Essentially weighted Rationalized Toeplitz Hankel operators

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Abstract

In this paper,the notion of Essentially weighted Rationalized Toeplitz Hankel operator is introduced and some properties of the set of all such kind of operators are discussed. Precisely if W-ERTHO($L^2(\beta)$) is the set of all Essentially weighted Rationalized Toeplitz Hankel operators, we throw some light on this set and attempt to investigate the properties of W-ERTHO($L^2(\beta)$).

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1 Introduction

Toeplitz and Hankel Operators are the subject of investigations for many researchers around the globe.In 1995, Mark.C.Ho[9], began the systematic study of the Slant Toeplitz operators on the Hardy Space. Since then, Slant Toeplitz operators have played outstanding roles in wavelet analysis, curve and surface modelling and dynamical systems. For instance, Villemoes [12] has associated the Besov regularity of solution of the refinement equation with the spectral radius of an associated Slant Toeplitz operators and has used the spectral radius of Slant Toeplitz Operators to characterize the $L_p(1 \leq p \leq \infty)$ regularity of refinable functions. All these operators have plenty of applications in prediction theory[8] and differential equations etc.Many generalizations of these operators like Slant Hankel, generalized Slant Toeplitz [2,7] have been studied by many mathematicians. The generalizations of all kinds of Toeplitz, Hankel, Slant Toeplitz, Slant Hankel operators is given in [4] as the Rationalized Toeplitz Hankel operator on the space L^2. Further in [6] the notion of weighted Rationalized Toeplitz Hankel operators is introduced on the generalized space $L^2(\beta)$. Also in [5], the notion of Essentially Rationalized Toeplitz Hankel operators is introduced and studied. Motivated by all the literature available, we here in this paper,introduce the notion of Essentially weighted Rationalized Toeplitz Hankel operators on the space $L^2(\beta)$. We begin with the following preliminaries.

Let $\beta = (\beta_n)_{n\in Z}$ be a sequence of positive numbers with $\beta_0 = 1, 0 < \frac{\beta_n}{\beta_{n+1}} \leq 1$, when $n \geq 0$ and $0 < \frac{\beta_n}{\beta_{n+1}} \leq 1$ when $n \leq 0$,where Z is the set of integers.Consider the space

$$L^2(\beta) = \{f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, a_n \in C\}$$

and

$$\|f\|_{\beta}^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 {\beta_n}^2 < \infty.$$

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Then $(L^2(\beta), \|.\|_\beta)$ is a Hilbert Space with an orthonormal basis

$$\left\{ e_n(z) = \frac{z^n}{\beta_n} \right\}_{n\in Z}$$

and with respect to inner product defined as

$$\left\langle \sum a_n z^n, \sum b_n z^n \right\rangle = \sum a_n \overline{b_n} {\beta_n}^2$$

$$L^\infty(\beta) = \left\{ \varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n : \varphi L^2(\beta) \subseteq L^2(\beta) \text{ and } \exists c \in R \text{ such that } \|\varphi f\|_{\beta} \leq c\|f\|_{\beta} \forall f \in L^2(\beta) \right\}$$

$L^\infty(\beta)$ is a Banach Space with a norm defined as

$$\|\varphi\|_\infty = \inf\{c : \|\varphi f\|_{\beta} \leq c\|f\|_{\beta} \; \forall f \in L^2(\beta)\}.$$

For φ in $L^\infty(\beta)$, let M^β_φ denote the multiplication operator on the space $L^2(\beta)$.Then

$$M^\beta_\varphi(f) = \varphi f \qquad \forall f \in L^2(\beta)$$

and

$$M_\varphi(e_k(z)) = \sum_{n=-\infty}^{\infty} a_{n-k} \frac{\beta_n}{\beta_k} e_n(z) \qquad \forall k \in Z$$

