The natural lift curve of the spherical indicatrix of a non-null curve according to Bishop frame in Minkowski 3-Space

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Abstract

In this study, we dealt with the natural lift curves of the spherical indicatrices of a non-null curve according to Bishop frame. Furthermore, some interesting results about the original curve were obtained depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$ and $T\left(H_0^2\right)$.

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1 Introduction and preliminary

Classical differential geometry began with the emergence of derivative and integral calculus, which helped solve geometric problems such as determining the tangents and curvatures of a curve. This process first started with the contribution of plane curves to the basic differential geometry in his work called derivative and integral calculus published by the French aristocrat Guillaume Francois Antoine de L'Hospital. In later years, very detailed contributions were made by famous scientists such as Leonhard Euler , Gaspard Monge , Joseph Louis Lagrange and Augustin Cauchy . In other words, classical differential geometry investigates the local properties of geometric shapes such as curves and surfaces. When local features are mentioned, a point of the investigated geometric shape.Knowing that there are traits that depend on the behavior in your neighborhood required.

According to the lorentzian character of the tangent vector field in the Lorentz-Minkowski space It should also be noted that other versions of the adapted movable roof are available. (Silva, [4]). One of the best-known examples of adapted moving roofs is Euclidean space. and a space curve in 3-dimensional embedded spaces such as the Lorentz-Minkowski space. It is the Serret-Frenet frame that formulated it. Frenet's equations are first used by space curves In order to bring simplicity and usefulness to his theory, Karl Eduard Senff and Johann Martin Bartels . Frenet the equations were reconstructed in Jean Frederic Frenet's doctoral dissertation published in 1852. It has been investigated. Also in 1851, these equations were independently Serret and is therefore often found in literature as Frenet-Serret It appears as equations. Although the Frenet frame is the most basic tool used to solve problems in areas such as biology, mechanics, and engineering, it loses its function in many areas such as robotics, camera, fluid flow, quantum mechanics, imaging due to these unnecessary rotations. Therefore, the question arises whether there is another adapted movable roof with minimal bending. Richard L. Bishop put forward the best answer to this as "there are 3 more than one way to crack a curve". Bishop observed that parallel vector fields on a C^2 regular curve form a 3-dimensional vector space. He revealed the equations of the Bishop roof, which is named after him; hence it is sometimes referred to as the Relatively Parallel Adapted Frame (Bishop, [1]).

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Fenchel W. [12] stated that a point $\gamma(t)$ on a curve, when plotting the curve, the Frenet vectors $\{T, N, B\}$ change and thus spherical signs are formed.

Thorpe J.A. [8], together with the geodesic spray concepts, gave the theorem that "for a curve γ to be an integral curve for the geodesic spray X of the natural lift γ, and only if γ is a geodesic over M". Çalışkan, Sivridağ and Hacısalihoğlu [9] , using these concepts and theorem given by Thorpe in E^3, have given that the curve should be a curve when the natural lift curve of the spherical indicators of a curve is an integral curve of the geodesic spray. Ergün and Çalışkan defined the concepts of the natural lift curve and geodesic spray in Minkowski 3-space.The anologue of the theorem of Thorpe was given in Minkowski 3-space by Ergün and Çalışkan [10].

Walrave [2] gave Frenet formulas of timelike, spacelike and null curves in $\mathbb{R}_1^3$ 3-dimensional Minkowski space and characterized curves of constant curvature.

Let Minkowski 3-space $\mathbb{R}_1^3$ be the vector space $\mathbb{R}^3$ equipped with the Lorentzian inner product g given by

$$g(X, X) = -x_1^2 + x_2^2 + x_3^2,$$

where $X = (x_1, x_2, x_3) \in \mathbb{R}^3$.

A vector $X = (x_1, x_2, x_3) \in \mathbb{R}_1^3$ is said to be timelike if $g(X, X) < 0$, spacelike if $g(X, X) > 0$ and lightlike (or null) if $g(X, X) = 0$. Similarly, an arbitrary curve $\gamma = \gamma(t)$ in $\mathbb{R}_1^3$ where t is a pseudo-arclength parameter, can be locally timelike, spacelike or null (lightlike), if all of its velocity vectors $\dot{\gamma}(t)$ are respectively timelike, spacelike or null (lightlike), for every $t \in I \subset \mathbb{R}$. A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ ($resp.$ $x_1 < 0$) and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ ($resp.$ $x_1 < 0$).The norm of a vector X is defined by [5]

$$\|X\|_{IL} = \sqrt{|g(X, X)|}.$$

The Lorentzian sphere and hyperbolic sphere of radius 1 in $\mathbb{R}_1^3$ are given by

$$S_1^2 = \left\{X = (x_1, x_2, x_3) \in \mathbb{R}_1^3 \;:\; g(X, X) = 1\right\}$$

and

$$H_0^2 = \left\{X = (x_1, x_2, x_3) \in \mathbb{R}_1^3 \;:\; g(X, X) = -1\right\},$$

respectively, [5].

