On the spectrum of singular Hahn-Sturm-Liouville operators

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Abstract

In this paper, we study the spectrum of Hahn-Sturm-Liouville operators. In this context, it is shown that the regular symmetric Hahn-Sturm-Liouville operator is semi-bounded from below. Moreover, we give some conditions for the self-adjoint singular Hahn-Sturm-Liouville operator to have a discrete spectrum. The method of proof is based on the splittings technique. Finally, we investigate the continuous spectrum of this operator.

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1 Introduction

The spectrum of an operator is one of the most popular problems in the operator theory. Especially, the spectrum of a self-adjoint differential operator is a very interesting subject. It consists of discrete spectrum and of continuous spectrum. Further, it depends on the behavior of the coefficients of the corresponding differential expression. Hence, many mathematicians investigated this subject (see [3]-[15]).

The Hahn difference operator theory was initiated by Hahn (see, e.g., [20], [21]) in order to construct a theory that can unify the quantum $q-$difference operator theory ([32]) and the forward difference operator theory ([34], [35]). The Hahn difference operator $D_{\omega,q}$ is defined by

$$D_{\omega,q}f\left(x\right)=\left(f\left(\omega+qx\right)-f\left(x\right)\right)\left(\omega+\left(q-1\right)x\right)^{-1},$$

where $q\in(0,1)$ and $\omega>0$. This operator has numerous applications in the construction of families of orthogonal polynomials, approximation problems (see [26], [27], [28], [29], [30], [31]). A proper inverse of $D_{\omega,q}$ was given [17], [18]. Next, in [23], Hamza et al. studied the theory of linear Hahn difference equations. They also investigate the existence and uniqueness of solutions for the initial value problems for Hahn difference equations, in addition, they proved Gronwall's and Bernoulli's inequalities with respect to the Hahn difference operator. Later, Hamza and Makharesh [24] studied Leibniz's rule and Fubini's theorem associated with the Hahn difference operator. Sitthiwirattham [33] investigated the boundary value problem for the nonlinear Hahn difference equation. Recently, in [25], Annaby et al. established a Sturm-Liouville theory associated with the Hahn difference operator in the regular setting. In [19], the authors study a couple of sampling theorems of Lagrange-type interpolation for $\omega,q-$integral transforms, whose kernels are either solutions or Green's function of the $\omega,q-$Hahn-Sturm-Liouville problem.

The aim of this paper is to investigate the spectrum of the Hahn-Sturm-Liouville operator associated with the difference expression

$$(Ty)(x):=-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\left(p(x)D_{\omega,q}y\left(x\right)\right)+r\left(x\right)y\left(x\right),\ \omega_{0}<x<a\leq\infty,\tag{1}$$

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where p, r are real-valued functions defined on $[\omega_0, (a-\omega)/q)$ and continuous at ω_0 $(p(x) \neq 0$, $x \in [\omega_0, (a-\omega)/q))$. Firstly, we shall prove that the regular symmetric Hahn-Sturm-Liouville operator is semi-bounded from below. Later, using the splitting method ([4]), we will give some conditions for this self-adjoint singular operator to have a discrete spectrum. We also investigate the continuous spectrum of this operator.

2 Preliminaries

In this section, we present some basic concepts concerning the theory of Hahn difference calculus. For more details, see [18], [20], [21], [25]. Throughout the paper, we let $q \in (0,1)$ and $\omega > 0$.

Define $\omega_0 := \omega/(1-q)$ and let I be a real interval containing ω_0.

Definition 2.1 ([20], [21]). Let $f : I \to \mathbb{R} := (-\infty, \infty)$ be a function. The Hahn difference operator is defined by

$$D_{\omega,q} f(x) = \begin{cases} D_{\omega,q} f(x) = (f(\omega + qx) - f(x))(\omega + (q-1)x)^{-1}, & x \neq \omega_0, \\ f'(\omega_0), & x = \omega_0, \end{cases}$$

provided that f is differentiable at ω_0. In this case, we call $D_{\omega,q} f$, the $\omega, q-$derivative of f.

Remark 2.2. The Hahn difference operator unifies two well-known operators. When $q \to 1$, we get the forward difference operator, which is defined by

$$\Delta_\omega f(x) := \frac{f(\omega + x) - f(x)}{(\omega + x) - x}, \ x \in \mathbb{R}.$$

When $\omega \to 0$, we get the Jackson $q-$difference operator, which is defined by

$$D_q f(x) := \frac{f(qx) - f(x)}{(qx) - x}, \ x \neq 0.$$

Furthermore, under appropriate conditions, we have

$$\lim_{\substack{q \to 1 \\ \omega \to 0}} D_{\omega,q} f(x) = f'(x).$$

In what follows, we will give several important properties of the $\omega, q-$ derivative.