We begin with the following definition
For $k \neq 0$ let $W^\beta_k : L^2(\beta) \to L^2(\beta)$ be defined as

$$W^\beta_k e_n(z) = \begin{cases} \frac{\beta_{n/k}}{\beta_k} e_{n/k}(z) & \text{if n is divisible by k} \\ 0 & \text{otherwise} \end{cases}$$

W^β_k is an bounded linear operator on $L^2(\beta)$.
In [4] the author has defined Rationalized Toeplitz Hankel operators of order (k_1, k_2) on the space L^2 as the operator $R_\varphi : L^2 \to L^2, \varphi \in L^\infty$

$$R_\varphi f = W_{k_1} M_\varphi W^*_{k_2} f \qquad \forall f \in L^2$$

where k_1 and k_2 are non-zero integers and for $k \neq 0, W_k$ on L^2 is defined as

$$W_k(z^i) = \begin{cases} z^{i/k} & \text{if i is divisible by k} \\ 0 & \text{otherwise} \end{cases}$$

It is also proved in [4] that if k_1 and k_2 are relatively prime integers then a bounded linear operator on L^2 is a Rationalized Toeplitz Hankel operator if and only if $M_{z^{k_2}} R = R M_{z^{k_1}}$.Further if k_1 and

k_2 are not relatively prime then if $d = gcd(k_1, k_2)$ i.e. $k_1 = dn$ and $k_2 = dm$, then a bounded operator R on L^2 is Rationalized Toeplitz Hankel operator if and only if

$$R|\tilde{N}_i = W_m M_{\tilde{\varphi}_i} W_n^* |\tilde{N}_i$$

for some $\tilde{\varphi}_i$ in L^∞ and for i=1,2,...d-1

$$L^2 = \tilde{N}_0 \oplus \tilde{N}_1 \oplus ... \oplus \tilde{N}_{d-1}$$

where

$$\tilde{N}_0 = N_0 \oplus N_1 \oplus ... \oplus N_{m-1}$$

$$\tilde{N}_1 = N_m \oplus N_{m+1} \oplus ... \oplus N_{2m-1}$$

$$\cdots$$

$$\tilde{N}_{d-1} = N_{(d-1)m} \oplus N_{(d-1)m+1} \oplus ... \oplus N_{dm-1}$$

where N_i =The closed linear span of $\{z^{k_1 t+1} : t \in Z\}$
Further in [6],the notion of weighted Rationalized Toeplitz Hankel operator is defined and it is proved that a bounded operator R^β on $L^2(\beta)$ is a weighted Rationalized Toeplitz Hankel operator of order (k_1, k_2) if and only if $M^\beta_{z^{k_2}} R = R M^\beta_{z^{k_1}}$ where $\beta = \{\beta_n\}_{n\in Z}$ is a sequence of positive numbers defined as above and $\frac{\beta_{k_2 n}}{\beta_n} = 1\, \forall n$.
Throughout here,we will consider the sequence (β_n) which satisfy the condition of $\frac{\beta_{k_2 n}}{\beta_n} = 1\, \forall n$ and also $\sup \left| \frac{\beta_{n+1}}{\beta_n} \right| < \infty$.
In [7] the notion of k^{th} order Essentially slant weighted Toeplitz Operators is introduced and studied. Further we have the properties of Essentially Rationalized Toeplitz Hankel Operators,motivated by all these,here in this paper,the notion of Essentially weighted Rationalized Toeplitz Hankel Operators is introduced and studied.

2 Essentially weighted rationalized Toeplitz Hankel operators

Let $\mathcal{K}(L^2(\beta))$ be the set of all compact operators on the weighted space $L^2(\beta)$.We begin with the following definition.

Definition 2.1. A bounded linear operator R on $L^2(\beta)$ is called Essentially weighted Rationalized Toeplitz Hankel Operator of order (k_1, k_2) if

$$M^\beta_{z^{k_2}} R - R M^\beta_{z^{k_1}} \in \mathcal{K}(L^2(\beta))$$

where k_1 and k_2 are non-zero integers. We denote the set of all Essentially weighted Rationalized Toeplitz Hankel Operators on $L^2(\beta)$ as W-ERTHO($L^2(\beta)$).