Lemma 1.1. Let $X$and Y be nonzero Lorentz orthogonal vectors in $\mathbb{R}_1^3$. If X is timelike, then Y is spacelike, *[3].*

Lemma 1.2. Let X and Y be pozitive (negative) timelike vectors in $\mathbb{R}_1^3$. Then

$$g(X, Y) \leq \|X\| \, \|Y\|$$

whit equality if and only if X and Y are linearly dependent, *[3].*

Lemma 1.3. i) Let X and Y be pozitive (negative) timelike vectors in $\mathbb{R}_1^3$. By the Lemma 2, there is unique nonnegative real number $\varphi(X, Y)$ such that

$$g(X, Y) = \|X\| \, \|Y\| \cosh \varphi(X, Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X, Y)$.

ii) *Let X and Y be spacelike vektors in $\mathbb{R}_1^3$ that span a spacelike vector subspace. Then we have*

$$|g(X,Y)| \leq \|X\| \, \|Y\| \, .$$

Hence, there is a unique real number $\varphi(X,Y)$ between 0 and π such that

$$g(X,Y) = \|X\| \, \|Y\| \cos \varphi (X,Y)$$

the Lorentzian spacelike angle between X and Y is defined to be $\varphi(X,Y)$.

iii) *Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a timelike vector subspace. Then we have*

$$g(X,Y) > \|X\| \, \|Y\| \, .$$

Hence, there is a unique pozitive real number $\varphi(X,Y)$ between 0 and π such that

$$|g(X,Y)| = \|X\| \, \|Y\| \cosh \varphi (X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$

iv) *Let X be a spacelike vector and Y be a pozitive timelike vector in $\mathbb{R}_1^3$. Then there is a unique nonnegative reel number $\varphi(X,Y)$ such that*

$$|g(X,Y)| = \|X\| \, \|Y\| \sinh \varphi (X,Y)$$

the Lorentzian timelike angle between X and Y is defined to be $\varphi(X,Y)$, [3].

We denote the moving Frenet frame along the curve γ by $\{T(t), N(t), B(t)\}$, where T, N and B are the tangent, the principal normal and the binormal vector of the curve γ, respectively.

(**i**) Let γ be a unit speed timelike space curve with curvature κ and torsion τ and Frenet vector fields of γ be $\{T, N, B\}$. In this trihedron, T is a timelike vector field, N and B are spacelike vector fields. Then, Frenet formulas are given by [2],

$$\begin{aligned} \dot{T} &= \kappa N, \\ \dot{N} &= \kappa T + \tau B, \\ \dot{B} &= -\tau N. \end{aligned}$$

(**ii**) Let γ be a unit speed spacelike space curve with a spacelike binormal. For the Frenet vector fields we assume that T and B are spacelike vector fields and N is a timelike vector field .Then, Frenet formulas are given by [2],

$$\begin{aligned} \dot{T} &= \kappa N, \\ \dot{N} &= \kappa T + \tau B, \\ \dot{B} &= \tau N. \end{aligned}$$

(**iii**) Let γ be a unit speed spacelike space curve with a timelike binormal. We assume that T and N are spacelike vector fields and B is a timelike vector field. Then, Frenet formulas are given by [2].

$$\begin{aligned} \dot{T} &= \kappa N, \\ \dot{N} &= -\kappa T + \tau B, \\ \dot{B} &= \tau N. \end{aligned}$$

Definition 1.4. Let $\gamma : I \to \mathbb{R}^3_1$ be a unit speed spacelike or timelik space curve. Let $T = \dot{\gamma}$ be the tangent vector defined at each point of the curve. In this case, M_1 and M_2 vectors are perpendicular to the tangent vector T at each point and any two vector fields in the normal plane, on the curve γ, $\{T, N, B\}$, there is always a frame $\{T, M_1, M_2\}$,as an alternative to the moving frame. $\{T, M_1, M_2\}$ is Bishop frame to this alternative frame, [7].

Let γ be a unit speed timelike space curve.In this trihedron, T is a timelike vector field, M_1 and M_2 are spacelike vector fields. Then, Frenet formulas are given by [7],

$$\begin{aligned} \dot{T} &= k_1 M_1 + k_2 M_2, \\ \dot{M}_1 &= k_1 T, \\ \dot{M}_2 &= k_2 T, \\ \kappa(t) &= \sqrt{|k_1^2 + k_2^2|},\ \varphi(t) = \arctan\left(\frac{k_1}{k_2}\right),\ \tau(t) = \dot{\varphi}, \\ k_1 &= \kappa \cos\varphi,\ k_2 = \kappa \sin\varphi, \\ T &= T, \\ M_1 &= \cos\varphi N - \sin\varphi B, \\ M_2 &= \sin\varphi N + \cos\varphi B \end{aligned}$$

where the differentiable functions k_1 and k_2 are the Bishop curvatures.

Let γ be a unit speed spacelike space curve with a spacelike binormal.In this trihedron, M_1 is a timelike vector field, T and M_2 are spacelike vector fields. Then, Frenet formulas are given by [7],

$$\begin{aligned} \dot{T} &= k_1 M_1 - k_2 M_2 \\ \dot{M}_1 &= k_1 T \\ \dot{M}_2 &= k_2 T \\ \kappa(t) &= \sqrt{|k_1^2 - k_2^2|},\ \varphi(t) = \arg\tanh\left(\frac{k_1}{k_2}\right),\ \tau(t) = \dot{\varphi}, \\ k_1 &= \kappa \cosh\varphi,\ k_2 = \kappa \sinh\varphi, \\ T &= T, \\ M_1 &= \cosh\varphi N - \sinh\varphi B, \\ M_2 &= -\sinh\varphi N + \cosh\varphi B \end{aligned}$$

where the differentiable functions k_1 and k_2 are the Bishop curvatures.

