Hemi-slant Riemannian submersions from cosymplectic manifolds

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Abstract

At this work, our main objective is to present the idea of hemi-slant Riemannian submersions from almost contact metric manifolds as a natural generalization of anti-invariant Riemannian submersions, semi-invariant Riemannian submersions and slant Riemannian submersions. We mostly examined on hemi-slant Riemannian submersions from cosymplectic manifolds onto Riemannian manifolds. During this way, we tend to study and investigate integrability conditions, the geometry of leaves of distributions which are emerged from the definition of the submersion. Besides, we tend to get new conditions for these submersions to be totally geodesic. Finally, we construct some quality examples of such submersion.

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1 Introduction

Differential geometry is one of the most popular branches of mathematics and physics from ancient days. Differential geometry has many topics that have very significant applications in both, mathematics and physics [3, 5, 6, 12, 13, 17]. Some of them are immersion and submersion. In both complex geometry and contact geometry, the characteristics of Riemannian submersions are a fascinating topic.

The theory of Riemannian submersions was firstly established by O′Neill [22] and Gray [9], in 1966 and 1967, respectively. In 1976, Watson [29] studied almost complex type of Riemannian submersions and defined almost Hermitian submersions between almost Hermitian manifolds. In 1985, D. Chinea [7] extended the idea of almost Hermitian submersion to different sub-classes of almost contact manifolds. There are so many important and interesting results about Riemannian and almost Hermitian submersions which are studied in [8, 20, 26].

As a natural generalization of holomorphic submersions and totally real submersions, B. Sahin introduced the notion of slant submersions [24] and semi-invariant submersions [23] from almost Hermitian manifolds onto arbitrary Riemannian manifolds in 2011 and 2013 respectively. The different kinds of Riemannian submersions between Riemannian manifolds endowed with different structures were studied by several geometers ([4], [10], [11], [14], [19],[15],[21], [25], [27]). As a generalization of invariant submersions and slant submersions, Park and Prasad [18] defined and studied the notion of semi-slant submersions from an almost Hermitian manifold onto a Riemannian manifold. In 2015, as a generalization of slant submersions and anti-invariant submersions, B. Sahin introduced the notion of hemi-slant Riemannian submersions [28] from almost Hermitian manifolds onto Riemannian manifolds. He gave a decomposition theorem for such submersions.

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Recently, Akyol et al. defined and studied semi-invariant $\xi^{\perp}-$Riemannian submersion and semi-slant $\xi^{\perp}-$Riemannian submersion from almost contact manifolds onto Riemannian manifold [1, 2].

In this research, we undertake our work as: In the second section, we present several main pieces of information relating to cosymplectic manifolds and Riemannian submersions. In the third section, certain interesting outcomes on hemi-slant Riemannian submersions from cosymplectic manifold onto Riemannian manifold are obtained and studied the geometry of leaves of distributions involved in above submersion. Finally, we obtain certain conditions for such submersions to be totally geodesic. In the last section, we have constructed some suitable examples for such submersions.

2 Preliminaries

In this section, we recall main definitions and properties of cosymplectic manifolds and Riemannian submersions.

We consider $\mathcal{M}_1$ is a $(2n+1)$-dimensional almost contact metric manifold [16] which carries a $(1,1)$ tensor field ϕ, $1-$form η, characteristic vector field ξ and Riemannian metric g satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \ \eta \circ \phi = 0, \ \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \eta(V_1)\eta(V_2), \tag{2.2}$$

for any vector fields $V_1, V_2 \in \Gamma(T\mathcal{M}_1)$, where $\Gamma(T\mathcal{M}_1)$ represents the Lie algebra of vector fields on $\mathcal{M}_1$ and $I : T\mathcal{M}_1 \longrightarrow T\mathcal{M}_1$ is the identity map. We have from definition $rank(\phi) = 2n$.

The immediate consequence of (2.2), we have

$$\eta(V_1) = g(V_1, \xi) \quad \text{and} \quad g(\phi V_1, V_2) + g(V_1, \phi V_2) = 0, \tag{2.3}$$

for all vector fields $V_1, V_2 \in \Gamma(T\mathcal{M}_1)$.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $\mathcal{M}_1 \times R$ is given by

$$J(U_1, \ f\frac{d}{dt}) = (\phi U_1 - f\xi, \ \eta(U_1)\frac{d}{dt}),$$

where $J^2 = -I$ and f is the differentiable function on $\mathcal{M}_1 \times R$ has no torsion i.e., J is integrable. The condition for normality in terms of ϕ, ξ and η is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on $\mathcal{M}_1$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Now, the fundamental $2-$form is defined by $\Phi(V_1, V_2) = g(V_1, \phi V_2)$.

An almost contact metric manifold is said to be a cosymplectic manifold if it is normal, and both Φ and η are closed. The structure equation of a cosymplectic manifold is given by

$$(\nabla_{V_1}\phi)V_2 = 0, \tag{2.4}$$

for all vector fields $V_1, V_2 \in \Gamma(T\mathcal{M}_1)$, where ∇ represents the Levi-Civita connection of $(\mathcal{M}_1, g)$. Moreover, for a cosymplectic manifold, we have

$$\nabla_{V_1}\xi = 0, \tag{2.5}$$

for every vector field $V_1 \in \Gamma(T\mathcal{M}_1)$.

Example 2.1. We consider R^{2k+1} with Cartesian coordinates (x_i, y_i, z) $(i = 1, ..., k)$ and its usual contact form $\eta = dz$.

The characteristic vector field ξ is given by $\frac{\partial}{\partial z}$, and its Riemannian metric g and tensor field ϕ are given by

$$g = \sum_{i=1}^{k}((dx_i)^2 + (dy_i)^2) + (dz)^2, \ \phi = \begin{bmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ i = 1, ...k.$$

This gives a cosymplectic structure on R^{2k+1}. The vector fields $E_i = \frac{\partial}{\partial y_i}, E_{k+i} = \frac{\partial}{\partial x_i}, \xi = \frac{\partial}{\partial z}$ form a $\phi-$basis for the cosymplectic structure. On the other hand, it can be shown that $(R^{2k+1}, \phi, \xi, \eta, g)$ is a cosymplectic manifold.

Let $(\mathcal{M}_1, g_{\mathcal{M}_1})$ and $(\mathcal{M}_2, g_{\mathcal{M}_2})$ be Riemannian manifolds, with $dim(\mathcal{M}_1) = m$ and $dim(\mathcal{M}_2) = n$ and $m > n$. A *Riemannian submersion* $h : \mathcal{M}_1 \to \mathcal{M}_2$ is a map from $\mathcal{M}_1$ onto $\mathcal{M}_2$ satisfying the following two axioms:

(A_1) h has maximal rank.

(A_2) The differential h_* preserves the lengths of the horizontal vectors.

For each $q \in \mathcal{M}_2$, $h^{-1}(q)$ is an $m-n$ dimensional submanifold of $\mathcal{M}_1$, so-called fiber. If a vector field on $\mathcal{M}_1$ is always tangent (or orthogonal) to fibers then it is called vertical (or horizontal) [26]. A vector field Z_1 on $\mathcal{M}_1$ is said to be basic if it is horizontal and h-related to a vector field $(Z_1)_*$ on $\mathcal{M}_2$, i.e., $h_*(Z_1)_p = (Z_1)_{*h(p)}$, for all $p \in \mathcal{M}_1$. We denote the projection morphisms on the vertical distribution $(kerh_*)$ and the horizontal distribution $(kerh_*)^{\perp}$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

A Riemannian submersion $h : \mathcal{M}_1 \to \mathcal{M}_2$ determines two $(1, 2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on $\mathcal{M}_1$. These tensor fields are called the fundamental tensor fields (O'Neill's tensors) or the invariants of h. For arbitrary vector fields E and L on $\mathcal{M}_1$, these tensor fields can be given by the formulas

$$\mathcal{A}_E L = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}L + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}L, \tag{2.6}$$

$$\mathcal{T}_E L = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}L + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}L, \tag{2.7}$$

where ∇ is the Levi-Civita connection of $g_{\mathcal{M}_1}$. It is easy to see that $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on the tangent bundle of $\mathcal{M}_1$ reversing the vertical and the horizontal distributions.