We observe that if R is a weighted Rationalized Toeplitz Hankel Operator of order (k_1, k_2) on $L^2(\beta)$ then $M^\beta_{z^{k_2}} R = R M^\beta_{z^{k_1}}$ and therefore $M^\beta_{z^{k_2}} R - R M^\beta_{z^{k_1}} = 0$, the zero operator.As the zero operator is compact,therefore this implies that

$$M^\beta_{z^{k_2}} R - R M^\beta_{z^{k_1}} \in \mathcal{K}(L^2(\beta))$$

That is,every weighted Rationalized Toeplitz Hankel Operator is a member of the set W-ERTHO($L^2(\beta)$).That is,W-RTHO($L^2(\beta)$) $\subseteq$W-ERTHO($L^2(\beta)$.
Also we observe that if,infact T is any compact perturbation of a weighted Rationalized Toeplitz Hankel Operator on $L^2(\beta)$,then $T \in$ W-ERTHO($L^2(\beta)$).
However in the following example we show that the converse need not be true.That is,we construct a non-compact operator in the class W-ERTHO($L^2(\beta)$) but not in the class W-RTHO($L^2(\beta)$).

Example 2.2. . Let $k_2 = 1$ and $k_1 = k, k \geq 2$ be fixed integer. Define

$$Ke_n = \begin{cases} e_1 & \text{if n=0} \\ 0 & \text{otherwise} \end{cases}$$

and

$$Te_n = \begin{cases} e_1 & \text{if n is not a multiple of k} \\ e_n & \text{if n is a multiple of k} \end{cases}$$

Define:$R = W_k T + K$
Then

$$\begin{aligned} M^{\beta}_{z^{k_2}} R - R M^{\beta}_{z^{k_1}} &= M^{\beta}_{z} R - R M^{\beta}_{z^k} \\ &= M^{\beta}_{z}(W_k T + K) - (W_k T + K) M^{\beta}_{z^k} \\ &= (M^{\beta}_{z} W_k T - W_k T M^{\beta}_{z^k}) + K' \end{aligned}$$

where K' is a compact operator.
We can see for all $n \in Z$,

$$(M^{\beta}_{z} W_k T - W_k T M^{\beta}_{z^k}) e_n(z) = 0 (mod\, \mathcal{K}(L^2(\beta))$$

and therefore

$$M^{\beta}_{z^{k_2}} R - R M^{\beta}_{z^{k_1}} \in \mathcal{K}(L^2(\beta)$$

So $R \in$ W-ERTHO($L^2(\beta)$) but $R \notin$ W-RTHO($L^2(\beta)$).

We have the following theorem:

Theorem 2.3. For non-zero integers k_1 and k_2 that are relatively prime the identity operator cannot be the Essentially weighted Rationalized Toeplitz Hankel Operator of order (k_1, k_2) on $L^2(\beta)$

Proof. Let I be identity operator on $L^2(\beta)$.So if we consider for each integer n,

$$(M^{\beta}_{z^{k_2}} I - I M^{\beta}_{z^{k_1}}) e_n(z) = \frac{\beta_{n+k_2}}{\beta_n} e_{n+k_2}(z) - \frac{\beta_{n+k_1}}{\beta_n} e_{n+k_1}(z)$$

which is not compact on $L^2(\beta)$
$\Longrightarrow$ The identity operator I is not a member of W-ERTHO($L^2(\beta)$)

Q.E.D.

Theorem 2.4. The space W-ERTHO($L^2(\beta)$) is a closed vector space of $B(L^2(\beta))$ under the $\star$-strong operator topology, where $B(L^2(\beta))$ is the set of all bounded linear operators on $(L^2(\beta))$.