Let γ be a unit speed spacelike space curve with a timelike binormal.In this trihedron, M_2 is a timelike vector field, T and M_1 are spacelike vector fields. Then, Frenet formulas are given by [7],

$$\begin{aligned}
\dot{T} &= k_1 M_1 - k_2 M_2 \\
\dot{M_1} &= -k_1 T \\
\dot{M_2} &= -k_2 T \\
\kappa(t) &= \sqrt{|k_1^2 - k_2^2|},\ \varphi(t) = \arg\tanh\left(\frac{k_1}{k_2}\right),\ \tau(t) = \dot{\varphi}, \\
k_1 &= \kappa \cosh\varphi,\ k_2 = \kappa \sinh\varphi, \\
T &= T, \\
M_1 &= \cosh\varphi N - \sinh\varphi B, \\
M_2 &= -\sinh\varphi N + \cosh\varphi B
\end{aligned}$$

where the differentiable functions k_1 and k_2 are the Bishop curvatures.

Definition 1.5. Let M be a hypersurface in $\mathbb{R}_1^3$ and let $\gamma : I \longrightarrow M$ be a parametrized curve. γ is called an integral curve of X if

$$\frac{d}{dt}(\gamma(t)) = X(\gamma(t)) \text{ (for all } t \in I)$$

where X is a smooth tangent vector field on M,[5].We have

$$TM = \bigcup_{P \in M} T_P M = \chi(M)$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields of M.

Definition 1.6. For any parametrized curve $\gamma : I \longrightarrow M$, $\overline{\gamma} : I \longrightarrow TM$ given by

$$\overline{\gamma}(t) = \left(\gamma(t), \dot{\gamma}(t)\right) = \dot{\gamma}(t)\,|_{\alpha(t)}$$

is called the natural lift of γ on TM,[10].Thus, we can write

$$\frac{d\overline{\gamma}}{dt} = \frac{d}{dt}\left(\dot{\gamma}(t)\,|_{\gamma(t)}\right) = \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)$$

where ∇ is the Levi-Civita connection on $\mathbb{R}_1^3$.

Definition 1.7. A $X \in \chi(TM)$ is called a geodesic spray if for $V \in TM$

$$X(V) = \varepsilon g(S(V), V) N\ , \varepsilon = g(N, N), [10].$$

Theorem 1.8. The natural lift $\overline{\gamma}$ of the curve γ is an integral curve of geodesic spray X if and only if γ is a geodesic on M,[10].

Theorem 1.9. Let $\gamma : I \longrightarrow E_1^3$ be a unit speed curve with nonzero k_1 and k_2 natural curvatures. If the curve γ is a slant helix if and only if $\frac{k_1}{k_2}$is constant, [7].

2 The natural lift curve of the spherical indicatrix of a timelike curve according to Bishop frame in Minkowski 3-Space

Let ∇, $\bar{\nabla}$ and $\bar{\bar{\nabla}}$ be Levi-Civita connections on $\mathbb{R}_1^3$, S_1^2 and H_0^2 respectively and ξ be a unit normal vector field of S_1^2 and H_0^2. Then Gauss Equations are given by the followings

$$\nabla_X Y = \bar{\nabla}_X Y + \varepsilon g\left(S\left(X\right), Y\right)\xi, \quad \nabla_X Y = \bar{\bar{\nabla}}_X Y + \varepsilon g\left(S\left(X\right), Y\right)\xi,$$

where $\varepsilon = g\left(\xi, \xi\right)$ and S is the shape operator of S_1^2 and H_0^2 and

$$S = I_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let γ_T be the spherical indicatrix of tangent vectors of γ and $\overline{\gamma}_T$ be the natural lift of the curve γ_T. In this case the equation for γ_T is $\gamma_T = T$. If $\overline{\gamma}_T$ is an integral curve of the geodesic spray, then from Theorem 1 we have

$$\bar{\bar{\nabla}}_{\dot{\gamma}_{_T}} \dot{\gamma}_{_T} = 0$$

that is

$$\nabla_{\dot{\gamma}_{_T}} \dot{\gamma}_{_T} = \bar{\bar{\nabla}}_{\dot{\gamma}_{_T}} \dot{\gamma}_{_T} + \varepsilon g\left(S\left(\dot{\gamma}_{_T}\right), \dot{\gamma}_{_T}\right)\xi$$

$$\nabla_{\dot{\gamma}_{_T}} \dot{\gamma}_{_T} = \varepsilon g\left(S\left(\dot{\gamma}_{_T}\right), \dot{\gamma}_{_T}\right) T$$

where $\varepsilon = g\left(T, T\right) = -1,\ S\left(\dot{\gamma}_{_T}\right) = -\dot{\gamma}_{_T},\ \ g\left(S\left(\dot{\gamma}_{_T}\right), \dot{\gamma}_{_T}\right) = -\left(k_1^2 + k_2^2\right),$