From equations (2.6) and (2.7), results in

$$\nabla_{Z_1} Z_2 = \mathcal{T}_{Z_1} Z_2 + \widehat{\nabla}_{Z_1} Z_2, \tag{2.8}$$

$$\nabla_{Z_1} V_1 = \mathcal{T}_{Z_1} V_1 + \mathcal{H}\nabla_{Z_1} V_1, \tag{2.9}$$

$$\nabla_{V_1} Z_1 = \mathcal{A}_{V_1} Z_1 + \mathcal{V}\nabla_{V_1} Z_1, \tag{2.10}$$

$$\nabla_{V_1} V_2 = \mathcal{H}\nabla_{V_1} V_2 + \mathcal{A}_{V_1} V_2, \tag{2.11}$$

for all $Z_1, Z_2 \in \Gamma(\ker h_*)$ and $V_1, V_2 \in \Gamma(\ker h_*)^{\perp}$, where $\widehat{\nabla}_{Z_1} Z_2 = \mathcal{V}\nabla_{Z_1} Z_2$ and $\mathcal{H}\nabla_{Z_1} V_1 = \mathcal{A}_{V_1} Z_1$, if V_1 is basic. It can be easily observed that $\mathcal{T}$ works at the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of the same distribution.

Clearly, for $q \in \mathcal{M}_1$, $X_1 \in \Gamma(\ker h_*)^\perp$ and $X_2 \in \Gamma(\ker h_*)$ the linear operators

$$\mathcal{A}_{X_1},\ \mathcal{T}_{X_2} : T_q\mathcal{M}_1 \to T_q\mathcal{M}_1,$$

are skew-symmetric, that is

$$g_{\mathcal{M}_1}(\mathcal{A}_{X_1}E, L) = -g_{\mathcal{M}_1}(E, \mathcal{A}_{X_1}L) \text{ and } g_{\mathcal{M}_1}(\mathcal{T}_{X_2}E, L) = -g_{\mathcal{M}_1}(E, \mathcal{T}_{X_2}L), \tag{2.12}$$

for each $E, L \in T_q\mathcal{M}_1$. Since $\mathcal{T}_{X_2}$ is skew-symmetric, we observe that h has totally geodesic fibres if and only if $\mathcal{T} \equiv 0$.

Suppose $(\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1})$ be a cosymplectic manifold, $(\mathcal{M}_2, g_{\mathcal{M}_2})$ be a Riemannian manifold and $h : \mathcal{M}_1 \to \mathcal{M}_2$ is smooth map. Therefore the second fundamental form of h is

$$(\nabla h_*)(Y_1, Y_2) = \nabla^h_{Y_1} h_* Y_2 - h_*(\nabla_{Y_1} Y_2), \quad \text{for all } Y_1, Y_2 \in \Gamma(T_p\mathcal{M}_1), \tag{2.13}$$

where we denote conveniently by ∇ the Levi-Civita connection of the metrics $g_{\mathcal{M}_1}$ and $g_{\mathcal{M}_2}$ and ∇^h is the pullback connection.

We recall that a differentiable map h between two Riemannian manifolds is said to be a harmonic map if $trace(\nabla h_*) = 0$ and h is called a totally geodesic if

$$(\nabla h_*)(Y_1, Y_2) = 0, \text{ for all } Y_1, Y_2 \in \Gamma(T\mathcal{M}_1). \tag{2.14}$$

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Now, we recall the following lemma from O'Neill [22], which is used throughout this paper.

Lemma 2.2. Let $h : \mathcal{M}_1 \to \mathcal{M}_2$ be a Riemannian submersion between Riemannian manifolds and Z_1, Z_2 be basic vector fields of $\mathcal{M}_1$. Then

(a) $g_{\mathcal{M}_1}(Z_1, Z_2) = g_{\mathcal{M}_2}(Z_{1*}, Z_{2*}) \circ h$,

(b) the horizontal part $[Z_1, Z_2]^{\mathcal{H}}$ of $[Z_1, Z_2]$ is a basic vector field and corresponds to $[Z_{1*}, Z_{2*}]$, i.e., $h_*([Z_1, Z_2]^{\mathcal{H}}) = [Z_{1*}, Z_{2*}]$,

(c) $[X_1, Z_1]$ is vertical for any vector field X_1 of $\ker h_*$,

(d) $(\nabla^{\mathcal{M}_1}_{Z_1} Z_2)^{\mathcal{H}}$ is the basic vector field corresponding to $(\nabla^{\mathcal{M}_2}_{Z_{1*}} Z_{2*})$,

where $\nabla^{\mathcal{M}_1}$ and $\nabla^{\mathcal{M}_2}$ are the Levi-Civita connection on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively.

Now the coming lemma can be proved as in [4].

Lemma 2.3. Let h be a Riemannian submersion from a Riemannian manifold $(\mathcal{M}_1, g_{\mathcal{M}_1})$ onto an other Riemannian manifold $(\mathcal{M}_2, g_{\mathcal{M}_2})$, then we have

(i) $(\nabla h_*)(U_1, U_2) = 0$,

(ii) $(\nabla h_*)(Z_1, Z_2) = -h_*(\mathcal{T}_{Z_1} Z_2) = -h_*(\nabla_{Z_1} Z_2)$,

(iii) $(\nabla h_*)(U_1, Z_1) = -h_*(\nabla_{U_1} Z_1) = -h_*(\mathcal{A}_{U_1} Z_1)$, where U_1 and U_2 are horizontal vector fields, and Z_1 and Z_2 are vertical vector fields.

3 Hemi-slant Riemannian submersions from cosymplectic manifolds

In this section, we define and study the concept of hemi-slant submersions from cosymplectic manifolds onto Riemannian manifolds.

Definition 3.1. [28] Let h be a Riemannian submersion from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1})$ onto a Riemannian manifold $(\mathcal{M}_2, g_{\mathcal{M}_2})$. Then we say that h is a hemi-slant submersion if the vertical distribution $\ker h_*$ of h admits three orthogonal complementary distributions D^θ, $D^\perp$ and $< \xi >$ such that D^θ is slant with angle θ and $D^\perp$ is anti-invariant, i.e.,

$$\ker h_* = D^\theta \oplus D^\perp \oplus < \xi > .$$

In this case, the angle θ is called the hemi-slant angle of the submersion.

We note that a hemi-slant submersion is proper if $\dim D^\perp \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$.

Let h be hemi-slant submersion from an almost contact metric manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1})$ onto a Riemannian manifold $(\mathcal{M}_2, g_{\mathcal{M}_2})$. Then, we have

$$T\mathcal{M}_1 = \ker h_* \oplus (\ker h_*)^\perp. \tag{3.1}$$

Now, for any vector field $W_1 \in \Gamma(\ker h_*)$, we choose

$$W_1 = \mathcal{P}W_1 + \mathcal{Q}W_1 + \eta(W_1)\xi, \tag{3.2}$$

where $\mathcal{P}$ and $\mathcal{Q}$ indicates to the projection morphisms of $\ker h_*$ onto D^θ and $D^\perp$, respectively.

For all $Y_1 \in \Gamma(\ker h_*)$, we set

$$\phi Y_1 = \psi Y_1 + \omega Y_1, \tag{3.3}$$

where $\psi Y_1 \in \Gamma(\ker h_*)$ and $\omega Y_1 \in \Gamma(\ker h_*)^\perp$.

From equations (3.2) and (3.3), we get

$$\phi W_1 = \psi(\mathcal{P}W_1) + \omega(\mathcal{P}W_1) + \psi(\mathcal{Q}W_1) + \omega(\mathcal{Q}W_1).$$

Since $\phi D^\perp \subset (\ker h_*)^\perp$, therefore $\psi(\mathcal{Q}W_1) = 0$. Hence we obtain

$$\phi W_1 = \psi(\mathcal{P}W_1) + \omega(\mathcal{P}W_1) + \omega(\mathcal{Q}W_1).$$

Thus we have

$$\phi(\ker h_*) = \psi D^\theta \oplus \omega D^\theta \oplus \omega D^\perp,$$

where $\oplus$ defines orthogonal direct sum.