Proof. Let $R_1, R_2 \in$W-ERTHO($L^2(\beta)$) and α_1, α_2 are complex numbers.Then

$$\begin{aligned} &M^\beta_{z^{k_2}}(\alpha_1 R_1 + \alpha_2 R_2) - (\alpha_1 R_1 + \alpha_2 R_2)M^\beta_{z^{k_1}} \\ = &\alpha_1(M^\beta_{z^{k_2}} R_1 - R_1 M^\beta_{z^{k_1}}) + \alpha_2(M^\beta_{z^{k_2}} R_2 - R_2 M^\beta_{z^{k_1}}) \in \mathcal{K}(L^2(\beta)) \end{aligned}$$

Hence $\alpha_1 R_1 + \alpha_2 R_2 \in$ W-ERTHO($L^2(\beta)$) and so it is a subspace of $B(L^2(\beta))$.For each n, consider the sequence of operators (R_n) and (R_n^*) in W-ERTHO($L^2(\beta)$) such that $R_n \to R$ and $R_n^* \to R^*$ uniformly on $B(L^2(\beta))$ as $n \to \infty$.
As $R_n, R_n^* \in$ W-ERTHO($L^2(\beta)$),therefore we have

$$M^\beta_{z^{k_2}} R_n - R_n M^\beta_{z^{k_1}} = T_n$$

$$M^\beta_{z^{k_2}} R_n^* - R_n^* M^\beta_{z^{k_1}} = C_n$$

where T_n and C_n are compact operators on $(L^2(\beta))$.So

$$\begin{aligned} \|M^\beta_{z^{k_2}} R - R M^\beta_{z^{k_1}} - T_n\|_\beta &= \|M^\beta_{z^{k_2}} R - R M^\beta_{z^{k_1}} - M^\beta_{z^{k_2}} R_n + R_n M^\beta_{z^{k_1}}\|_\beta \\ &\leq (\|M^\beta_{z^{k_2}}\|_\beta + \|M^\beta_{z^{k_2}}\|_\beta)(\|R_n - R\|_\beta) \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

Thus the sequence $T_n \to M^\beta_{z^{k_2}} R - R M^\beta_{z^{k_1}}$ uniformly.As T_n is uniformly closed,therefore $M^\beta_{z^{k_2}} R - R M^\beta_{z^{k_1}}$ is compact and hence $R \in$W-ERTHO($L^2(\beta)$). Similarly,by using the same arguments we get

$$\|M^\beta_{z^{k_2}} R^* - R^* M^\beta_{z^{k_1}} - C_n\|_\beta \to 0 \text{ as } n \to \infty$$

and hence $R^* \in$W-ERTHO($L^2(\beta)$). Thus the set W-ERTHO($L^2(\beta)$) is a closed subspace of $B(L^2(\beta))$ under a $\star$-strong operator topology. Q.E.D.

Let us now begin the following result for the product.

Theorem 2.5. If $R_1, R_2 \in$W-ERTHO($L^2(\beta)$),then the operator $R_1 R_2 \in$W-ERTHO($L^2(\beta)$) if and only if $R_1 M^\beta_{z^{k_1} - z^{k_2}} R_2 \in \mathcal{K}(L^2(\beta))$

Proof. Let $R_1, R_2 \in$W-ERTHO($L^2(\beta)$).Then

$$M^\beta_{z^{k_2}} R_1 - R_1 M^\beta_{z^{k_1}} = K_1$$

and

$$M^\beta_{z^{k_2}} R_2 - R_2 M^\beta_{z^{k_1}} = K_2$$

where K_1 and K_2 are compact operators on $(L^2(\beta))$.
Consider the product

$$\begin{aligned} M^{\beta}_{z^{k_2}} R_1 R_2 - R_1 R_2 M^{\beta}_{z^{k_1}} &= R_1 M^{\beta}_{z^{k_1}} R_2 - R_1 R_2 M^{\beta}_{z^{k_1}} \\ &= (R_1 M^{\beta}_{z^{k_1}} + K_1) R_2 - R_1 (M^{\beta}_{z^{k_2}} R_2 - K_2) \\ &= (R_1 M^{\beta}_{z^{k_1}} - R_1 M^{\beta}_{z^{k_2}}) R_2 + K_1 R_2 + R_1 K_2 \\ &= R_1 M^{\beta}_{z^{k_1} - z^{k_2}} R_2 + K' \end{aligned}$$