$$\nabla_{\dot{\gamma}_{_T}} \dot{\gamma}_{_T} = \left(k_1^2 + k_2^2\right) T$$

$$\frac{d}{ds_T}\left(k_1 M_1 + k_2 M_2\right) = \left(k_1^2 + k_2^2\right) T$$

$$\frac{d}{ds}\left(k_1 M_1 + k_2 M_2\right)\frac{ds}{ds_T} - \left(k_1^2 + k_2^2\right) T = 0$$

$$\left(\dot{k}_1 M_1 + k_1^2 T + \dot{k}_2 M_2 + k_2^2 T\right)\frac{ds}{ds_T} - \left(k_1^2 + k_2^2\right) T = 0$$

$$\left(\frac{k_1^2 + k_2^2}{\sqrt{|k_1^2 + k_2^2|}} - \left(k_1^2 + k_2^2\right)\right) T + \left(\frac{\dot{k}_1}{\sqrt{|k_1^2 + k_2^2|}}\right) M_1 + \left(\frac{\dot{k}_2}{\sqrt{|k_1^2 + k_2^2|}}\right) M_2 = 0$$

Since $\{T, M_1, M_2\}$ Bishop frame is linearly independent system , we have

$$\begin{aligned}\left(\frac{k_1^2+k_2^2}{\sqrt{|k_1^2+k_2^2|}}-\left(k_1^2+k_2^2\right)\right) &= 0,\\ \left(\frac{\dot{k_1}}{\sqrt{|k_1^2+k_2^2|}}\right) &= 0,\\ \left(\frac{\dot{k_2}}{\sqrt{|k_1^2+k_2^2|}}\right) &= 0.\end{aligned}$$

Hence we have

$$\begin{aligned}k_1^2+k_2^2 &= 1,\\ i)\ k_1 &= 1,\ k_2=0\\ ii)\ k_1 &= 0,\ k_2=1\end{aligned}$$

Corollary 2.1. If the natural lift $\overline{\gamma}_T$ of γ_T is an integral curve of the geodesic on the tangent bundle $T\left(H_0^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows,

$$\begin{cases} T=T,\ M_1=N,\ \ M_2=B,\ (k_1=1,\ k_2=0)\\ T=T,\ M_1=-B,\ \ M_2=N, (\ k_1=0,\ k_2=1)\end{cases}$$

Let γ_{M_1} be the spherical indicatrix of γ relative to M_1 and $\overline{\gamma}_{M_1}$ be the natural lift of the curve γ_{M_1}. In this case the equation for γ_{M_1} is $\gamma_{M_1}=M_1$. If $\overline{\gamma}_{M_1}$ is an integral curve of the geodesic spray, then from Theorem 1 we have

$$\overline{\nabla}_{\dot{\gamma}_{M_1}}\dot{\gamma}_{M_1}=0$$

that is

$$\nabla_{\dot{\gamma}_{M_1}}\dot{\gamma}_{M_1}=\overline{\nabla}_{\dot{\gamma}_{M_1}}\dot{\gamma}_{M_1}+\varepsilon g\left(S\left(\dot{\gamma}_{M_1}\right),\dot{\gamma}_{M_1}\right)\xi$$

$$\nabla_{\dot{\gamma}_{M_1}}\dot{\gamma}_{M_1}=\varepsilon g\left(S\left(\dot{\gamma}_{M_1}\right),\dot{\gamma}_{M_1}\right)M_1$$

where $\varepsilon=g\left(M_1,M_1\right)=+1,\ S\left(\dot{\gamma}_{M_1}\right)=-\dot{\gamma}_{M_1},\ \ g\left(S\left(\dot{\gamma}_{M_1}\right),\dot{\gamma}_{M_1}\right)=k_1^2,$

$$\nabla_{\dot{\gamma}_{M_1}}\dot{\gamma}_{M_1}=k_1^2M_1$$

$$\frac{d}{ds_{M_1}}\left(k_1T\right)=k_1^2M_1$$

$$\frac{d}{ds}\left(k_1T\right)\frac{ds}{ds_{M_1}}-k_1^2M_1=0$$

$$\left(\dot{k_1}T+k_1^2M_1+k_1k_2M_2\right)\frac{ds}{ds_{M_1}}-k_1^2M_1=0$$

$$\left(\frac{\dot{k_1}}{k_1}\right)T+\left(k_1-k_1^2\right)M_1+\left(k_2\right)M_2=0$$

Since $\{T,M_1,M_2\}$ Bishop frame is linearly independent system , we have

$$\begin{aligned}\frac{\dot{k_1}}{k_1}&=0,\\ \left(k_1-k_1^2\right)&=0,\\ k_2&=0.\end{aligned}$$