Since $\omega D^\theta \subseteq (\ker h_*)^\perp$, $\omega D^\perp (= \phi D^\perp) \subseteq (\ker h_*)^\perp$ then, the horizontal distribution $(\ker h_*)^\perp$ is decomposed as

$$(\ker h_*)^\perp = \omega D^\theta \oplus \phi D^\perp \oplus \mu,$$

where μ is the orthogonal complementary of $\omega D^\theta \oplus \phi D^\perp$ in $(\ker h_*)^\perp$ and it is invariant with respect to ϕ.

Also for all $X_1 \in \Gamma(\ker h_*)^\perp$, we get

$$\phi X_1 = \mathcal{B}X_1 + \mathcal{C}X_1, \tag{3.4}$$

where $\mathcal{B}X_1 \in \Gamma(\ker h_*)$ and $\mathcal{C}X_1 \in \Gamma(\ker h_*)^\perp$.

Lemma 3.2. Let h be a hemi-slant submersion from an almost contact metric manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1})$ onto a Riemannian manifold $(\mathcal{M}_2, g_{\mathcal{M}_2})$. Then, we have

$$\begin{aligned} (i)\ \ \psi D^{\perp} &= \{0\},\ \ (ii)\ \ \psi D^{\theta} = D^{\theta}, \\ (iii)\ \ \mathcal{B}\omega D^{\theta} &= D^{\theta},\ (iv)\ \ \mathcal{B}\phi D^{\perp} = D^{\perp}. \end{aligned}$$

Lemma 3.3. Let h be a hemi-slant submersion from an almost contact metric manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1})$ onto a Riemannian manifold $(\mathcal{M}_2, g_{\mathcal{M}_2})$. Then, we have

$$(i)\ \psi^2 Z_1 + \mathcal{B}\omega Z_1 = -Z_1 + \eta(Z_1) \otimes \xi,\ (ii)\ \omega\psi Z_1 + \mathcal{C}\omega Z_1 = 0,$$

$$(iii)\ \omega\mathcal{B}Z_2 + \mathcal{C}^2 Z_2 = -Z_2,\ (iv)\ \psi\mathcal{B}Z_2 + \mathcal{B}\mathcal{C}Z_2 = 0,$$

for all $Z_1 \in \Gamma(\ker h_*)$ and $Z_2 \in \Gamma(\ker h_*)^{\perp}$.

Proof. By making use of equations (2.1), (3.3) and (3.4), lemma follows. ∎

Lemma 3.4. Let h be a hemi-slant submersion from an almost contact metric manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1})$ onto a Riemannian manifold $(\mathcal{M}_2, g_{\mathcal{M}_2})$. Then, we have

$$\psi^2 U_1 = -(\cos^2\theta) U_1,$$

for all $U_1 \in \Gamma(D^{\theta})$.

Proof. For a non-zero vector field $U_1 \in \Gamma(D^{\theta})$, we have

$$\cos\theta = \frac{\| \psi U_1 \|}{\| \phi U_1 \|}, \tag{3.5}$$

and

$$\cos\theta = \frac{g_{\mathcal{M}_1}(\phi U_1, \psi U_1)}{\| \phi U_1 \| \| \psi U_1 \|}.$$

By using equations (2.1), (2.2) and (3.3), we have

$$\begin{aligned} \cos\theta &= \frac{g_{\mathcal{M}_1}(\psi U_1, \psi U_1)}{\| U_1 \| \| \psi U_1 \|}, \\ &= \frac{-g_{\mathcal{M}_1}(U_1, \psi^2 U_1)}{\| U_1 \| \| \psi U_1 \|}. \end{aligned} \tag{3.6}$$

From equations (3.5) and (3.6), we get

$$\cos^2\theta = \frac{-g_{\mathcal{M}_1}(U_1, \psi^2 U_1)}{\| U_1 \| \| \psi U_1 \|} \frac{\| \psi U_1 \|}{\| \phi U_1 \|}.$$

Then, we obtain

$$g_{\mathcal{M}_1}(\cos^2\theta\ U_1, U_1) = -g_{\mathcal{M}_1}(U_1, \psi^2 U_1),$$

which gives the assertion. ∎

From lemma 3.4, one can easily obtain coming lemma:

Lemma 3.5. Let h be a hemi-slant submersion from an almost contact metric manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1})$ onto a Riemannian manifold $(\mathcal{M}_2, g_{\mathcal{M}_2})$. Then, we have

$$g_{\mathcal{M}_1}(\psi Z_1, \psi Z_2) = \cos^2\theta g_{\mathcal{M}_1}(Z_1, Z_2),$$

and

$$g_{\mathcal{M}_1}(\omega Z_1, \omega Z_2) = \sin^2\theta g_{\mathcal{M}_1}(Z_1, Z_2),$$

for all $Z_1, Z_2 \in \Gamma(D^\theta)$.

Next, throughout this section, we take $(\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1})$ as a cosymplectic manifold and $(\mathcal{M}_2, g_{\mathcal{M}_2})$ as a Riemannian manifold.

Lemma 3.6. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion. Then, we have

$$\mathcal{V}\nabla_{Z_1}\psi Z_2 + \mathcal{T}_{Z_1}\omega Z_2 = \psi\mathcal{V}\nabla_{Z_1}Z_2 + \mathcal{B}\mathcal{T}_{Z_1}Z_2, \tag{3.7}$$

$$\mathcal{T}_{Z_1}\psi Z_2 + \mathcal{H}\nabla_{Z_1}\omega Z_2 = \omega\mathcal{V}\nabla_{Z_1}Z_2 + \mathcal{C}\mathcal{T}_{Z_1}Z_2, \tag{3.8}$$

$$\mathcal{V}\nabla_{W_1}\mathcal{B}W_2 + \mathcal{A}_{W_1}\mathcal{C}W_2 = \psi\mathcal{A}_{W_1}W_2 + \mathcal{B}\mathcal{H}\nabla_{W_1}W_2, \tag{3.9}$$

$$\mathcal{A}_{W_1}\mathcal{B}W_2 + \mathcal{H}\nabla_{W_1}\mathcal{C}W_2 = \omega\mathcal{A}_{W_1}W_2 + \mathcal{C}\mathcal{H}\nabla_{W_1}W_2, \tag{3.10}$$

$$\mathcal{V}\nabla_{Z_1}\mathcal{B}W_1 + \mathcal{T}_{Z_1}\mathcal{C}W_1 = \psi\mathcal{T}_{Z_1}W_1 + \mathcal{B}\mathcal{H}\nabla_{Z_1}W_1, \tag{3.11}$$

$$\mathcal{T}_{Z_1}\mathcal{B}W_1 + \mathcal{H}\nabla_{Z_1}\mathcal{C}W_1 = \omega\mathcal{T}_{Z_1}W_1 + \mathcal{C}\mathcal{H}\nabla_{Z_1}W_1, \tag{3.12}$$

$$\mathcal{V}\nabla_{W_2}\psi Z_1 + \mathcal{A}_{W_2}\omega Z_1 = \mathcal{B}\mathcal{A}_{W_2}Z_1 + \psi\mathcal{V}\nabla_{W_2}Z_1, \tag{3.13}$$

$$\mathcal{A}_{W_2}\psi Z_1 + \mathcal{H}\nabla_{W_2}\omega Z_1 = \mathcal{C}\mathcal{A}_{W_2}Z_1 + \omega\mathcal{V}\nabla_{W_2}Z_1, \tag{3.14}$$

for any $Z_1, Z_2 \in \Gamma(\ker h_*)$ and $W_1, W_2 \in \Gamma(\ker h_*)^\perp$.