where $K' = K_1 R_2 + R_1 K_2$ is a compact operator.
Thus the product $R_1 R_2 \in$W-ERTHO$(L^2(\beta))$ if and only if $M^{\beta}_{z^{k_2}} R_1 R_2 - R_1 R_2 M^{\beta}_{z^{k_1}} \in \mathcal{K}(L^2(\beta))$.
That is,if and only if $R_1 M^{\beta}_{z^{k_1} - z^{k_2}} R_2 \in \mathcal{K}(L^2(\beta))$.
This completes the proof. Q.E.D.

In general we can prove the following.

Theorem 2.6. Let $R \in$W-ERTHO$(L^2(\beta))$ and $s \in N, s > 1$.If $s = p_i + q_i$ where $p_i, q_i \in N$ for $i = 1, 2, \cdots n(s)$ and $R^{p_i}, R^{q_i} \in$W-ERTHO$(L^2(\beta))$
Then
i)$R^s \in$W-ERTHO$(L^2(\beta))$
ii)$R^{p_i} M_{z^{k_1} - z^{k_2}} R^{q_i} \in \mathcal{K}(L^2(\beta))\ i = 1, 2, \cdots n(s)$
iii)$R^{q_i} M_{z^{k_1} - z^{k_2}} R^{p_i} \in \mathcal{K}(L^2(\beta))\ i = 1, 2, \cdots n(s)$

We have the following theorem

Theorem 2.7. If $R_1, R_2 \in$W-ERTHO$(L^2(\beta))$ such that either R_1 commutes essentially with $M^{\beta}_{z^{k_2}}$ and R_2 commutes essentially with $M^{\beta}_{z^{k_1}}$,then the product $R_1 R_2 \in$W-ERTHO$(L^2(\beta))$

Proof. Let $R_1, R_2 \in$W-ERTHO$(L^2(\beta))$

Case I: Let R_1 commute essentially with $M^{\beta}_{z^{k_2}}$,then

$$R_1 M^{\beta}_{z^{k_2}} = M^{\beta}_{z^{k_2}} R_1 (mod(\mathcal{K}(L^2(\beta))$$

So we have,

$$\begin{aligned} M^{\beta}_{z^{k_2}} R_1 R_2 - R_1 R_2 M^{\beta}_{z^{k_1}} &= (M^{\beta}_{z^{k_2}} R_1 R_2 - R_1 M^{\beta}_{z^{k_2}} R_2)(mod(\mathcal{K}(L^2(\beta)) \\ &= (R_1 M^{\beta}_{z^{k_2}} R_2 - R_1 M^{\beta}_{z^{k_2}} R_2)(mod(\mathcal{K}(L^2(\beta)) \\ &= 0(mod(\mathcal{K}(L^2(\beta)) \end{aligned}$$

$\implies R_1 R_2 \in$W-ERTHO$(L^2(\beta))$.
Case II: Let

$$R_2 M^{\beta}_{z^{k_1}} = M^{\beta}_{z^{k_1}} R_2 (mod(\mathcal{K}(L^2(\beta)).$$

Then

$$\begin{aligned} M^{\beta}_{z^{k_2}} R_1 R_2 - R_1 R_2 M^{\beta}_{z^{k_1}} &= (R_1 M^{\beta}_{z^{k_1}} R_2 - R_1 R_2 M^{\beta}_{z^{k_1}})(mod(\mathcal{K}(L^2(\beta)) \\ &= (R_1 M^{\beta}_{z^{k_1}} R_2 - R_1 M^{\beta}_{z^{k_1}} R_2)(mod(\mathcal{K}(L^2(\beta)) \\ &= 0(mod(\mathcal{K}(L^2(\beta)) \end{aligned}$$

$\implies R_1R_2 \in$W-ERTHO$(L^2(\beta))$. Q.E.D.

