x)$ generates four parameters as $2\cos\frac{\pi}{12}$, $2\cos\frac{\pi}{6}$, $2\cos\frac{\pi}{4}$ and $2\cos\frac{\pi}{3}$.

Observation 2.6. We set a general way to get the relationship between the parameter of the extended Hecke groups and Fibonacci polynomial $F_n(x)$. All the roots of Fibonacci polynomial $F_n(x)$ are known as $2i\cos\frac{k\pi}{n}$ for $k = 1, 2, ...n-1$ from Equation 1.6. $F_n(x)$ generates the parameter for the extended Hecke group every provided condition that k divides n except for $k = \frac{n}{2}$ and $k = n$. For example, $F_9(x) = x^8 + 7x^6 + 15x^4 + 10x^2 + 1$ generates exactly two parameters for extended Hecke groups denoted $\overline{H}_9$ and $\overline{\Gamma}$.

Theorem 2.7. (Birol-Extended Hecke-Fibonacci Theorem)
The number of the parameters for the extended Hecke groups generated by $F_n(x)$ is calculated by the formula

$$B(n) = \begin{cases} \prod_{i=1}^{t}(a_i+1) - 2 & \text{if } n \text{ even} \\ \prod_{i=1}^{t}(a_i+1) - 1 & \text{if } n \text{ odd} \end{cases}$$

where $n = \prod_{i=1}^{t} p_i^{a_i}$ for p_i distinct prime numbers and a_i positive integers.

Proof. It can be proved using the fundamental theorem of arithmetic, the formula for the total number of divisors of a number considering the root formula of Fibonacci polynomials and the parameter of the extended Hecke group as $\lambda_q = 2\cos\frac{\pi}{q}, q \in \mathbb{N}, q \geq 3$. Q.E.D.

Corollary 2.8. Considering the polynomial space, the $\{F_n(x) : n \geq 3\}$ set of Fibonacci polynomials is a relation with the ability to generate parameter for the extended Hecke groups. This relation has reflection and symmetry properties.

Remark 2.9. We call the above relation as α. Notice that α is not reflexive relation. We give a counterexample to prove that. $\Big(F_4(x), F_{12}(x)\Big) \in \alpha$ via $\overline{H}_4$ and $\Big(F_{12}(x), F_3(x)\Big) \in \alpha$ via $\overline{\Gamma}$ but $\Big(F_4(x), F_3(x)\Big) \notin \alpha$. Although $F_3(x)$ and $F_4(x)$ generate one parameter for the extended Hecke groups, these polynomials do not generate a common parameter for any extended Hecke group. $F_3(x)$ and $F_4(x)$ generate a parameter for the extended Hecke groups $\overline{\Gamma}$ and $\overline{H}_4$, respectively.

Definition 2.10. (Birol-Extended Hecke-Fibonacci Number Sequence)
We define a new number sequence derived from the Theorem 2.7. This sequence amazingly shows the relationship between the root of the Fibonacci polynomial and the extended Hecke groups. The first, the second, and the third terms of the number sequence are obtained as 1 from Fibonacci polynomials $F_3(x)$, $F_4(x)$, and $F_5(x)$, respectively. This sequence is as follows.
1, 1, 1, 2, 1, 2, 2, 2, 1, 4, 1, 2, 3, 3, 1, 4, 1, 4, 3,...

3 A study concerning Fibonacci polynomials

In this section, we explain some properties of the linear groups, generating matrices, and some elements associated with the roots of Fibonacci polynomials.

Proposition 3.1. If the matrix A_x and B_x are defined by

$$A_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \text{ and } B_x = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$$

then, we have

$$(B_x A_x)^n = \begin{bmatrix} F_{2n-1}(x) & F_{2n}(x) \\ F_{2n}(x) & F_{2n+1}(x) \end{bmatrix}.$$

Proof. We can prove the proposition using the mathematical induction method, Equation 1.1, and Equation 1.2. For n=1, we have

$$(B_x A_x)^n = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ x & 1+x^2 \end{bmatrix} = \begin{bmatrix} F_1(x) & F_2(x) \\ F_2(x) & F_3(x) \end{bmatrix}$$

Now assume that the propositon holds for n=k, that is

$$(B_x A_x)^k = \begin{bmatrix} F_{2k-1}(x) & F_{2k}(x) \\ F_{2k}(x) & F_{2k+1}(x) \end{bmatrix}$$

Then, for n=$k+1$, we obtain

$$\begin{aligned} (B_x A_x)^{k+1} &= \begin{bmatrix} F_{2k-1}(x) & F_{2k}(x) \\ F_{2k}(x) & F_{2k+1}(x) \end{bmatrix} \begin{bmatrix} 1 & x \\ x & 1+x^2 \end{bmatrix} \\ &= \begin{bmatrix} xF_{2k}(x) + F_{2k-1}(x) & (1+x^2)F_{2k}(x) + xF_{2k-1}(x) \\ xF_{2k+1}(x) + F_{2k}(x) & xF_{2k}(x) + (1+x^2)F_{2k+1}(x) \end{bmatrix} = \begin{bmatrix} F_{2k+1}(x) & F_{2k+2}(x) \\ F_{2k+2}(x) & F_{2k+3}(x) \end{bmatrix} \end{aligned}$$

Thus, we get the desired result. Q.E.D.