Proof. Using equations (2.1) – (2.3) and (2.8) – (2.11), lemma follows. ∎

Now, we define

$$(\nabla_{U_1}\psi)U_2 = \mathcal{V}\nabla_{U_1}\psi U_2 - \psi\mathcal{V}\nabla_{U_1}U_2, \tag{3.15}$$

$$(\nabla_{U_1}\omega)U_2 = \mathcal{H}\nabla_{U_1}\omega U_2 - \omega\mathcal{V}\nabla_{U_1}U_2, \tag{3.16}$$

$$(\nabla_{Y_1}\mathcal{B})Y_2 = \mathcal{V}\nabla_{Y_1}\mathcal{B}Y_2 - \mathcal{B}\mathcal{H}\nabla_{Y_1}Y_2, \tag{3.17}$$

$$(\nabla_{Y_1}\mathcal{C})Y_2 = \mathcal{H}\nabla_{Y_1}\mathcal{C}Y_2 - \mathcal{C}\mathcal{H}\nabla_{Y_1}Y_2, \tag{3.18}$$

for any $U_1, U_2 \in \Gamma(\ker h_*)$ and $Y_1, Y_2 \in \Gamma(\ker h_*)^\perp$.

Lemma 3.7. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion. Then, we have

$$(\nabla_{U_1}\psi)U_2 = \mathcal{B}\mathcal{T}_{U_1}U_2 - \mathcal{T}_{U_1}\omega U_2,$$

$$(\nabla_{U_1}\omega)U_2 = \mathcal{C}\mathcal{T}_{U_1}U_2 - \mathcal{T}_{U_1}\psi U_2,$$

$$(\nabla_{Z_1}\mathcal{B})Z_2 = \psi\mathcal{A}_{Z_1}Z_2 - \mathcal{A}_{Z_1}\mathcal{C}Z_2,$$

$$(\nabla_{Z_1}\mathcal{C})Z_2 = \omega\mathcal{A}_{Z_1}Z_2 - \mathcal{A}_{Z_1}\mathcal{B}Z_2,$$

for any vectors $U_1, U_2 \in \Gamma(\ker h_*)$ and $Z_1, Z_2 \in \Gamma(\ker h_*)^\perp$.

Proof. By the use of equations (3.7) – (3.10) and (3.15) – (3.18), lemma follows. ∎

From lemma 3.7, we have the coming result:

Corollary 3.8. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion. If the tensors ψ and ω are parallel with respect to the linear connection ∇ on $\mathcal{M}_1$, then

$$\mathcal{B}\mathcal{T}_{V_1}V_2 = \mathcal{T}_{V_1}\omega V_2,$$

and

$$\mathcal{C}\mathcal{T}_{V_1}V_2 = \mathcal{T}_{V_1}\psi V_2,$$

respectively, for any $V_1, V_2 \in \Gamma(\ker h_*)$.

Let h be a proper hemi-slant submersion from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1})$ onto a Riemannian manifold $(\mathcal{M}_2, g_{\mathcal{M}_2})$. Then we say that the fibers of h are mixed geodesic, if $\mathcal{T}_{V_1}V_2 = 0$, for $V_1 \in D^\theta$ and $V_2 \in D^\perp$.

Lemma 3.9. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ is a proper hemi-slant submersion. If ω is parallel with respect to ∇ on $\ker h_*$, then

$$\mathcal{T}_{W_3}W_1 = 0, \quad \text{if } \ \mathcal{C} \equiv 0$$

and

$$\mathcal{T}_{W_3}W_2 = 0, \quad \text{if } \ \mathcal{C} \neq 0,$$

for all $W_1 \in D^\theta$, $W_2 \in D^\perp$ and $W_3 \in \ker h_*$.

Proof. If $\mathcal{C} \equiv 0$, then for all $W_1 \in D^\theta$ and $W_3 \in \ker h_*$, corollary 3.8 gives

$$\mathcal{T}_{W_3}\psi W_1 = 0,$$

putting $W_1 = \psi W_1$ and using lemma 3.4 in above equation, we obtain

$$\cos^2\theta \mathcal{T}_{W_3}W_1 = 0.$$

Since h is proper hemi-slant submersion, we have first assertion.

Next, for all $W_2 \in D^\perp$ and $W_3 \in \ker h_*$, from corollary 3.8 and lemma 3.2(i), we obtain

$$\mathcal{C}\mathcal{T}_{W_3}W_2 = \mathcal{T}_{W_3}\psi W_2 = 0.$$

Since $\mathcal{C} \neq 0$, we have

$$\mathcal{T}_{W_3}W_2 = 0,$$

which completes the proof. ∎

Moreover, from lemma 3.9, one can easily see that the fibers of h are mixed geodesic.

Lemma 3.10. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion. If ω is parallel with respect to ∇ on $\mathcal{D}^\theta$, then we have

$$\mathcal{T}_{\psi W_1}\psi W_1 = -\cos^2\theta.\mathcal{T}_{W_1}W_1, \text{ for every } W_1 \in \Gamma(\mathcal{D}^\theta).$$

Proof. If ω is parallel, then from Lemma 3.7, we have

$$\mathcal{C}\mathcal{T}_{W_1}W_2 - \mathcal{T}_{W_1}\psi W_2 = 0,$$

$$\mathcal{C}\mathcal{T}_{W_1}W_2 = \mathcal{T}_{W_1}\psi W_2, \text{ for all } W_1, W_2 \in \Gamma(\mathcal{D}^\theta). \tag{3.19}$$

Interchanging the role of W_1 and W_2 in the above equation, we have

$$\mathcal{C}\mathcal{T}_{W_2}W_1 = \mathcal{T}_{W_2}\psi W_1, \text{ for all } W_1, W_2 \in \Gamma(\mathcal{D}^\theta) \tag{3.20}$$

Since $\mathcal{T}$ is symmetric, from equations (3.19), (3.20), and lemma 3.4, assertion follows. ∎

Theorem 3.11. Let h be a Riemannian submersion from an almost contact metric manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1})$ onto a Riemannian manifold $(\mathcal{M}_2, g_{\mathcal{M}_2})$. Then h is a hemi-slant submersion if and only if there exists a constant $\lambda \in [-1, 0]$ and a distribution D on $\ker h_*$ such that
(a) $D = \{Z_1 \in \ker h_* \mid \psi^2 Z_1 = \lambda Z_1\}$,
(b) for any $Z_1 \in \ker h_*$ orthogonal to D, we have $\psi Z_1 = 0$.

Theorem 3.12. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion with the slant angle θ. Then the slant distribution $\mathcal{D}^\theta$ is integrable if and only if

$$g_{\mathcal{M}_1}(\mathcal{T}_{U_1}\omega\psi U_2 - \mathcal{T}_{U_2}\omega\psi U_1, V_1) = g_{\mathcal{M}_1}(\mathcal{H}\nabla_{U_1}\omega U_2 - \mathcal{H}\nabla_{U_2}\omega U_1, \phi V_1),$$

for all $U_1, U_2 \in \Gamma(\mathcal{D}^\theta)$ and $V_1 \in \Gamma(\mathcal{D}^\perp)$.

Proof. We note that $\mathcal{D}^\theta$ is integrable if and only if $g_{\mathcal{M}_1}([U_1, U_2], V_1) = 0$, $g_{\mathcal{M}_1}([U_1, U_2], V_2) = 0$, and $g_{\mathcal{M}_1}([U_1, U_2], \xi) = 0$, for all $U_1, U_2 \in \Gamma(\mathcal{D}^\theta), V_1 \in \Gamma(\mathcal{D}^\perp)$ and $V_2 \in (\ker h_*)^\perp$. Since $\ker h_*$ is integrable then $g_{\mathcal{M}_1}([U_1, U_2], V_2) = 0$. Thus, $\mathcal{D}^\theta$ is integrable if and only if $g_{\mathcal{M}_1}([U_1, U_2], V_1) = 0$ and $g_{\mathcal{M}_1}([U_1, U_2], \xi) = 0$.

Now, after a straightforward computation, we obtain $g_{\mathcal{M}_1}([U_1, U_2], \xi) = 0$.