Theorem 2.3 ([18]). Let $f, g : I \to \mathbb{R}$ be $\omega, q-$ differentiable at $x \in I$ and $h(x) := \omega + qx$, then we have for all $x \in I$:

i) $D_{\omega,q}(af + bg)(x) = aD_{\omega,q} f(x) + bD_{\omega,q} g(x)$, $a, b \in I$,
ii) $D_{\omega,q}(fg)(x) = D_{\omega,q}(f(x)) g(x) + f(\omega + xq) D_{\omega,q} g(x)$,
iii) $D_{\omega,q}\left(\frac{f}{g}\right)(x) = \frac{D_{\omega,q}(f(x))g(x) - f(x)D_{\omega,q}g(x)}{g(x)g(\omega + xq)}$,
iv) $D_{\omega,q} f\left(h^{-1}(x)\right) = D_{-\omega q^{-1}, q^{-1}} f(x)$, $h^{-1}(x) = (x - \omega)/q$.

Now, we will give the $\omega, q-$integral of the function f.

Definition 2.4 (Jackson-Nörlund Integral [18]). Let $f : I \to \mathbb{R}$ be a function and $a, b, \omega_0 \in I$. We define $\omega, q-$integral of the function f from a to b by

$$\int_a^b f(x)\, d_{\omega,q}(x) := \int_{\omega_0}^b f(x)\, d_{\omega,q}(x) - \int_{\omega_0}^a f(x)\, d_{\omega,q}(x),$$

where

$$\int_{\omega_0}^x f(t)\, d_{\omega,q}(t) := ((1-q)x - \omega) \sum_{n=0}^{\infty} q^n f\left(\omega \frac{1-q^n}{1-q} + xq^n\right), x \in I$$

provided that the series converges at $x = a$ and $x = b$. In this case, f is called $\omega, q-$integrable on $[a, b]$.

Similarly, one can define the $\omega, q-$integral of the function f over $[\omega_0, \infty)$ by

$$\int_{\omega_0}^{\infty} f(x)\, d_{\omega,q}(x) := \lim_{b\to\infty} \int_{\omega_0}^b f(x)\, d_{\omega,q}(x).$$

The following properties of $\omega, q-$integration can be found in [18].

Lemma 2.5 ([18]). Let $f, g : I \to \mathbb{R}$ be $\omega, q-$integrable on I, $a, b, c \in I$, $a < c < b$ and $\alpha, \beta \in \mathbb{R}$. Then the following formulas hold:

i) $\int_a^b \{\alpha f(x) + \beta g(x)\}\, d_{\omega,q}(x) = \alpha \int_a^b f(x)\, d_{\omega,q}(x) + \beta \int_a^b g(x)\, d_{\omega,q}(x)$,
ii) $\int_a^a f(x)\, d_{\omega,q}(x) = 0$,
iii) $\int_a^b f(x)\, d_{\omega,q}(x) = \int_a^c f(x)\, d_{\omega,q}(x) + \int_c^b f(x)\, d_{\omega,q}(x)$,
iv) $\int_a^b f(x)\, d_{\omega,q}(x) = -\int_b^a f(x)\, d_{\omega,q}(x)$,
Next, we present the $\omega, q-$integration by parts.

Lemma 2.6 ([18]). Let $f, g : I \to \mathbb{R}$ be $\omega, q-$integrable on I, $a, b \in I$, and $a < b$. Then the following formula holds:

$$\int_a^b f(x)\, D_{\omega,q} g(x)\, d_{\omega,q}(x) + \int_a^b g(\omega + qx)\, D_{\omega,q} f(x)\, d_{\omega,q}(x)$$
$$= f(b)\, g(b) - f(a)\, g(a).$$

The next result is the fundamental theorem of Hahn calculus.