We know the following corollary from [41].

Corollary 3.2. If a is a root of Fibonacci polynomial $F_{2n}(x)$ if and only if $F_{2n+1}(a) = F_{2n-1}(a) = \pm 1$.

We get the following proposition using with Proposition 3.1 and Equation 1.3.

Proposition 3.3. If the matrices A_x and B_x are defined by

$$A_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \text{ and } B_x = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$$

then, we have

$$(A_x B_x)^n (B_x A_x)^n = \begin{bmatrix} F_{2n-1}(x)F_{2n+1}(x) + F_{2n}^2(x) & 2F_{2n}(x)F_{2n+1}(x) \\ 2F_{2n-1}(x)F_{2n}(x) & F_{2n-1}(x)F_{2n+1}(x) + F_{2n}^2(x) \end{bmatrix}.$$

In the following remarks and corollaries, we investigate some properties of the obtained matrices.

Remark 3.4. If a is a root of Fibonacci polynomial $F_{2n+1}(x)$, that is, $F_{2n+1}(a) = 0$ then, the matrix $(A_aB_a)^n(B_aA_a)^n$ does not commute with the matrix A_a. If the matrix $(A_aB_a)^n(B_aA_a)^n$ commutes with the matrix A_a then it should be $2aF_{2n-1}(a)F_{2n}(a) = 0$. We know the case is impossible from [6]. Because, if a is root of the Fibonacci polynomial $F_{2n+1}(x)$ i.e. $F_{2n+1}(x) = 0$ then, we have $F_{2n}(x)F_{2n-1}(x) \neq 0$. This can be checked using root formula of the Fibonacci polynomial $F_{2n+1}(x)$ via Equation 1.6 and considering root interval of the Fibonacci polynomial $F_{2n+1}(x)$. Also, let us observe this information with an example. For instance, let us consider the roots of the polynomial $F_5(x)$. Then, we obtain the set of the roots as follows:

$$\left\{ i\sqrt{\frac{(3-\sqrt{5})}{2}}, -i\sqrt{\frac{(3-\sqrt{5})}{2}}, i\sqrt{\frac{(3+\sqrt{5})}{2}}, -i\sqrt{\frac{(3+\sqrt{5})}{2}} \right\}.$$

If we take $a = i\sqrt{\frac{(3-\sqrt{5})}{2}}$ then, we get $2aF_3(a)F_4(a) = 2a(a^2+1)(a^3+2a) \neq 0$, which contradicts with our hypothesis. Therefore, the matrix $(A_aB_a)^n(B_aA_a)^n$ does not commute with the matrix A_a.

Corollary 3.5. If a is a root of Fibonacci polynomial $F_{2n-1}(x)$ that is, $F_{2n-1}(a) = 0$. Then, $(A_aB_a)^n(B_aA_a)^n$ is an upper triangular matrix.

Remark 3.6. Using Corollary 3.5, we get the following statements:

(1) The matrix $(A_aB_a)^n(B_aA_a)^n$ commutes with the matrix A_a.

(2) The matrix $(A_aB_a)^n(B_aA_a)^n$ does not commute with the matrix B_a. It can be checked using the similar arguments given in Remark 3.4 and considering Equation 1.8.

Corollary 3.7. If a is a root of Fibonacci polynomial $F_{2n}(x)$ that is, $F_{2n}(a) = 0$. Then, using Corollary 3.2 and Proposition 3.3 we get

$$(A_aB_a)^n(B_aA_a)^n = I.$$

Remark 3.8. Using Corollary 3.7, we get the following statements:

(1) The matrix $(A_aB_a)^n(B_aA_a)^n$ commutes with the matrix A_a.

(2) The matrix $(A_aB_a)^n(B_aA_a)^n$ commutes with the matrix B_a.

We get following proposition using with Proposition 3.1 and Equation 1.3:

Proposition 3.9. If the matrices A_x and B_x respectively are defined by

$$A_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \text{ and } B_x = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$$

then, we have

$$(B_xA_x)^n(A_xB_x)^n = \begin{bmatrix} F_{2n-1}(x)F_{2n+1}(x) + F_{2n}^2(x) & 2F_{2n-1}(x)F_{2n}(x) \\ 2F_{2n}(x)F_{2n+1}(x) & F_{2n-1}(x)F_{2n+1}(x) + F_{2n}^2(x) \end{bmatrix}.$$

Corollary 3.10. If a is a root of Fibonacci polynomial $F_{2n+1}(x)$ that is, $F_{2n+1}(a) = 0$. Then, $(B_aA_a)^n(A_aB_a)^n$ is an upper triangular matrix.