Next, from equations (2.2), (2.4), (3.3) and Lemma 3.4, we have

$$\begin{aligned} g_{\mathcal{M}_1}([U_1, U_2], V_1) &= g_{\mathcal{M}_1}(\nabla_{U_1}U_2, V_1) - g_{\mathcal{M}_1}(\nabla_{U_2}U_1, V_1), \\ &= g_{\mathcal{M}_1}(\nabla_{U_1}\psi U_2, \phi V_1) + g_{\mathcal{M}_1}(\nabla_{U_1}\omega U_2, \phi V_1) - g_{\mathcal{M}_1}(\nabla_{U_2}\psi U_1, \phi V_1) - \\ &\quad g_{\mathcal{M}_1}(\nabla_{U_2}\omega U_1, \phi V_1), \\ &= g_{\mathcal{M}_1}(\nabla_{U_1}(\cos^2\theta)U_2, V_1) - g_{\mathcal{M}_1}(\nabla_{U_1}\omega\psi U_2, V_1) + g_{\mathcal{M}_1}(\nabla_{U_1}\omega U_2, \phi V_1) \\ &\quad - g_{\mathcal{M}_1}(\nabla_{U_2}(\cos^2\theta)U_1, V_1) + g_{\mathcal{M}_1}(\nabla_{U_2}\omega\psi U_1, V_1) - g_{\mathcal{M}_1}(\nabla_{U_2}\omega U_1, \phi V_1), \\ &= \cos^2\theta g_{\mathcal{M}_1}([U_1, U_2], V_1) - g_{\mathcal{M}_1}(\nabla_{U_1}\omega\psi U_2, V_1) + g_{\mathcal{M}_1}(\nabla_{U_2}\omega\psi U_1, V_1) \\ &\quad + g_{\mathcal{M}_1}(\nabla_{U_1}\omega U_2, \phi V_1) - g_{\mathcal{M}_1}(\nabla_{U_2}\omega U_1, \phi V_1). \end{aligned}$$

Next, using equation (2.9), we have

$$\begin{aligned} &\sin^2\theta g_{\mathcal{M}_1}([U_1, U_2], V_1) \\ = &-g_{\mathcal{M}_1}(\mathcal{T}_{U_1}\omega\psi U_2, V_1) + g_{\mathcal{M}_1}(\mathcal{T}_{U_2}\omega\psi U_1, V_1) \\ &+ g_{\mathcal{M}_1}(\mathcal{H}\nabla_{U_1}\omega U_2, \phi V_1) - g_{\mathcal{M}_1}(\mathcal{H}\nabla_{U_2}\omega U_1, \phi V_1), \end{aligned}$$

which completes the proof. ∎

From Theorem 3.12, we have the following sufficient conditions for slant distribution $\mathcal{D}^\theta$ to be integrable:

Corollary 3.13. Let h be a hemi-slant submersion from a cosymplectic manifold $(\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1})$ onto a Riemannian manifold $(\mathcal{M}_2, g_{\mathcal{M}_2})$. If, for any $Z_1, Z_2 \in \Gamma(\mathcal{D}^\theta)$

$$\mathcal{T}_{Z_1}\omega\psi Z_2 - \mathcal{T}_{Z_2}\omega\psi Z_1 \in \Gamma(\mathcal{D}^\theta)$$

and

$$\mathcal{H}\nabla_{Z_1}\omega Z_2 - \mathcal{H}\nabla_{Z_2}\omega Z_1 \in \omega\mathcal{D}^\theta \oplus \mu,$$

then slant distribution $\mathcal{D}^\theta$ is integrable.

Theorem 3.14. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion. Then the anti-invariant distribution $\mathcal{D}^\perp$ is always integrable.

Proof. For any $U_1 \in \Gamma(\ker h_*)$ and $U_2, U_3 \in \Gamma(\mathcal{D}^\perp)$, we have $\Phi(U_1, U_2) = 0$, $\Phi(U_1, U_3) = 0$ and $\Phi(U_2, U_3) = 0$. Since, on a cosymplectic manifold fundamental $2-$form Φ is closed, i.e., $d\Phi = 0$. Now, using integrability of $\ker h_*$, we have

$$\begin{aligned}
0 &= 3d\Phi(U_1, U_2, U_3), \\
&= U_1\Phi(U_2, U_3) - U_2\Phi(U_1, U_3) + U_3\Phi(U_1, U_2), \\
&\quad -\Phi([U_1, U_2], U_3) + \Phi([U_1, U_3], U_2) - \Phi([U_2, U_3], U_1), \\
&= -\Phi([U_2, U_3], U_1), \\
&= -g([U_2, U_3], \psi U_1).
\end{aligned}$$

Since U_1 is an arbitrary vector field in $\ker h_*$ and $[U_2, U_3]$ is tangent to $\mathcal{M}_1$, therefore $[U_2, U_3] \in \mathcal{D}^\perp$, which completes the proof. ∎

Corollary 3.15. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion. Therefore, we have

$$\mathcal{T}_{Y_1}\phi Y_2 = \mathcal{T}_{Y_2}\phi Y_1, \tag{3.21}$$

for any $Y_1, Y_2 \in \Gamma(\mathcal{D}^\perp)$.

Proof. For any $Y_1, Y_2 \in \Gamma(\mathcal{D}^\perp)$, using equation (3.7) and lemma 3.2 (i), we have

$$\mathcal{T}_{Y_1}\omega Y_2 = \psi\widehat{\nabla}_{Y_1}Y_2 + \mathcal{B}\mathcal{T}_{Y_1}Y_2 \tag{3.22}$$

Interchanging the role of Y_1 and Y_2 in the above equation, we have

$$\mathcal{T}_{Y_2}\omega Y_1 = \psi\widehat{\nabla}_{Y_2}Y_1 + \mathcal{B}\mathcal{T}_{Y_2}Y_1 \tag{3.23}$$

Since $\mathcal{T}$ is symmetric, from equations (3.22) and (3.23), we get

$$\mathcal{T}_{Y_1}\omega Y_2 - \mathcal{T}_{Y_2}\omega Y_1 = \psi[Y_1, Y_2].$$

By making the use of Theorem 3.14, Lemma 3.2 (i) and the fact that $\omega Y_1 = \phi Y_1$ for any $Y_1 \in \Gamma(\mathcal{D}^\perp)$, we have the assertion. ∎

Theorem 3.16. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion with the slant angle θ. Then the horizontal distribution $(\ker h_*)^\perp$ is totally geodesic foliation on $\mathcal{M}_1$ if and only if

$$\cos^2\theta g_{\mathcal{M}_1}(\mathcal{A}_{Y_1}Y_2, \mathcal{P}Y_3) = g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Y_1}Y_2, \omega\psi\mathcal{P}Y_3) - g_{\mathcal{M}_1}(\mathcal{A}_{Y_1}\mathcal{B}Y_2 + \mathcal{H}\nabla_{Y_1}\mathcal{C}Y_2, \omega Y_3), \tag{3.24}$$

for all $Y_1, Y_2 \in \Gamma(\ker h_*)^\perp$ and $Y_3 \in \Gamma(\ker h_*)$.