Theorem 2.7 ([18]). Let $f : I \to \mathbb{R}$ be continuous at ω_0. Define

$$F(x) := \int_{\omega_0}^x f(t)\, d_{\omega,q}(t),\ x \in I.$$

Then F is continuous at ω_0. Moreover, $D_{\omega,q} F(x)$ exists for every $x \in I$ and $D_{\omega,q} F(x) = f(x)$. Conversely,

$$\int_a^b D_{\omega,q} F(x)\, d_{\omega,q}(x) = F(b) - F(a).$$

Let $L^2_{\omega,q}(\omega_0,\infty)$ be the space of all complex-valued functions defined on $[\omega_0,\infty)$ such that

$$\|f\| := \left(\int_{\omega_0}^{\infty} |f(x)|^2 \, d_{\omega,q}x\right)^{1/2} < \infty.$$

The space $L^2_{\omega,q}(\omega_0,\infty)$ is a separable Hilbert space with the inner product

$$(f,g) := \int_{\omega_0}^{\infty} f(x)\,\overline{g(x)} d_{\omega,q}x,\ \ f,g \in L^2_{\omega,q}(\omega_0,\infty)$$

(see [25]).

Definition 2.8. Let D_L denote a subset of the complex Hilbert space H. A linear operator L is said to be Hermitian if, for all $x,y \in D_L$, $(Lx,y) = (x,Ly)$ holds. A Hermitian operator with a domain of definition dense in H is called a symmetric operator. An operator L^* defined on $D_{L^*} \subseteq H$ is called the adjoint of the symmetric operator L if for all $x \in D_L,\ y \in D_{L^*},\ (Lx,y) = (x, L^*y)$. An operator with a domain of definition dense set in H is said to be self-adjoint if $L = L^*$. An operator L is said to be compact if it maps every bounded set into a compact set (see [2]).

Definition 2.9. A complex number λ is called a regular point of the linear operator L acting in a complex Hilbert space H if

(R1) *the inverse* $R_\lambda(L) = (L - \lambda I)^{-1}$ (*where* I *is the identity operator in* H) *exists, and*
(R2) $R_\lambda(L)$ *is a bounded operator defined on the whole space* H.
Let
(R3) $R_\lambda(L)$ *is defined on a set which dense* H.

The operator $R_\lambda(L)$ is then called the *resolvent* of the operator L. All non-regular points λ are called points of the *spectrum* of the operator L.

The point spectrum or discrete spectrum $\sigma_p(L)$ is the set such that $R_\lambda(L)$ does not exist. A $\lambda \in \sigma_p(L)$ is called an eigenvalue of L. The spectrum of the operator L is said to be *purely discrete* if it consists of a denumerable set of eigenvalues with no finite point of accumulation.

The *continuous spectrum* $\sigma_c(L)$ is the set such that $R_\lambda(L)$ exists and satisfies **(R3)** but not **(R2)**.

The *residual spectrum* $\sigma_r(L)$ is the set such that $R_\lambda(L)$ exists but does not satisfy **(R3)** (see [16]).

Theorem 2.10 ([16])**.** The residual spectrum $\sigma_r(L)$ of a self-adjoint linear operator acting on a complex Hilbert space H is empty.

Theorem 2.11 ([2])**.** All self-adjoint extensions of a closed, symmetric operator which has equal and finite deficiency indices have one and same continuous spectrum.

Definition 2.12 ([2])**.** The direct sum $L_1 \oplus L_2$ of two operators L_1, L_2 in the spaces $H_1,\ H_2$ is an operator in the space $H_1 \oplus H_2$ of all ordered pairs $\{x_1,x_2\},\ x_1 \in H_1, x_2 \in H_2$; its domain of definition is the set of all ordered pairs $\{x_1,x_2\},\ x_1 \in D_{L_1},\ x_2 \in D_{L_2}$, and

$$(L_1 \oplus L_2)\{x_1,x_2\} = \{L_1x_1, L_2x_2\}.$$

It is easily seen that if L_1 and L_2 are each self-adjoint operators, then their direct sum $L_1 \oplus L_2$ is also a self-adjoint operator.

Definition 2.13 ([2])**.** A symmetric operator L is said to be semi-bounded from below if there is a number m such that, for all $x \in D_L$, the inequality

$$(Lx, x) \geq m \left\| x \right\|^2$$

holds. Similarly, if for all $x \in D_L$, there is a number M such that the inequality

$$(Lx, x) \leq M \left\| x \right\|^2$$

holds, then L is said to be semi-bounded from above.

Theorem 2.14 ([2])**.** If a symmetric operator L with finite deficiency indices (n, n) satisfies the condition

$$(Lx, x) \geq m \left\| x \right\|^2, \quad x \in D_L,$$

or the condition

$$(Lx, x) \leq M \left\| x \right\|^2, \quad x \in D_L,$$

then the part of the spectrum of every self-adjoint extension of L which lies to the left of m or to the right of M can consist of only a finite number of eigenvalues and the sum of their multiplicities does not exceed n.