Remark 3.11. Using Corollary 3.10, we obtain the following statements:
(1) The matrix $(B_aA_a)^n(A_aB_a)^n$ commutes with the matrix A_a.
(2) The matrix $(B_aA_a)^n(A_aB_a)^n$ does not commute with the matrix B_a. It can be checked using the similar arguments given in Remark 3.4 and considering Equation 1.7.

Corollary 3.12. If a is a root of Fibonacci polynomial $F_{2n-1}(x)$ that is, $F_{2n-1}(a) = 0$. Then, $(B_aA_a)^n(A_aB_a)^n$ is a lower triangular matrix.

Remark 3.13. Using Corollary 3.12, we obtain the following statements:
(1) The matrix $(B_aA_a)^n(A_aB_a)^n$ does not commute with with the matrix A_a. It can be checked using the similar arguments given in Remark 3.4 and considering Equation 1.8.
(2) The matrix $(B_aA_a)^n(A_aB_a)^n$ commutes with the matrix B_a.

Corollary 3.14. If a is a root of Fibonacci polynomial $F_{2n}(x)$, that is, $F_{2n}(a) = 0$. Then, using Corollary 3.2 we get

$$(B_aA_a)^n(A_aB_a)^n = I.$$

Remark 3.15. Using Corollary 3.14, we obtain the following statements:
(1) The matrix $(B_aA_a)^n(A_aB_a)^n$ commutes with the matrix A_a.
(2) The matrix $(B_aA_a)^n(A_aB_a)^n$ commutes with the matrix B_a.

4 A view the group concerning Fibonacci polynomials

In this section, we explicitly express the relationships between the generators and elements of the groups associated with the roots of the Fibonacci polynomial. We take advantage of some facts we have obtained about the roots of Fibonacci polynomials.

Theorem 4.1. If a is a root of Fibonacci polynomial $F_{2n}(x)$, that is, $F_{2n}(a) = 0$. Then, we obtain $(B_aA_a)^n = \pm \mathrm{I}$.

Proof. Using Corollary 3.2, we obtain the equality. Q.E.D.

Theorem 4.2. If a is a root of Fibonacci polynomial $F_{2n+1}(x)$, that is, $F_{2n+1}(a) = 0$. Then, we have
(1) $(A_aB_a)^nA_a = B_a(A_aB_a)^n$
(2) $(B_aA_a)^nB_a = A_a(B_aA_a)^n$

Proof. Using the Equation 1.7, we obtain above. Q.E.D.

Theorem 4.3. If a is a root of Fibonacci polynomial $F_{2n+1}(x)$, that is, $F_{2n+1}(a) = 0$. Then, we have
(1) $(A_aB_a)^n(B_aA_a)^n = \begin{bmatrix} -1 & 0 \\ 2a & -1 \end{bmatrix}$
(2) $(A_aB_a)^n(B_aA_a)^nA_a = \begin{bmatrix} -1 & -a \\ 2a & 2a^2-1 \end{bmatrix}$
(3) $A_a(A_aB_a)^n(B_aA_a)^n = \begin{bmatrix} 2a^2-1 & -a \\ 2a & -1 \end{bmatrix}$
(4) $(A_aB_a)^n(B_aA_a)^nB_a = \begin{bmatrix} -1 & 0 \\ a & -1 \end{bmatrix}$

(5) $(B_aA_a)^n(A_aB_a)^n = \begin{bmatrix} -1 & 2a \\ 0 & -1 \end{bmatrix}$

(6) $(B_aA_a)^n(A_aB_a)^nA_a = \begin{bmatrix} -1 & a \\ 0 & -1 \end{bmatrix}$

(7) $(B_aA_a)^n(A_aB_a)^nB_a = \begin{bmatrix} 2a^2-1 & 2a \\ -a & -1 \end{bmatrix}$

(8) $B_a(B_aA_a)^n(A_aB_a)^n = \begin{bmatrix} -1 & 2a \\ -a & 2a^2-1 \end{bmatrix}$

Proof. Using the Equation 1.7, we obtain above. Q.E.D.

Theorem 4.4. If a is a root of Fibonacci polynomial $F_{2n}(x)$, that is, $F_{2n}(a) = 0$. Then, we get

(1) $(A_aB_a)^n(B_aA_a)^n = I$

(2) $(B_aA_a)^n(A_aB_a)^n = I$

Proof. Using Corollary 3.2, we obtain above. Q.E.D.