Proof. For all $Y_1, Y_2 \in \Gamma(\ker h_*)^\perp$ and $Y_3 \in \Gamma(\ker h_*)$, using equations (2.1) – (2.4), (3.2), (3.3), (3.4) and Lemma 3.4, we have

$$\begin{aligned} g_{\mathcal{M}_1}(\nabla_{Y_1}Y_2, Y_3) &= g_{\mathcal{M}_1}(\nabla_{Y_1}Y_2, \mathcal{P}Y_3 + \mathcal{Q}Y_3 + \eta(Y_3)\xi), \\ &= g_{\mathcal{M}_1}(\nabla_{Y_1}\phi Y_2, \psi\mathcal{P}Y_3) + g_{\mathcal{M}_1}(\nabla_{Y_1}\phi Y_2, \omega\mathcal{P}Y_3) + g_{\mathcal{M}_1}(\nabla_{Y_1}\phi Y_2, \phi\mathcal{Q}Y_3), \\ &= \cos^2\theta g_{\mathcal{M}_1}(\nabla_{Y_1}Y_2, \mathcal{P}Y_3) - g_{\mathcal{M}_1}(\nabla_{Y_1}Y_2, \omega\psi\mathcal{P}Y_3) + g_{\mathcal{M}_1}(\nabla_{Y_1}\mathcal{B}Y_2, \omega\mathcal{P}Y_3) \\ &\quad + g_{\mathcal{M}_1}(\nabla_{Y_1}\mathcal{C}Y_2, \omega\mathcal{P}Y_3) + g_{\mathcal{M}_1}(\nabla_{Y_1}\mathcal{B}Y_2, \phi\mathcal{Q}Y_3) + g_{\mathcal{M}_1}(\nabla_{Y_1}\mathcal{C}Y_2, \phi\mathcal{Q}Y_3), \end{aligned}$$

Now, using (2.10) and (2.11) , we obtain

$$\begin{aligned} g_{\mathcal{M}_1}(\nabla_{Y_1}Y_2, Y_3) &= \cos^2\theta g_{\mathcal{M}_1}(\mathcal{A}_{Y_1}Y_2, \mathcal{P}Y_3) - g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Y_1}Y_2, \omega\psi\mathcal{P}Y_3) \\ &\quad + g_{\mathcal{M}_1}(\mathcal{A}_{Y_1}\mathcal{B}Y_2, \omega\mathcal{P}Y_3 + \phi\mathcal{Q}Y_3) + g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Y_1}\mathcal{C}Y_2, \omega\mathcal{P}Y_3 + \phi\mathcal{Q}Y_3). \end{aligned}$$

Since $\omega Y_3 = \omega\mathcal{P}Y_3 + \phi\mathcal{Q}Y_3$, we have

$$\begin{aligned} g_{\mathcal{M}_1}(\nabla_{Y_1}Y_2, Y_3) &= \cos^2\theta g_{\mathcal{M}_1}(\mathcal{A}_{Y_1}Y_2, \mathcal{P}Y_3) - g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Y_1}Y_2, \omega\psi\mathcal{P}Y_3) \\ &\quad + g_{\mathcal{M}_1}(\mathcal{A}_{Y_1}\mathcal{B}Y_2, \omega Y_3) + g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Y_1}\mathcal{C}Y_2, \omega Y_3), \end{aligned}$$

which completes the proof. ∎

Theorem 3.17. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion with the slant angle θ. Then the vertical distribution $\ker h_*$ is a totally geodesic foliation on $\mathcal{M}_1$ if and only if

$$\begin{aligned} &\cos^2\theta g_{\mathcal{M}_1}(\mathcal{T}_{Z_1}\mathcal{P}Z_2, Z_3) \\ = \; &g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Z_1}\omega\psi\mathcal{P}Z_2, Z_3) - g_{\mathcal{M}_1}(\mathcal{T}_{Z_1}\omega Z_2, \mathcal{B}Z_3) - g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Z_1}\omega Z_2, \mathcal{C}Z_3), \end{aligned}$$

for all $Z_1, Z_2 \in \Gamma(\ker h_*)$ and $Z_3 \in \Gamma(\ker h_*)^\perp$.

Proof. For all $Z_1, Z_2 \in \Gamma(\ker h_*)$ and $Z_3 \in \Gamma(\ker h_*)^\perp$, using equations (2.1) – (2.4), (3.2), (3.3), (3.4) and Lemma 3.4, we have

$$\begin{aligned} g_{\mathcal{M}_1}(\nabla_{Z_1}Z_2, Z_3) &= g_{\mathcal{M}_1}(\nabla_{Z_1}\mathcal{P}Z_2, Z_3) + g_{\mathcal{M}_1}(\nabla_{Z_1}\mathcal{Q}Z_2, Z_3) + g_{\mathcal{M}_1}(\nabla_{Z_1}(\eta(Z_2)\xi), Z_3), \\ &= g_{\mathcal{M}_1}(\nabla_{Z_1}\psi\mathcal{P}Z_2, \phi Z_3) + g_{\mathcal{M}_1}(\nabla_{Z_1}\omega\mathcal{P}Z_2, \phi Z_3) + g_{\mathcal{M}_1}(\nabla_{Z_1}\phi\mathcal{Q}Z_2, \phi Z_3), \\ &= \cos^2\theta g_{\mathcal{M}_1}(\nabla_{Z_1}\mathcal{P}Z_2, Z_3) - g_{\mathcal{M}_1}(\nabla_{Z_1}\omega\psi\mathcal{P}Z_2, Z_3) \\ &\quad + g_{\mathcal{M}_1}(\nabla_{Z_1}(\omega\mathcal{P}Z_2 + \phi\mathcal{Q}Z_2), \mathcal{B}Z_3) + g_{\mathcal{M}_1}(\nabla_{Z_1}(\omega\mathcal{P}Z_2 + \phi\mathcal{Q}Z_2), \mathcal{C}Z_3). \end{aligned}$$

Since $\omega\mathcal{P}Z_2 + \phi\mathcal{Q}Z_2 = \omega Z_2$, and using (2.8) and (2.9) , we obtain

$$\begin{aligned} g_{\mathcal{M}_1}(\nabla_{Z_1}Z_2, Z_3) &= \cos^2\theta g_{\mathcal{M}_1}(\mathcal{T}_{Z_1}\mathcal{P}Z_2, Z_3) - g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Z_1}\omega\psi\mathcal{P}Z_2, Z_3) \\ &\quad + g_{\mathcal{M}_1}(\mathcal{T}_{Z_1}\omega Z_2, \mathcal{B}Z_3) + g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Z_1}\omega Z_2, \mathcal{C}Z_3), \end{aligned}$$

which gives us the proof. ∎

From Theorem 3.16 and Theorem 3.17, we have the following result:

Corollary 3.18. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion with the slant angle θ. Then, $\mathcal{M}_1$ is a locally product manifold of the form $(\mathcal{M}_1)_{(\ker h_*)} \times (\mathcal{M}_1)_{(\ker h_*)^\perp}$, where $(\mathcal{M}_1)_{(\ker h_*)}$ and $(\mathcal{M}_1)_{(\ker h_*)^\perp}$ are integral manifolds of the distributions $\ker h_*$ and $(\ker h_*)^\perp$ respectively, if and only if

$$\cos^2\theta g_{\mathcal{M}_1}(\mathcal{A}_{W_1}W_2, \mathcal{P}Z_1) = g_{\mathcal{M}_1}(\mathcal{H}\nabla_{W_1}W_2, \omega\psi\mathcal{P}Z_1) - g_{\mathcal{M}_1}(\mathcal{A}_{W_1}\mathcal{B}W_2 + \mathcal{H}\nabla_{W_1}\mathcal{C}W_2, \omega Z_1),$$

and

$$\cos^2\theta g_{\mathcal{M}_1}(\mathcal{T}_{Z_1}\mathcal{P}Z_2, W_2) = g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Z_1}\omega\psi\mathcal{P}Z_2, W_2) - g_{\mathcal{M}_1}(\mathcal{T}_{Z_1}\omega Z_2, \mathcal{B}W_2) - g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Z_1}\omega Z_2, \mathcal{C}W_2),$$

for any $Z_1, Z_2 \in \Gamma(\ker h_*)$ and $W_1, W_2 \in \Gamma(\ker h_*)^\perp$.

Theorem 3.19. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion. Then, the anti-invariant distribution $\mathcal{D}^\perp$ defines a totally geodesic foliation on $\mathcal{M}_1$ if and only if

$$g_{\mathcal{M}_1}(\mathcal{T}_{U_1}U_2, \omega\psi W_1) = g_{\mathcal{M}_1}(\mathcal{H}\nabla_{U_1}\omega U_2, \omega W_1),$$

and

$$g_{\mathcal{M}_1}(\mathcal{T}_{U_1}\phi U_2, \mathcal{B}W_2) = -g_{\mathcal{M}_1}(\mathcal{H}\nabla_{U_1}\phi U_2, \mathcal{C}W_1),$$

for all $U_1, U_2 \in \Gamma(\mathcal{D}^\perp)$, $W_1 \in \Gamma(\mathcal{D}^\theta)$ and $W_2 \in \Gamma(\ker h_*)^\perp$.