With this result, we have the following corollary.

Corollary 2.8. If the operator R_1(or R_2) commutes essentially with $M^{\beta}_{z^{k_2}}$(or $M^{\beta}_{z^{k_1}}$) and R_2(or R_1) $\in$W-ERTHO$(L^2(\beta))$,then the product $R_1R_2 \in$W-ERTHO$(L^2(\beta))$.
However the following example shows that in general the product R_1R_2 may not be in W-ERTHO$(L^2(\beta))$.

Example 2.9. Let $R_1 = R_2 = W_kT + K$ on $L^2(\beta)$ where T and K are defined as

$$Ke_n = \begin{cases} e_1 & \text{if n=0} \\ 0 & \text{otherwise} \end{cases}$$

and

$$Te_n = \begin{cases} e_1 & \text{if n is a multiple of k} \\ e_n & \text{if n is not a multiple of k} \end{cases}$$

and if $k \neq 0$

$$W^{\beta}_k e_n(z) = \begin{cases} \frac{\beta_{n/k}}{\beta_k} e_{n/k}(z) & \text{if n is divisible by k} \\ 0 & \text{otherwise} \end{cases}$$

Then both $R_1, R_2 \in$W-ERTHO$(L^2(\beta))$ but we can see the product $R_1R_2 \notin$W-ERTHO$(L^2(\beta))$ as

$$M^{\beta}_z R_1R_2 - R_1R_2M^{\beta}_{z^k} = M^{\beta}_z(W_kT)^2 - (W_kT)^2M^{\beta}_{z^k} \notin \mathcal{K}(L^2(\beta))$$

Hence, $R_1R_2 \notin$W-ERTHO$(L^2(\beta))$. Also in general for any $R \in$W-ERTHO$(L^2(\beta))$ the adjoint of R^* may not be a member of W-ERTHO$(L^2(\beta))$,we can see in the following example.

Example 2.10. For $k_2 = 1$ and $k_1 = k(\geq 2)$ be a fixed integer.
Let $R = W_{k_1} = W_k, k \geq 2$ then

$$\begin{aligned}(M^{\beta}_{z^{k_2}}R - RM^{\beta}_{z^{k_1}})e_n(z) &= M^{\beta}_z W_k e_n(z) - W_k M^{\beta}_{z^k} e_n(z) \\ &= 0 \,\forall n \in Z\end{aligned}$$

However,

$$(M^{\beta}_{z^{k_2}}R^* - R^*M^{\beta}_{z^{k_1}})e_n = (M^{\beta}_z R^* - R^*M^{\beta}_{z^k})e_n \notin \text{W-ERTHO}(L^2(\beta))$$

Thus $R \in$ W-ERTHO$(L^2(\beta))$ but $R^* \notin$ W-ERTHO$(L^2(\beta))$.

However when both the operators R and its adjoint R^* are in the class W-ERTHO$(L^2(\beta))$,then the operator RT essentially equals to T^*R on $L^2(\beta)$ where $T = M^{\beta}_{\bar{z}^{k_2}} + M^{\beta}_{z^{k_1}}$

Theorem 2.11. If $R, R^* \in$W-ERTHO$(L^2(\beta))$ then $RT = T^*R(mod(\mathcal{K}(L^2(\beta))$ where $T = M^{\beta}_{\bar{z}^{k_2}} + M^{\beta}_{z^{k_1}}$.