Theorem 4.5. If a is a root of Fibonacci polynomial $F_{2n-1}(x)$, that is, $F_{2n-1}(a) = 0$. Then, we obtain

(1) $(A_aB_a)^n(B_aA_a)^n = \begin{bmatrix} -1 & -2a \\ 0 & -1 \end{bmatrix}$

(2) $(A_aB_a)^n(B_aA_a)^nA_a = \begin{bmatrix} -1 & -3a \\ 0 & -1 \end{bmatrix}$

(3) $(A_aB_a)^n(B_aA_a)^nB_a = \begin{bmatrix} -2a^2-1 & -2a \\ -a & -1 \end{bmatrix}$

(4) $B_a(A_aB_a)^n(B_aA_a)^n = \begin{bmatrix} -1 & -2a \\ -a & -2a^2-1 \end{bmatrix}$

(5) $(B_aA_a)^n(A_aB_a)^n = \begin{bmatrix} -1 & 0 \\ -2a & -1 \end{bmatrix}$

(6) $(B_aA_a)^n(A_aB_a)^nA_a = \begin{bmatrix} -1 & -a \\ -2a & -2a^2-1 \end{bmatrix}$

(7) $A_a(B_aA_a)^n(A_aB_a)^n = \begin{bmatrix} -2a^2-1 & -a \\ -2a & -1 \end{bmatrix}$

(8) $(B_aA_a)^n(A_aB_a)^nB_a = \begin{bmatrix} -1 & 0 \\ -3a & -1 \end{bmatrix}$

Proof. Using the Equation 1.8, we get above. Q.E.D.

Corollary 4.6. If r is a root of Fibonacci polynomial $F_{2n+1}(x)$, that is, $F_{2n+1}(r) = 0$. Then, we obtain

(1) $[(\mathbf{A_rB_r})^{\mathbf{n}}(\mathbf{B_rA_r})^{\mathbf{n}}\mathbf{B_r}]^\top = [\mathbf{B_r}(\mathbf{A_rB_r})^{\mathbf{n}}(\mathbf{B_rA_r})^{\mathbf{n}}]^\top = (B_rA_r)^n(A_rB_r)^nA_r = A_r(B_rA_r)^n(A_rB_r)^n$

(2) $[(\mathbf{A_rB_r})^{\mathbf{n}}(\mathbf{B_rA_r})^{\mathbf{n}}\mathbf{A_r}]^\top = B_r(B_rA_r)^n(A_rB_r)^n$

(3) $[\mathbf{A_r}(\mathbf{A_rB_r})^{\mathbf{n}}(\mathbf{B_rA_r})^{\mathbf{n}}]^\top = (B_rA_r)^n(A_rB_r)^nB_r$

Corollary 4.7. If r is a root of Fibonacci polynomial $F_{2n}(x)$, that is, $F_{2n}(r) = 0$. Then, we get

(1) $[(\mathbf{A_rB_r})^{\mathbf{n}}(\mathbf{B_rA_r})^{\mathbf{n}}\mathbf{B_r}]^\top = [\mathbf{B_r}(\mathbf{A_rB_r})^{\mathbf{n}}(\mathbf{B_rA_r})^{\mathbf{n}}]^\top = A_r(B_rA_r)^n(A_rB_r)^n = (B_rA_r)^n(A_rB_r)^nA_r$

(2) $[(\mathbf{A_rB_r})^{\mathbf{n}}(\mathbf{B_rA_r})^{\mathbf{n}}\mathbf{A_r}]^\top = [\mathbf{A_r}(\mathbf{A_rB_r})^{\mathbf{n}}(\mathbf{B_rA_r})^{\mathbf{n}}]^\top = B_r(B_rA_r)^n(A_rB_r)^n = (B_rA_r)^n(A_rB_r)^nB_r$

Corollary 4.8. If r is a root of Fibonacci polynomial $F_{2n-1}(x)$, that is, $F_{2n-1}(r) = 0$. Then, we have

(1) $[(\mathbf{A_r B_r})^{\mathbf{n}}(\mathbf{B_r A_r})^{\mathbf{n}}\mathbf{A_r}]^{\top} = [\mathbf{A_r}(\mathbf{A_r B_r})^{\mathbf{n}}(\mathbf{B_r A_r})^{\mathbf{n}}]^{\top} = B_r(B_rA_r)^n(A_rB_r)^n = (B_rA_r)^n(A_rB_r)^nB_r$

(2) $[(\mathbf{A_r B_r})^{\mathbf{n}}(\mathbf{B_r A_r})^{\mathbf{n}}\mathbf{B_r}]^{\top} = A_r(B_rA_r)^n(A_rB_r)^n$

(3) $[\mathbf{B_r}(\mathbf{A_r B_r})^{\mathbf{n}}(\mathbf{B_r A_r})^{\mathbf{n}}]^{\top} = (B_rA_r)^n(A_rB_r)^nA_r$

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