Proof. For all $U_1, U_2 \in \Gamma(\mathcal{D}^\perp)$, $W_1 \in \Gamma(\mathcal{D}^\theta)$, and $W_2 \in \Gamma(\ker h_*)^\perp$ using equations (2.2), (2.4), (3.3), (2.9) and Lemma 3.4, we have

$$\begin{aligned} & g_{\mathcal{M}_1}(\nabla_{U_1}U_2, W_1) \\ = \; & g_{\mathcal{M}_1}(\nabla_{U_1}\phi U_2, \psi W_1) + g_{\mathcal{M}_1}(\nabla_{U_1}\phi U_2, \omega W_1), \\ = \; & \cos^2\theta g_{\mathcal{M}_1}(\nabla_{U_1}U_2, W_1) - g_{\mathcal{M}_1}(\mathcal{T}_{U_1}U_2, \omega\psi W_1) + g_{\mathcal{M}_1}(\mathcal{H}\nabla_{U_1}\omega U_2, \omega W_1). \end{aligned}$$

Now, we have

$$\sin^2\theta g_{\mathcal{M}_1}(\nabla_{U_1}U_2, W_1) = -g_{\mathcal{M}_1}(\mathcal{T}_{U_1}U_2, \omega\psi W_1) + g_{\mathcal{M}_1}(\mathcal{H}\nabla_{U_1}\omega U_2, \omega W_1).$$

Next, using equations (2.2), (2.4), (3.4) and (2.9), we have

$$\begin{aligned} g_{\mathcal{M}_1}(\nabla_{U_1}U_2, W_2) &= g_{\mathcal{M}_1}(\nabla_{U_1}\phi U_2, \phi W_2) \\ &= g_{\mathcal{M}_1}(\mathcal{T}_{U_1}\phi U_2, \mathcal{B}W_2) + g_{\mathcal{M}_1}(\mathcal{H}\nabla_{U_1}\phi U_2, \mathcal{C}W_2). \end{aligned}$$

Also, $g_{\mathcal{M}_1}(\nabla_{U_1}U_2, \xi) = 0$. Therefore, the proof completed. ∎

Theorem 3.20. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion. Then the slant distribution $\mathcal{D}^\theta$ defines a totally geodesic foliation on $\mathcal{M}_1$ if and only if

$$g_{\mathcal{M}_1}(\mathcal{T}_{Y_1}\omega\psi Y_2, Z_1) = g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Y_1}\omega Y_2, \omega Z_1),$$

and

$$g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Y_1}\omega\psi Y_2, Z_2) = g_{\mathcal{M}_1}(\mathcal{T}_{Y_1}\omega Y_2, \mathcal{B}Z_2) + g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Y_1}\omega Y_2, \mathcal{C}Z_2),$$

for all $Y_1, Y_2 \in \Gamma(\mathcal{D}^\theta)$, $Z_1 \in \Gamma(\mathcal{D}^\perp)$ and $Z_2 \in \Gamma(\ker h_*)^\perp$.

Proof. For all $Y_1, Y_2 \in \Gamma(\mathcal{D}^\theta)$, $Z_1 \in \Gamma(\mathcal{D}^\perp)$ and $Z_2 \in \Gamma(\ker h_*)^\perp$, using equations (2.2), (2.4), (3.3), and Lemma 3.4, we have

$$\begin{aligned} g_{\mathcal{M}_1}(\nabla_{Y_1} Y_2, Z_1) &= g_{\mathcal{M}_1}(\nabla_{Y_1}\psi Y_2, \phi Z_1) + g_{\mathcal{M}_1}(\nabla_{Y_1}\omega Y_2, \phi Z_1), \\ &= \cos^2\theta g_{\mathcal{M}_1}(\nabla_{Y_1} Y_2, Z_1) - g_{\mathcal{M}_1}(\nabla_{Y_1}\omega\psi Y_2, Z_1) + g_{\mathcal{M}_1}(\nabla_{Y_1}\omega Y_2, \omega Z_1). \end{aligned}$$

Now using (2.9), we have

$$\sin^2\theta g_{\mathcal{M}_1}(\nabla_{Y_1} Y_2, Z_1) = -g_{\mathcal{M}_1}(\mathcal{T}_{Y_1}\omega\psi Y_2, Z_1) + g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Y_1}\omega Y_2, \omega Z_1).$$

Next, using equations (2.2), (2.4), (3.3), (3.4) , and Lemma 3.4, we have

$$\begin{aligned} g_{\mathcal{M}_1}(\nabla_{Y_1} Y_2, Z_2) &= g_{\mathcal{M}_1}(\nabla_{Y_1}\psi Y_2, \phi Z_2) + g_{\mathcal{M}_1}(\nabla_{Y_1}\omega Y_2, \phi Z_2), \\ &= \cos^2\theta g_{\mathcal{M}_1}(\nabla_{Y_1} Y_2, Z_2) - g_{\mathcal{M}_1}(\nabla_{Y_1}\omega\psi Y_2, Z_2) + g_{\mathcal{M}_1}(\nabla_{Y_1}\omega Y_2, \phi Z_2), \\ &= \cos^2\theta g_{\mathcal{M}_1}(\nabla_{Y_1} Y_2, Z_2) - g_{\mathcal{M}_1}(\nabla_{Y_1}\omega\psi Y_2, Z_2) \\ &\quad + g_{\mathcal{M}_1}(\nabla_{Y_1}\omega Y_2, \mathcal{B}Z_2) + g_{\mathcal{M}_1}(\nabla_{Y_1}\omega Y_2, \mathcal{C}Z_2). \end{aligned}$$

Now using (2.9), we have

$$\begin{aligned} \sin^2\theta g_{\mathcal{M}_1}(\nabla_{Y_1} Y_2, Z_2) &= -g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Y_1}\omega\psi Y_2, Z_2) + g_{\mathcal{M}_1}(\mathcal{T}_{Y_1}\omega Y_2, \mathcal{B}Z_2) \\ &\quad + g_{\mathcal{M}_1}(\mathcal{H}\nabla_{Y_1}\omega Y_2, \mathcal{C}Z_2). \end{aligned}$$

Also, $g_{\mathcal{M}_1}(\nabla_{Y_1} Y_2, \xi) = 0$. Therefore, the proof is completed. ∎

Theorem 3.21. Let $h : (\mathcal{M}_1, \phi, \xi, \eta, g_{\mathcal{M}_1}) \to (\mathcal{M}_2, g_{\mathcal{M}_2})$ be a hemi-slant submersion. Then, h is a totally geodesic map on $\mathcal{M}_1$ if and only if

$$\omega(\widehat{\nabla}_{U_1}\psi U_2 + \mathcal{T}_{U_1}\omega U_2) + \mathcal{C}(\mathcal{T}_{U_1}\psi U_2 + \mathcal{H}\nabla_{U_1}\omega U_2) = 0$$

and

$$\omega(\widehat{\nabla}_{U_1}\mathcal{B}W_1 + \mathcal{T}_{U_1}\mathcal{C}W_1) + \mathcal{C}(\mathcal{T}_{U_1}\mathcal{B}W_1 + \mathcal{H}\nabla_{U_1}\mathcal{C}W_1) = 0,$$

for all $U_1, U_2 \in \Gamma(\ker h_*)$, $W_1 \in \Gamma(\ker h_*)^\perp$.