Hence we have

$$k_1=1,\ k_2=0$$

Corollary 2.2. If the natural lift $\overline{\gamma}_{M_1}$ of γ_{M_1} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^2\right)$, then there is a relationship between frames $\{T,N,B\}$ and $\{T,M_1,M_2\}$ as follows,

$$T=T,\ M_1=N,\quad M_2=B,\ \left(k_1=1,\ k_2=0\right)$$

Let γ_{M_2} be the spherical indicatrix of γ relative to M_2 and $\overline{\gamma}_{M_2}$ be the natural lift of the curve γ_{M_2}. In this case the equation for γ_{M_2} is $\gamma_{M_2}=M_2$. If $\overline{\gamma}_{M_2}$ is an integral curve of the geodesic spray, then from Theorem 1 we have

$$\bar{\nabla}_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2}=0$$

that is

$$\nabla_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2}=\bar{\nabla}_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2}+\varepsilon g\left(S\left(\dot{\gamma}_{M_2}\right),\dot{\gamma}_{M_2}\right)\xi$$

$$\nabla_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2}=\varepsilon g\left(S\left(\dot{\gamma}_{M_2}\right),\dot{\gamma}_{M_2}\right)M_2$$

where $\varepsilon=g\left(M_2,M_2\right)=+1,\ S\left(\dot{\gamma}_{M_2}\right)=-\dot{\gamma}_{M_2},\quad g\left(S\left(\dot{\gamma}_{M_2}\right),\dot{\gamma}_{M_2}\right)=k_2^2,$

$$\nabla_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2}=k_2^2M_2$$

$$\frac{d}{ds_{M_2}}\left(k_2T\right)=k_2^2M_2$$

$$\frac{d}{ds}(k_2T)\frac{ds}{ds_{M_2}} - k_2^2M_2 = 0$$

$$\left(\dot{k}_1T + k_1k_2M_1 + k_2^2M_2\right)\frac{ds}{ds_{M_2}} - k_2^2M_2 = 0$$

$$\left(\frac{\dot{k}_2}{k_2}\right)T + (k_1)M_1 + \left(k_2 - k_2^2\right)M_2 = 0$$

Since $\{T, M_1, M_2\}$ Bishop frame is linearly independent system , we have

$$\begin{aligned} \frac{\dot{k}_2}{k_2} &= 0, \\ k_1 &= 0 \\ \left(k_2 - k_2^2\right) &= 0. \end{aligned}$$

Hence we have

$$k_1 = 0,\ k_2 = 1$$

Corollary 2.3. If the natural lift $\overline{\gamma}_{M_2}$ of γ_{M_2} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows,

$$T = T,\ M_1 = -B,\ \ M_2 = N,\ (k_1 = 0,\ k_2 = 1)$$

3 The natural lift curve of the spherical indicatrix of a spacelike curve with spacelike binormal according to Bishop frame in Minkowski 3-Space

Let ∇, $\overline{\nabla}$ and $\overline{\overline{\nabla}}$ be Levi-Civita connections on $\mathbb{R}_1^3$, S_1^2 and H_0^2 respectively and ξ be a unit normal vector field of S_1^2 and H_0^2. Then Gauss Equations are given by the followings

$$\nabla_XY = \overline{\nabla}_XY + \varepsilon g\left(S\left(X\right), Y\right)\xi, \quad \nabla_XY = \overline{\overline{\nabla}}_XY + \varepsilon g\left(S\left(X\right), Y\right)\xi,$$

where $\varepsilon = g\left(\xi, \xi\right)$ and S is the shape operator of S_1^2 and H_0^2 and

$$S = I_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let γ_T be the spherical indicatrix of tangent vectors of γ and $\overline{\gamma}_T$ be the natural lift of the curve γ_T. In this case the equation for γ_T is $\gamma_T = T$. If $\overline{\gamma}_T$ is an integral curve of the geodesic spray, then from Theorem 1 we have

$$\overline{\nabla}_{\dot{\gamma}_T}\dot{\gamma}_T = 0$$

that is

$$\nabla_{\dot{\gamma}_T}\dot{\gamma}_T = \bar{\nabla}_{\dot{\gamma}_T}\dot{\gamma}_T + \varepsilon g\left(S\left(\dot{\gamma}_T\right), \dot{\gamma}_T\right)\xi$$

$$\nabla_{\dot{\gamma}_T}\dot{\gamma}_T = \varepsilon g\left(S\left(\dot{\gamma}_T\right), \dot{\gamma}_T\right)T$$

where $\varepsilon = g(T,T) = 1,\ S\left(\dot{\gamma}_T\right) = -\dot{\gamma}_T,\ \ g\left(S\left(\dot{\gamma}_T\right), \dot{\gamma}_T\right) = \left(k_1^2 - k_2^2\right),$

$$\nabla_{\dot{\gamma}_T}\dot{\gamma}_T = \left(k_1^2 - k_2^2\right)T$$

$$\frac{d}{ds_T}\left(k_1 M_1 - k_2 M_2\right) = \left(k_1^2 + k_2^2\right)T$$

$$\frac{d}{ds}\left(k_1 M_1 - k_2 M_2\right)\frac{ds}{ds_T} - \left(k_1^2 + k_2^2\right)T = 0$$

$$\left(\dot{k}_1 M_1 + k_1^2 T - \dot{k}_2 M_2 - k_2^2 T\right)\frac{ds}{ds_T} - \left(k_1^2 + k_2^2\right)T = 0$$

$$\left(\frac{k_1^2 - k_2^2}{\sqrt{|k_1^2 - k_2^2|}} - \left(k_1^2 - k_2^2\right)\right)T + \left(\frac{\dot{k}_1}{\sqrt{|k_1^2 - k_2^2|}}\right)M_1 + \left(\frac{\dot{k}_2}{\sqrt{|k_1^2 - k_2^2|}}\right)M_2 = 0$$