3 Symmetric operator

In this section, we construct a symmetric Hahn-Sturm-Liouville operator. Let us consider the linear set $D_{\max}$ consisting of all vectors $y \in L^2_{\omega,q}(\omega_0, a)$ such that y and $pD_{\omega,q}y$ are continuous at ω_0 and $Ty \in L^2_{\omega,q}(\omega_0, a)$. We define the *maximal operator* $T_{\max}$ on $D_{\max}$ by the equality $T_{\max}y = Ty$.

The $\omega, q-$Wronskian of $y(.), \ z(.)$ is defined to be

$$W_q(y, z)(x) := y(x)D_{\omega,q}z(x) - z(x)D_{\omega,q}y(x), \ x \in [\omega_0, a). \tag{2}$$

For every $y, z \in D_{\max}$ we have $\omega, q-$Green's formula

$$\int_{\omega_0}^{a} (Ty)(x)\overline{z(x)}d_{\omega,q}x - \int_{\omega_0}^{a} y(x)\overline{(Tz)(x)}d_{\omega,q}x = [y, z](a) - [y, z](\omega_0), \tag{3}$$

where

$$[y, z](x) := p(x)\left\{y(x)\overline{D_{-\omega q^{-1},q^{-1}}z(x)} - D_{-\omega q^{-1},q^{-1}}y(x)\overline{z(x)}\right\}$$

(see [25]).

Let $\omega_0 < a < \infty$ and $D_{\min}$ be the linear set of all vectors $y \in D_{\max}$ satisfying the conditions

$$y(\omega_0) = (pD_{-\omega q^{-1},q^{-1}}y)(\omega_0) = y(a) = (pD_{-\omega q^{-1},q^{-1}}y)(a) = 0. \tag{4}$$

The operator $T_{\min}$, that is the restriction of the operator $T_{\max}$ to $D_{\min}$ is called the *minimal operator.*

Theorem 3.1. The operator $T_{\min}$ is Hermitian.

Proof. By applying $\omega, q-$Green's formula (3) to the functions y, z in $D_{\min}$, we have

$$\int_{\omega_0}^{a} (Ty)(x)\overline{z(x)}d_{\omega,q}x = \int_{\omega_0}^{a} y(x)\overline{(Tz)(x)}d_{\omega,q}x,$$

i.e.,

$$(T_{\min}y, z) = (y, T_{\min}z).$$

Q.E.D.

Theorem 3.2. Let $f \in L^2_{\omega,q}(\omega_0, a)$. Then, the equation

$$T(y) = f \tag{5}$$

has a solution $y(x)$ satisfying the conditions

$$y(\omega_0) = (pD_{-\omega q^{-1},q^{-1}}y)(\omega_0) = y(a) = (pD_{-\omega q^{-1},q^{-1}}y)(a) = 0, \tag{6}$$

if and only if the function f is orthogonal to all solutions of the equation

$$T(y) = 0.$$

Proof. Let $y(x)$ be the solution of the equation $T(y) = f$ satisfying the conditions

$$y(\omega_0) = (pD_{-\omega q^{-1},q^{-1}}y)(\omega_0) = 0. \tag{7}$$

There exists one such solution (see [25]). Let us denote by z_1 and z_2, a fundamental system of solutions of the equation $T(z) = 0$ satisfying the conditions

$$\begin{aligned} z_1(a) = 1, (pD_{-\omega q^{-1},q^{-1}}z_1)(a) = 0, \\ z_2(a) = 0, (pD_{-\omega q^{-1},q^{-1}}z_2)(a) = 1. \end{aligned} \tag{8}$$

Applying $\omega, q-$Green's formula (3) to the functions $y(x)$ and $z_i(x)$ $(i = 1, 2)$, we conclude that

$$(f, z_i) = (T(y), z_i) = [y, z_i](a) - [y, z_i](\omega_0) + (y, T(z_i)). \tag{9}$$

By the conditions (7), we deduce that $[y, z_i](\omega_0) = 0$. It follows from $T(z_i) = 0$ that

$$(f, z_i) = [y, z_i](a) = \begin{cases} -\left(pD_{-\omega q^{-1},q^{-1}}y\right)(a) & \text{for } i = 1 \\ y(a) & \text{for } i = 2, \end{cases} \tag{10}$$

and this is precisely the assertion of the theorem. Q.E.D.