Proof. Since h is Riemannian submersion, for any $W_1, W_2 \in \Gamma(\ker h_*)^\perp$, from Lemma 2.3, we have

$$\nabla h_*(W_1, W_2) = 0.$$

Next, for $U_1, U_2 \in \Gamma(\ker h_*)$, we have

$$\begin{aligned} (\nabla h_*)(U_1, U_2) &= \nabla^h_{U_1} h_* U_2 - h_*(\nabla_{U_1} U_2), \\ &= -h_*(\nabla_{U_1} U_2). \end{aligned}$$

From equations (2.1), (2.4), (3.3), (2.8) , (2.9) and (3.4) , we have

$$\begin{aligned} (\nabla h_*)(U_1, U_2) &= h_*(\phi^2\nabla_{U_1} U_2 - \eta(\nabla_{U_1} U_2)\xi), \\ &= h_*(\phi(\nabla_{U_1}\psi U_2 + \nabla_{U_1}\omega U_2)), \\ &= h_*(\phi(\mathcal{T}_{U_1}\psi U_2 + \widehat{\nabla}_{U_1}\psi U_2 + \mathcal{T}_{U_1}\omega U_2 + \mathcal{H}\nabla_{U_1}\omega U_2)), \\ &= h_*(\mathcal{B}\mathcal{T}_{U_1}\psi U_2 + \mathcal{C}\mathcal{T}_{U_1}\psi U_2 + \psi\widehat{\nabla}_{U_1}\psi U_2 + \omega\widehat{\nabla}_{U_1}\psi U_2 \\ &\quad + \psi\mathcal{T}_{U_1}\omega U_2 + \omega\mathcal{T}_{U_1}\omega U_2 + \mathcal{B}\mathcal{H}\nabla_{U_1}\omega U_2 + \mathcal{C}\mathcal{H}\nabla_{U_1}\omega U_2), \\ &= h_*(\mathcal{C}\mathcal{T}_{U_1}\psi U_2 + \omega\widehat{\nabla}_{U_1}\psi U_2 + \omega\mathcal{T}_{U_1}\omega U_2 + \mathcal{C}\mathcal{H}\nabla_{U_1}\omega U_2). \end{aligned}$$

Thus, $(\nabla h_*)(U_1, U_2) = 0 \iff \omega(\widehat{\nabla}_{U_1}\psi U_2 + \mathcal{T}_{U_1}\omega U_2) + \mathcal{C}(\mathcal{T}_{U_1}\psi U_2 + \mathcal{H}\nabla_{U_1}\omega U_2) = 0.$

Now, for $U_1 \in \Gamma(\ker h_*)$, $W_1 \in \Gamma(\ker h_*)^{\perp}$, we have

$$\begin{aligned}(\nabla h_*)(U_1, W_1) &= \nabla^h_{U_1} h_* W_1 - h_*(\nabla_{U_1} W_1),\\ &= -h_*(\nabla_{U_1} W_1).\end{aligned}$$

From equations (2.1), (2.4), (3.3), (2.8) , (2.9) and (3.4) , we have

$$\begin{aligned}(\nabla h_*)(U_1, W_1) &= h_*(\phi^2\nabla_{U_1} W_1 - \eta(\nabla_{U_1} W_1)\xi),\\ &= h_*(\phi\nabla_{U_1}\phi W_1),\\ &= h_*(\phi(\nabla_{U_1}\mathcal{B}W_1 + \nabla_{U_1}\mathcal{C}W_1)),\\ &= h_*(\phi(\mathcal{T}_{U_1}\mathcal{B}W_1 + \widehat{\nabla}_{U_1}\mathcal{B}W_1 + \mathcal{T}_{U_1}\mathcal{C}W_1 + \mathcal{H}\nabla_{U_1}\mathcal{C}W_1)),\\ &= h_*(\mathcal{B}\mathcal{T}_{U_1}\mathcal{B}W_1 + \mathcal{C}\mathcal{T}_{U_1}\mathcal{B}W_1 + \psi\widehat{\nabla}_{U_1}\mathcal{B}W_1 + \omega\widehat{\nabla}_{U_1}\mathcal{B}W_1\\ &\quad +\psi\mathcal{T}_{U_1}\mathcal{C}W_1 + \omega\mathcal{T}_{U_1}\mathcal{C}W_1 + \mathcal{B}\mathcal{H}\nabla_{U_1}\mathcal{C}W_1 + \mathcal{C}\mathcal{H}\nabla_{U_1}\mathcal{C}W_1),\\ &= h_*(\mathcal{C}\mathcal{T}_{U_1}\mathcal{B}W_1 + \omega\widehat{\nabla}_{U_1}\mathcal{B}W_1 + \omega\mathcal{T}_{U_1}\mathcal{C}W_1 + \mathcal{C}\mathcal{H}\nabla_{U_1}\mathcal{C}W_1).\end{aligned}$$

Thus, $(\nabla h_*)(U_1, W_1) = 0 \iff \omega(\widehat{\nabla}_{U_1}\mathcal{B}W_1 + \mathcal{T}_{U_1}\mathcal{C}W_1) + \mathcal{C}(\mathcal{T}_{U_1}\mathcal{B}W_1 + \mathcal{H}\nabla_{U_1}\mathcal{C}W_1) = 0.$
This proof is completed. ∎

4 Examples

Example 4.1. Every anti-invariant Riemannian submersion from a cosymplectic manifold onto a Riemannian manifold is a hemi-slant submersion with slant distribution $\mathcal{D}^{\theta} = \{0\}$.

Example 4.2. Every slant Riemannian submersion from a cosymplectic manifold onto a Riemannian manifold is a hemi-slant submersion with $\mathcal{D}^{\perp} = \{0\}$.

Example 4.3. Every semi-invariant Riemannian submersion from a cosymplectic manifold onto a Riemannian manifold is a hemi-slant submersion with slant angle $\theta = 0$.

Example 4.4. Let $\mathbb{R}^7$ has a cosymplectic structure as in Example 2.1. Define a map $h : \mathbb{R}^7 \to \mathbb{R}^3$ by

$$h(x_1, x_2, x_3, y_1, y_2, y_3, z) = (y_2, y_1 \sin\alpha + x_2 \cos\alpha, y_3),$$

where $\alpha \in (0, \frac{\pi}{2})$. Then, by direct calculations, we obtain the Jacobian matrix of h as:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & \cos\alpha & 0 & \sin\alpha & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ \end{pmatrix}.$$

Since, the rank of Jacobian matrix is 3, therefore the map h is a submersion. After some straightforward computations, we obtain

$$(\ker h_*) = span\{X_1 = \frac{\partial}{\partial x_1}, X_2 = \cos\alpha\frac{\partial}{\partial y_1} - \sin\alpha\frac{\partial}{\partial x_2}, X_3 = \frac{\partial}{\partial x_3}, X_4 = \frac{\partial}{\partial z}\},$$

and

$$(\ker h_*)^\perp = span\{V_1 = \sin\alpha\frac{\partial}{\partial y_1} + \cos\alpha\frac{\partial}{\partial x_2}, V_2 = \frac{\partial}{\partial y_2}, V_3 = \frac{\partial}{\partial y_3}\}.$$

Then it follows that

$$\begin{aligned}\mathcal{D}^\theta &= span\{X_1 = \frac{\partial}{\partial x_1}, X_2 = \cos\alpha\frac{\partial}{\partial y_1} - \sin\alpha\frac{\partial}{\partial x_2}\},\\ \mathcal{D}^\perp &= span\{X_3 = \frac{\partial}{\partial x_3}\}, \xi = \frac{\partial}{\partial z}.\end{aligned}$$

Thus the map h is a hemi-slant submersion with the slant angle α.

Example 4.5. Define a submersion $h : \mathbb{R}^7 \to \mathbb{R}^3$ by

$$h(x_1, x_2, x_3, y_1, y_2, y_3, z) = (\frac{x_1 - y_2}{2}, y_1, y_3),$$

Then it follows that

$$(\ker h_*) = span\{X_1 = \frac{1}{2}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_2}), X_2 = \frac{\partial}{\partial x_2}, X_3 = \frac{\partial}{\partial x_3}, X_4 = \frac{\partial}{\partial z}\},$$

and

$$(\ker h_*)^\perp = span\{V_1 = \frac{1}{2}(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_2}), V_2 = \frac{\partial}{\partial y_1}, V_3 = \frac{\partial}{\partial y_3}\}.$$

Thus, we have

$$\mathcal{D}^\theta = span\{X_1, X_2\}, \mathcal{D}^\perp = span\{X_3\} \text{ and } \quad \xi = \frac{\partial}{\partial z}.$$

Thus the map h is a hemi-slant submersion with the slant angle $\theta = \frac{\pi}{4}$.

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