Since $\{T, M_1, M_2\}$ Bishop frame is linearly independent system , we have

$$\begin{aligned}\left(\frac{k_1^2 - k_2^2}{\sqrt{|k_1^2 - k_2^2|}} - \left(k_1^2 - k_2^2\right)\right) &= 0,\\ \left(\frac{\dot{k}_1}{\sqrt{|k_1^2 - k_2^2|}}\right) &= 0,\\ \left(\frac{\dot{k}_2}{\sqrt{|k_1^2 - k_2^2|}}\right) &= 0.\end{aligned}$$

Hence we have

$$k_1 = constant,\ k_2 = constant$$

Corollary 3.1. If the natural lift $\overline{\gamma}_T$ of γ_T is an integral curve of the geodesic on the tangent bundle $T\left(S_1^2\right)$, then the curve γ is a slant helix.

Let γ_{M_1} be the spherical indicatrix of γ relative to M_1 and $\overline{\gamma}_{M_1}$ be the natural lift of the curve γ_{M_1}. In this case the equation for γ_{M_1} is $\gamma_{M_1} = M_1$. If $\overline{\gamma}_{M_1}$ is an integral curve of the geodesic spray, then from Theorem 1 we have

$$\bar{\bar{\nabla}}_{\dot{\gamma}_{M_1}}\dot{\gamma}_{M_1} = 0$$

that is

$$\nabla_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} = \overline{\overline{\nabla}}_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} + \varepsilon g\left(S\left(\dot{\gamma}_{M_1}\right), \dot{\gamma}_{M_1}\right)\xi$$

$$\nabla_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} = \varepsilon g\left(S\left(\dot{\gamma}_{M_1}\right), \dot{\gamma}_{M_1}\right) M_1$$

where $\varepsilon = g\left(M_1, M_1\right) = -1,\ S\left(\dot{\gamma}_{M_1}\right) = -\dot{\gamma}_{M_1},\ \ g\left(S\left(\dot{\gamma}_{M_1}\right), \dot{\gamma}_{M_1}\right) = -k_1^2,$

$$\nabla_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} = k_1^2 M_1$$

$$\frac{d}{ds_{M_1}}\left(k_1 T\right) = k_1^2 M_1$$

$$\frac{d}{ds}\left(k_1 T\right)\frac{ds}{ds_{M_1}} - k_1^2 M_1 = 0$$

$$\left(\dot{k}_1 T + k_1^2 M_1 - k_1 k_2 M_2\right)\frac{ds}{ds_{M_1}} - k_1^2 M_1 = 0$$

$$\left(\frac{\dot{k}_1}{k_1}\right) T + \left(k_1 - k_1^2\right) M_1 + \left(-k_2\right) M_2 = 0$$

Since $\{T, M_1, M_2\}$ Bishop frame is linearly independent system , we have

$$\begin{aligned} \frac{\dot{k}_1}{k_1} &= 0, \\ \left(k_1 - k_1^2\right) &= 0, \\ k_2 &= 0. \end{aligned}$$

Hence we have

$$k_1 = 1,\ k_2 = 0$$

Corollary 3.2. If the natural lift $\overline{\gamma}_{M_1}$ of γ_{M_1} is an integral curve of the geodesic on the tangent bundle $T\left(H_0^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows,

$$T = T,\ M_1 = N,\ \ M_2 = B,\ \left(k_1 = 1,\ k_2 = 0\right)$$

Let γ_{M_2} be the spherical indicatrix of γ relative to M_2 and $\overline{\gamma}_{M_2}$ be the natural lift of the curve γ_{M_2}. In this case the equation for γ_{M_2} is $\gamma_{M_2} = M_2$. If $\overline{\gamma}_{M_2}$ is an integral curve of the geodesic spray, then from Theorem 1 we have

$$\overline{\nabla}_{\dot{\gamma}_{M_2}} \dot{\gamma}_{M_2} = 0$$

that is

$$\nabla_{\dot{\gamma}_{M_2}} \dot{\gamma}_{M_2} = \bar{\nabla}_{\dot{\gamma}_{M_2}} \dot{\gamma}_{M_2} + \varepsilon g\left(S\left(\dot{\gamma}_{M_2}\right), \dot{\gamma}_{M_2}\right) \xi$$

$$\nabla_{\dot{\gamma}_{M_2}} \dot{\gamma}_{M_2} = \varepsilon g\left(S\left(\dot{\gamma}_{M_2}\right), \dot{\gamma}_{M_2}\right) M_2$$

where $\varepsilon = g\left(M_2, M_2\right) = +1$, $S\left(\dot{\gamma}_{M_2}\right) = -\dot{\gamma}_{M_2}$, $g\left(S\left(\dot{\gamma}_{M_2}\right), \dot{\gamma}_{M_2}\right) = -k_2^2$,

$$\nabla_{\dot{\gamma}_{M_2}} \dot{\gamma}_{M_2} = -k_2^2 M_2$$

$$\frac{d}{ds_{M_2}}\left(k_2 T\right) = -k_2^2 M_2 \frac{d}{ds}\left(k_2 T\right)$$

$$\frac{ds}{ds_{M_2}} - k_2^2 M_2 = 0$$

$$\left(\dot{k}_1 T + k_1 k_2 M_1 + k_2^2 M_2\right) \frac{ds}{ds_{M_2}} + k_2^2 M_2 = 0$$

$$\left(\frac{\dot{k}_2}{k_2}\right) T + \left(k_1\right) M_1 + \left(-k_2 + k_2^2\right) M_2 = 0$$