Denote by Ω the set of all solutions of the equation $T(z) = 0$. Further, we denote by $\digamma$ the range of the operator $T_{\min}$. It follows from Theorem 16 that

$$L^2_{\omega,q}(\omega_0, a) = \Omega \oplus \digamma. \tag{11}$$

Theorem 3.3. For arbitrary complex numbers $\alpha_1, \alpha_2, \alpha_3$ and α_4, there exists a function $y \in D_{\max}$ satisfying the conditions

$$\begin{aligned} y(\omega_0) = \alpha_1,\ (pD_{-\omega q^{-1},q^{-1}}y)(\omega_0) = \alpha_2, \\ y(a) = \alpha_3,\ (pD_{-\omega q^{-1},q^{-1}}y)(a) = \alpha_4. \end{aligned} \tag{12}$$

Proof. Firstly, we will prove the theorem for the special case when α_1 and α_2 are zero. Let f be an arbitrary vector in $L^2_{\omega,q}(\omega_0, a)$ satisfying the conditions

$$(f, z_i) = \begin{cases} -\alpha_4 & \text{for } i = 1 \\ \alpha_3 & \text{for } i = 2. \end{cases} \tag{13}$$

Here z_1 and z_2 are a fundamental system of solutions of the equation $T(z) = 0$. There exists such a vector f. If we put

$$f = c_1 z_1 + c_2 z_2,$$

then the conditions (13) provide a system of equations in the constants c_i $(i = 1, 2)$ whose determinant is the same as the Gram determinant for the linearly independent functions z_1, z_2, and does not vanish.

Let ξ denote the solution of the equation

$$T(\xi) = f$$

satisfying the conditions

$$\xi(\omega_0) = (pD_{-\omega q^{-1}, q^{-1}}\xi)(\omega_0) = 0. \tag{14}$$

Then we have

$$\xi(a) = \alpha_3, \ (pD_{-\omega q^{-1}, q^{-1}}\xi)(a) = \alpha_4.$$

Applying $\omega, q-$Green's formula (3) to the functions $\xi(x)$ and $z_i(x)$ $(i = 1, 2)$, we get

$$(f, z_i) = (T(\xi), z_i) = [\xi, z_i](a) - [\xi, z_i](\omega_0) + (\xi, T(z_i)). \tag{15}$$

It follows from $T(z_i) = 0$ and (14) that $[\xi, z_i](a) = 0$. From conditions (8) and (10), we get

$$[\xi, z_i](a) = \begin{cases} -\left(pD_{-\omega q^{-1}, q^{-1}}\xi\right)(a) & \text{for } i = 1 \\ \xi(a) & \text{for } i = 2. \end{cases}$$

From (13) and (15), we conclude that

$$\xi(a) = \alpha_3, \ (pD_{-\omega q^{-1}, q^{-1}}\xi)(a) = \alpha_4.$$

Thus, we have constructed a function $\xi \in D_{\max}$ such that

$$\xi(\omega_0) = (pD_{-\omega q^{-1}, q^{-1}}\xi)(\omega_0) = 0,$$
$$\xi(a) = \alpha_3, \ (pD_{-\omega q^{-1}, q^{-1}}\xi)(a) = \alpha_4.$$

Similarly, one can construct a function $\eta \in D_{\max}$ such that

$$\eta(\omega_0) = \alpha_1, \ (pD_{-\omega q^{-1}, q^{-1}}\eta)(\omega_0) = \alpha_2,$$
$$\eta(a) = 0, \ (pD_{-\omega q^{-1}, q^{-1}}\eta)(a) = 0.$$

Then the function $y = \xi + \eta \in D_{\max}$ satisfies the conditions (12). Q.E.D.

Theorem 3.4. $D_{\min}$ is dense in $L^2_{\omega,q}(\omega_0, a)$.

Proof. We will show that every vector ζ orthogonal to $D_{\min}$ is zero. Let ζ be such a vector, i.e.,

$$(\zeta, y) = 0, \text{ for all } y \in D_{\min}.$$

Let ν be any particular solution of the equation $T(\nu) = \zeta$. For an arbitrary vector $y \in D_{\min}$, we have

$$(\nu, T_{\min} y) = (T_{\max}\nu, y) = (T(\nu), y) = (\zeta, y) = 0.$$

An application of Theorem 16 yields $\zeta = 0$. Q.E.D.

It follows from Theorem 15 and Theorem 18 that $T_{\min}$ is a symmetric operator.