Proof. Let $R, R^* \in$W-ERTHO$(L^2(\beta))$.Then we have

$$M^{\beta}_{z^{k_2}}R - RM^{\beta}_{z^{k_1}} = K_1 \tag{1.1}$$

$$M^{\beta}_{z^{k_2}}R^* - R^*M^{\beta}_{z^{k_1}} = K_2 \tag{1.2}$$

where K_1 and K_2 are compact operators on $(L^2(\beta))$.
On taking adjoints on both sides of equation (2.2) and subtracting from the equation (2.1) we get

$$R(M^{\beta}_{\bar{z}^{k_2}} + M^{\beta}_{z^{k_1}}) - (M^{\beta}_{\bar{z}^{k_1}} + M^{\beta}_{z^{k_2}})R = K_3$$

where K_3 is a compact operator on $L^2(\beta)$. Thus we have if $R, R^* \in$W-ERTHO$(L^2(\beta))$ then $RT = T^*R(mod(\mathcal{K}(L^2(\beta))$ where $T = M^{\beta}_{\bar{z}^{k_2}} + M^{\beta}_{z^{k_1}}$.
This completes the proof. Q.E.D.

Thus we conclude that the necessary condition for any operator $R \in$ W-ERTHO$(L^2(\beta))$ to be self-adjoint is that $RT = T^*R(mod(\mathcal{K}(L^2(\beta))$ where $T = M^{\beta}_{\bar{z}^{k_2}} + M^{\beta}_{z^{k_1}}$.However if R is essentially normal operator,then we have the following.

Theorem 2.12. Let $R \in$ W-ERTHO$(L^2(\beta))$ be an essentially normal operator.Then the operators RR^* and R^*R are both in W-ERTHO$(L^2(\beta))$ if and only if R^* commutes essentially with $RM^{\beta}_{z^{k_1}}$.

Proof. Let $R \in$ W-ERTHO$(L^2(\beta))$ be such that $RR^* = R^*R(mod(\mathcal{K}(L^2(\beta))$. Then we get

$$\begin{aligned} M^{\beta}_{z^{k_2}}R^*R - R^*RM^{\beta}_{z^{k_1}} &= M^{\beta}_{z^{k_2}}R^*R - R^*RM^{\beta}_{z^{k_1}}(mod(\mathcal{K}(L^2(\beta)) \\ &= (RM^{\beta}_{z^{k_1}})R^* - R^*(RM^{\beta}_{z^{k_1}})(mod(\mathcal{K}(L^2(\beta)) \end{aligned}$$

Therefore,$R^*R \in$ W-ERTHO$(L^2(\beta))$ if and only if R^* commutes essentially with $RM^{\beta}_{z^{k_1}}$.
Similarly we have

$$\begin{aligned} M^{\beta}_{z^{k_2}}RR^* - RR^*M^{\beta}_{z^{k_1}} &= RM^{\beta}_{z^{k_1}}R^* - RR^*M^{\beta}_{z^{k_1}}(mod(\mathcal{K}(L^2(\beta)) \\ &= (RM^{\beta}_{z^{k_1}})R^* - R^*(RM^{\beta}_{z^{k_1}})(mod(\mathcal{K}(L^2(\beta)) \end{aligned}$$

Thus,$RR^* \in$ W-ERTHO$(L^2(\beta))$ if and only if R^* commutes essentially with $RM^{\beta}_{z^{k_1}}$. Q.E.D.

Theorem 2.13. Let $R \in$ W-ERTHO$(L^2(\beta))$.Then $0 \in \sigma_e(R)$,the essential spectrum of R.

Proof. Let $R \in$ W-ERTHO$(L^2(\beta))$. Let if possible $0 \notin \sigma_e(R)$. Then R is essentially invertible in $L^2(\beta)$.As the operator $R \in$ W-ERTHO$(L^2(\beta))$,so $M^{\beta}_{z^{k_2}}R - RM^{\beta}_{z^{k_1}} \in \mathcal{K}(L^2(\beta)$. This implies that $M^{\beta}_{z^{k_2}} - RM^{\beta}_{z^{k_1}}R^{-1} \in \mathcal{K}(L^2(\beta)$ and so for $k_1 \neq k_2$ $M^{\beta}_{z^{k_1}}$ and $M^{\beta}_{z^{k_2}}$ are essentially similar.This is not possible as their Fredholm indexes are not same. Hence our assumption was wrong and $0 \in \sigma_e(R)$.
This completes the proof. Q.E.D.

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