Since $\{T, M_1, M_2\}$ Bishop frame is linearly independent system , we have

$$\begin{aligned} \frac{\dot{k}_2}{k_2} &= 0, \\ k_1 &= 0 \\ \left(k_2 - k_2^2\right) &= 0. \end{aligned}$$

Hence we have

$$k_1 = 0,\ k_2 = 1$$

Corollary 3.3. If the natural lift $\overline{\gamma}_{M_2}$ of γ_{M_2} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows,

$$T = T,\ M_1 = -B,\quad M_2 = -N,\ \left(k_1 = 0,\ k_2 = 1\right)$$

4 The natural lift curve of the spherical indicatrix of a spacelike curve with timelike binormal according to Bishop frame in Minkowski 3-Space

Let ∇, $\bar{\nabla}$ and $\bar{\bar{\nabla}}$ be Levi-Civita connections on $\mathbb{R}_1^3$, S_1^2 and H_0^2 respectively and ξ be a unit normal vector field of S_1^2 and H_0^2. Then Gauss Equations are given by the followings

$$\nabla_X Y = \bar{\nabla}_X Y + \varepsilon g\left(S\left(X\right), Y\right)\xi, \quad \nabla_X Y = \bar{\bar{\nabla}}_X Y + \varepsilon g\left(S\left(X\right), Y\right)\xi,$$

where $\varepsilon = g\left(\xi, \xi\right)$ and S is the shape operator of S_1^2 and H_0^2 and

$$S = I_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let γ_T be the spherical indicatrix of tangent vectors of γ and $\overline{\gamma}_T$ be the natural lift of the curve γ_T. In this case the equation for γ_T is $\gamma_T = T$. If $\overline{\gamma}_T$ is an integral curve of the geodesic spray, then from Theorem 1 we have

$$\bar{\nabla}_{\dot{\gamma}_T} \dot{\gamma}_T = 0$$

that is

$$\nabla_{\dot{\gamma}_T} \dot{\gamma}_T = \bar{\nabla}_{\dot{\gamma}_T} \dot{\gamma}_T + \varepsilon g\left(S\left(\dot{\gamma}_T\right), \dot{\gamma}_T\right)\xi$$

$$\nabla_{\dot{\gamma}_T} \dot{\gamma}_T = \varepsilon g\left(S\left(\dot{\gamma}_T\right), \dot{\gamma}_T\right)T$$

where $\varepsilon = g\left(T, T\right) = +1,\ S\left(\dot{\gamma}_T\right) = -\dot{\gamma}_T, \quad g\left(S\left(\dot{\gamma}_T\right), \dot{\gamma}_T\right) = \left(k_1^2 - k_2^2\right),$

$$\nabla_{\dot{\gamma}_T} \dot{\gamma}_T = \left(k_1^2 - k_2^2\right)T$$

$$\frac{d}{ds_T}\left(k_1 M_1 - k_2 M_2\right) = \left(k_1^2 + k_2^2\right)T$$

$$\frac{d}{ds}\left(k_1 M_1 - k_2 M_2\right)\frac{ds}{ds_T} - \left(k_1^2 + k_2^2\right)T = 0$$

$$\left(\dot{k}_1 M_1 + k_1^2 T - \dot{k}_2 M_2 - k_2^2 T\right)\frac{ds}{ds_T} - \left(k_1^2 + k_2^2\right)T = 0$$

$$\left(\frac{k_1^2 - k_2^2}{\sqrt{\left|k_1^2 - k_2^2\right|}} - \left(k_1^2 - k_2^2\right)\right)T + \left(\frac{\dot{k}_1}{\sqrt{\left|k_1^2 - k_2^2\right|}}\right)M_1 + \left(\frac{\dot{k}_2}{\sqrt{\left|k_1^2 - k_2^2\right|}}\right)M_2 = 0$$

Since $\{T, M_1, M_2\}$ Bishop frame is linearly independent system , we have

$$\begin{aligned}\left(\frac{k_1^2-k_2^2}{\sqrt{|k_1^2-k_2^2|}}-\left(k_1^2-k_2^2\right)\right) &= 0,\\ \left(\frac{\dot{k}_1}{\sqrt{|k_1^2-k_2^2|}}\right) &= 0,\\ \left(\frac{\dot{k}_2}{\sqrt{|k_1^2-k_2^2|}}\right) &= 0.\end{aligned}$$

Hence we have

$$k_1 = constant,\ k_2 = constant$$

Corollary 4.1. If the natural lift $\overline{\gamma}_T$ of γ_T is an integral curve of the geodesic on the tangent bundle $T\left(S_1^2\right)$, then the curve γ is a slant helix.