Theorem 3.5. The equality $T_{\max} = T_{\min}^{*}$ holds.

Proof. For arbitrary vectors $y \in D_{\min}$ and $z \in D_{\max}$, we have

$$(T_{\min} y, z) = (y, T_{\max} z),$$

i.e., $T_{\max} \subset T_{\min}^{*}$. Hence, we have to prove the converse. Let ζ be an arbitrary vector in the domain of definition $D_{\min}^{*}$ of the operator $T_{\min}^{*}$ and $T_{\min}^{*}\zeta = \nu$. Further, we denote by $\xi(x)$ any particular solution of the equation $T(\xi) = \nu$. Then we have

$$(\nu, y) = (T(\xi), y) = (T_{\max}\xi, y) = (\xi, T_{\min} y), \text{ for every } y \in D_{\min}. \tag{16}$$

By definition of the adjoint operator, we get

$$(\nu, y) = (T_{\min}^{*}\zeta, y) = (\zeta, T_{\min} y). \tag{17}$$

Subtracting (17) from (16), we have

$$(\xi - \zeta, T_{\min} y) = 0,$$

i.e., $\xi - \zeta \in \digamma^{\perp}$. By virtue of (11), we deduce that $\xi - \zeta \in \Omega$. Thus, $T(\xi - \zeta) = 0$, i.e, $T\zeta = T\xi = \nu = T_{\min}^{*}\zeta$. Q.E.D.

Theorem 3.6. The equality $T_{\max}^{*} = T_{\min}$ holds.

Proof. From Theorem 19, we have

$$T_{\max}^{*} = T_{\min}^{**} \supset T_{\min}.$$

Thus, we have to show the opposite inclusion. Since $T_{\min} \subset T_{\max}$, we arrive at

$$T_{\max}^{*} \subset T_{\min}^{*} = T_{\max}. \tag{18}$$

Let ξ be a vector in the domain of definition $D_{\max}^{*}$ of the operator $T_{\max}^{*}$. From (18), we have $\xi \in D_{\max}$ and $T_{\max}^{*}\xi = T_{\max}\xi$. Then we get

$$(T_{\max}^{*}\xi, y) = (\xi, T_{\max} y),$$

$$(T_{\max}\xi, y) = (\xi, T_{\max} y) \text{ for all } y \in D_{\max}.$$

Using the $\omega, q-$Green's formula (3), we conclude that

$$[\xi, y](a) - [\xi, y](\omega_0) = 0 \text{ for all } y \in D_{\max}. \tag{19}$$

It follows from Theorem 17 that the equation (19) is possible if

$$\xi(\omega_0) = (pD_{-\omega q^{-1}, q^{-1}}\xi)(\omega_0) = \xi(a) = (pD_{-\omega q^{-1}, q^{-1}}\xi)(a) = 0,$$

i.e., $\xi \in D_{\min}$. Q.E.D.

It follows from Theorem 20 that $T_{\min}$ is a closed, symmetric operator. Furthermore, the deficiency indices of the operator $T_{\min}$ is $(2, 2)$.

4 The spectrum of the self-adjoint operator

In this section, we will study the spectrum of the self-adjoint Hahn-Sturm-Liouville operators.

Theorem 4.1. Let $p(x) > 0$ $(x \in [\omega_0, h^{-1}(a)])$, $\omega_0 < a < \infty)$. Then the operator $T_{\min}$ is semi-bounded from below. Moreover, the negative part of the spectrum of every self-adjoint extension of $T_{\min}$ consists of not more than a finite number of negative eigenvalues of finite multiplicity.

Proof. Using $\omega, q-$integration by parts, we get, for $y \in D_{\min}$,

$$\begin{aligned}(T_{\min}y, y) &= \int_{\omega_0}^{a} T(y)\,\overline{y} d_{\omega,q}x \\ &= \int_{\omega_0}^{a} \left[-\frac{1}{q} D_{-\omega q^{-1}, q^{-1}}(p(x)D_{\omega,q}y(x)) + r(x)y(x)\right] \overline{y} d_{\omega,q}x \\ &= \int_{\omega_0}^{a} \left[-\frac{1}{q} D_{-\omega q^{-1}, q^{-1}}(p(x)D_{\omega,q}y(x))\,\overline{y}(x) + r(x)|y(x)|^2\right] d_{\omega,q}x \\ &= \int_{\omega_0}^{a} [p(x)|D_{\omega,q}y(x)|^2 + r(x)|y(x)|^2] d_{\omega,q}x.\end{aligned}$$

Define

$$v(x, \xi) = \begin{cases} 1, & \xi \le x \\ 0, & \xi > x, \end{cases}$$

and

$$H(\xi, \eta) = -\int_{\omega_0}^{a} r(x)\, v(x, \xi)\, v(x, \eta)\, d_{\omega,q}x.$$

For $y \in D_{\min}$ we conclude that

$$y(x) = \int_{\omega_0}^{a} \frac{v(x, \xi)(pD_{\omega,q}y)(\xi)}{p(\xi)} d_{\omega,q}\xi.$$

Thus we have

$$\begin{aligned}&(T_{\min}y,y)\\&=\int_{\omega_0}^{a}\frac{\left|(pD_{\omega,q}y)(\xi)\right|^2}{p(\xi)}d_{\omega,q}\xi\\&-\int_{\omega_0}^{a}\int_{\omega_0}^{a}\frac{H(\xi,\eta)(pD_{\omega,q}y)(\xi)(pD_{\omega,q}\overline{y})(\eta))}{p(\xi)p(\eta)}d_{\omega,q}\xi d_{\omega,q}\eta.\end{aligned} \tag{20}$$

Let $L^2_{\omega,q,p}(\omega_0,a)$ be the Hilbert space of all complex-valued functions defined on $[\omega_0,a]$ with the inner product

$$(f_1,f_2)_1=\int_{\omega_0}^{a}f_1(x)\overline{f_2(x)}\frac{1}{p(x)}d_{\omega,q}x.$$

In $L^2_{\omega,q,p}(\omega_0,a)$ we consider the integral operator $\mathcal{K}$ with the symmetric kernel $H(\xi,\eta)$:

$$\mathcal{K}f=\int_{\omega_0}^{a}\frac{H(\xi,\eta)}{p(\eta)}f(\eta)d_{\omega,q}\eta,$$

where

$$\int_{\omega_0}^{a}\int_{\omega_0}^{a}\frac{\left|H(\xi,\eta)\right|^2}{p(\xi)p(\eta)}d_{\omega,q}\xi d_{\omega,q}\eta<\infty.$$

Then $\mathcal{K}$ is a compact operator in the space $L^2_{\omega,q,p}(\omega_0,a)$ ([25]).

Let $\varphi_1,\varphi_2,\varphi_3,\ldots$ be a complete orthonormal system of eigenfunctions of the operator $\mathcal{K}$ and $\lambda_1,\lambda_2,\lambda_3,\ldots$ be the corresponding eigenvalues. Thus we have

$$(\mathcal{K}f,f)_1=\sum_{k=1}^{\infty}\lambda_k\left|(f,\varphi_k)_1\right|^2.$$

As $k\to\infty$, we have $\lambda_k\to 0$. Then there exists a certain number N such that $\lambda_k<1$ for $k>N$. For $(f,\varphi_k)_1=0,\ k=1,2,\ldots,N$, we have

$$(\mathcal{K}f,f)_1=\sum_{k=N+1}^{\infty}\lambda_k\left|(f,\varphi_k)_1\right|^2\leq\sum_{k=N+1}^{\infty}\left|(f,\varphi_k)_1\right|^2,$$

i.e.,

$$(\mathcal{K}f,f)_1\leq(f,f)_1. \tag{21}$$

Let $\mathcal{D}$ denote the manifold of all functions $y\in D_{\min}$ which satisfy the conditions

$$(pD_{\omega,q}f,\varphi_k)_1=0,\ k=1,2,\ldots,N,\ y\in D_{\min}.$$

By virtue of (21), we deduce that, for $y \in \mathcal{D}$,

$$\int_{\omega_0}^{a}\int_{\omega_0}^{a}\frac{H(\xi,\eta)\,(pD_{\omega,q}y)(\xi))\,(pD_{\omega,q}\overline{y}(\eta))}{p(\xi)\,p(\eta)}d_{\omega,q}\xi d_{\omega,q}\eta$$

$$\leq (\mathcal{K}pD_{\omega,q}y, pD_{\omega,q}y)_1 \leq (pD_{\omega,q}y, pD_{\omega,q}y)_1$$

$$= \int_{\omega_0}^{a}\frac{|(pD_{\omega,q}y)(\xi)|^2}{p(\xi)}d_{\omega,q}\xi.$$

It follows from equality (20) that

$$(T_{\min}y, y) \geq 0.$$

On the other hand, the dimension of the manifold $D_{\min}$ modulo $\mathcal{D}$ is N, and therefore, the operator $T_{\min}$ is semi-bounded from below on the whole manifold $D_{\min}$. By Theorem 14, we get the desired conclusion. Q.E.D.