Let γ_{M_1} be the spherical indicatrix of γ relative to M_1 and $\overline{\gamma}_{M_1}$ be the natural lift of the curve γ_{M_1}. In this case the equation for γ_{M_1} is $\gamma_{M_1} = M_1$. If $\overline{\gamma}_{M_1}$ is an integral curve of the geodesic spray, then from Theorem 1 we have

$$\bar{\nabla}_{\dot{\gamma}_{M_1}}\dot{\gamma}_{M_1} = 0$$

that is

$$\nabla_{\dot{\gamma}_{M_1}}\dot{\gamma}_{M_1} = \bar{\nabla}_{\dot{\gamma}_{M_1}}\dot{\gamma}_{M_1} + \varepsilon g\left(S\left(\dot{\gamma}_{M_1}\right), \dot{\gamma}_{M_1}\right)\xi$$

$$\nabla_{\dot{\gamma}_{M_1}}\dot{\gamma}_{M_1} = \varepsilon g\left(S\left(\dot{\gamma}_{M_1}\right), \dot{\gamma}_{M_1}\right)M_1$$

where $\varepsilon = g\left(M_1, M_1\right) = 1,\ S\left(\dot{\gamma}_{M_1}\right) = -\dot{\gamma}_{M_1},\ \ g\left(S\left(\dot{\gamma}_{M_1}\right), \dot{\gamma}_{M_1}\right) = -k_1^2,$

$$\nabla_{\dot{\gamma}_{M_1}}\dot{\gamma}_{M_1} = -k_1^2 M_1$$

$$\frac{d}{ds_{M_1}}\left(-k_1 T\right) = -k_1^2 M_1$$

$$\frac{d}{ds}\left(-k_1 T\right)\frac{ds}{ds_{M_1}} + k_1^2 M_1 = 0$$

$$\left(-\dot{k}_1 T - k_1^2 M_1 + k_1 k_2 M_2\right)\frac{ds}{ds_{M_1}} + k_1^2 M_1 = 0$$

$$\left(\frac{\dot{k}_1}{k_1}\right)T + \left(k_1 - k_1^2\right)M_1 + k_2 M_2 = 0$$

Since $\{T, M_1, M_2\}$ Bishop frame is linearly independent system , we have

$$\begin{aligned} \frac{\dot{k_1}}{k_1} &= 0, \\ \left(-k_1 + k_1^2\right) &= 0, \\ k_2 &= 0. \end{aligned}$$

Hence we have

$$k_1 = 1,\ k_2 = 0$$

Corollary 4.2. If the natural lift $\overline{\gamma}_{M_1}$ of γ_{M_1} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows,

$$T = T,\ M_1 = N,\ \ M_2 = B,\ (k_1 = 1,\ k_2 = 0)$$

Let γ_{M_2} be the spherical indicatrix of γ relative to M_2 and $\overline{\gamma}_{M_2}$ be the natural lift of the curve γ_{M_2}. In this case the equation for γ_{M_2} is $\gamma_{M_2} = M_2$. If $\overline{\gamma}_{M_2}$ is an integral curve of the geodesic spray, then from Theorem 1 we have

$$\overline{\overline{\nabla}}_{\dot{\gamma}_{M_2}} \dot{\gamma}_{M_2} = 0$$

that is

$$\nabla_{\dot{\gamma}_{M_2}} \dot{\gamma}_{M_2} = \overline{\overline{\nabla}}_{\dot{\gamma}_{M_2}} \dot{\gamma}_{M_2} + \varepsilon g\left(S\left(\dot{\gamma}_{M_2}\right), \dot{\gamma}_{M_2}\right)\xi$$

$$\nabla_{\dot{\gamma}_{M_2}} \dot{\gamma}_{M_2} = \varepsilon g\left(S\left(\dot{\gamma}_{M_2}\right), \dot{\gamma}_{M_2}\right) M_2$$

where $\varepsilon = g\left(M_2, M_2\right) = -1,\ S\left(\dot{\gamma}_{M_2}\right) = -\dot{\gamma}_{M_2},\ \ g\left(S\left(\dot{\gamma}_{M_2}\right), \dot{\gamma}_{M_2}\right) = -k_2^2,$

$$\nabla_{\dot{\gamma}_{M_2}} \dot{\gamma}_{M_2} = k_2^2 M_2$$

$$\frac{d}{ds_{M_2}}\left(-k_2 T\right) = k_2^2 M_2$$

$$\frac{d}{ds}\left(-k_2 T\right)\frac{ds}{ds_{M_2}} - k_2^2 M_2 = 0$$

$$\left(-\dot{k}_2 T - k_1 k_2 M_1 + k_2^2 M_2\right)\frac{ds}{ds_{M_2}} - k_2^2 M_2 = 0$$

$$\left(-\frac{\dot{k_2}}{k_2}\right) T + \left(-k_1\right) M_1 + \left(+k_2 - k_2^2\right) M_2 = 0$$

Since $\{T, M_1, M_2\}$ Bishop frame is linearly independent system , we have

$$\begin{aligned}\frac{\dot{k}_2}{k_2} &= 0,\\ k_1 &= 0\\ \left(k_2 - k_2^2\right) &= 0.\end{aligned}$$

Hence we have

$$k_1 = 0,\ k_2 = 1$$

Corollary 4.3. If the natural lift $\overline{\gamma}_{M_2}$ of γ_{M_2} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows,

$$T = T,\ M_1 = -B,\quad M_2 = -N,\ (k_1 = 0,\ k_2 = 1)$$

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