Let $a = \infty$ and $D_{\min}$ be the linear set of all vectors $y \in D_{\max}$ satisfying the conditions

$$y(\omega_0) = (pD_{-\omega q^{-1},q^{-1}}y)(\omega_0) = 0,\ [y,z](\infty) = 0,$$

for arbitrary $z \in D_{\max}$. The operator $T_{\min}$, that is the restriction of the operator $T_{\max}$ to $D_{\min}$ is called the *minimal operator.*

Let H' denotes the set of all functions f from $L^2_{\omega,q}(\omega_0,\infty)$ which vanish outside a finite interval $[\alpha,\beta] \subset [\omega_0,\infty)$ and $D'_{\min} = H' \cap D_{\min}$. Moreover, let $T'_{\min}$ denote the restriction of the operator $T_{\min}$ to $D'_{\min}$. Then $T_{\min}$ is the closure of the operator $T'_{\min}$, i.e., $\widetilde{T'_{\min}} = T_{\min}$ ([2]).

Now we restrict $D'_{\min}$ by imposing the additional conditions

$$y(c) = (pD_{-\omega q^{-1},q^{-1}}y)(c) = 0,$$

where c is a fixed point of the interval (ω_0,∞). By this restriction, we obtain the manifold $D''_{\min}$.

The restriction $T''_{\min}$ of the operator $T'_{\min}$ to $D''_{\min}$ is called the splitting of the operator $T'_{\min}$ at the point c of the interval (ω_0,∞). It is clear that

$$T''_{\min} = T'_1 \oplus T'_2, \tag{22}$$

i.e., the operator $T''_{\min}$ is the direct sum of two operators T'_1 and T'_2 in the spaces $L^2_{\omega,q}(\omega_0,c)$ and $L^2_{\omega,q}(c,\infty)$, where T'_1 and T'_2 are generated in these spaces from the Hahn-Sturm-Liouville expression T in the same way as $T'_{\min}$ was.

If $T_1 = \widetilde{T'_1}$ and $T_2 = \widetilde{T'_2}$ are the closures of the operators T'_1 and T'_2, then (22) implies that

$$\widetilde{T''_{\min}} = T_1 \oplus T_2.$$

If we extend the symmetric operators T_1 and T_2 into self-adjoint operators $T_{1,s}$ and $T_{2,s}$ in the spaces $L^2_{\omega,q}(\omega_0,c)$ and $L^2_{\omega,q}(c,\infty)$ respectively, then the direct sum

$$L = T_{1,s} \oplus T_{2,s}$$

will be a self-adjoint extension of the symmetric operator $\widetilde{T''_{\min}}$. The spectrum of the operator L is the set-theoretic sum of the spectra of $T_{1,s}$ and $T_{2,s}$.

It follows from Theorem 11 that all self-adjoint extensions of the operator $\widetilde{T''_{\min}}$ have one and the same continuous spectrum, since its deficiency indices are finite. Both the operator L and also each self-adjoint extension T_s of the operator $T_{\min}$ are such extensions. Hence, the continuous parts of the spectrum of the two operators L and T_s coincide.

Then we have the following theorem.

Theorem 4.2. The continuous parts of the spectrum of every self-adjoint extension of the operator $T_{\min}$ is the set-theoretic sum of the continuous parts of the spectra of $T_{1,s}$ and $T_{2,s}$, where $T_{1,s}$ and $T_{2,s}$ have been obtained by the splitting of the operator $T_{\min}$.

Theorem 4.3. If

$$\lim_{x\to\infty} r(x) = +\infty \tag{23}$$

and

$$p(x) > 0, \ x \in [\omega_0, \infty) \tag{24}$$

then every self-adjoint extension T_s of the singular operator $T_{\min}$ has a purely discrete spectrum.

Proof. Let $N > 0$ be an arbitrary number. From (23), one can choose a number c such that

$$|r(x)| > N \text{ for } c < x < \infty. \tag{25}$$

By the condition (24), via $\omega, q-$integration by parts, we obtain ($y \in D_{T_2'}$)

$$(T_2'y, y) = \int_c^\infty Ty\overline{y}d_{\omega,q}x = \int_c^\infty \left[-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}(pD_{\omega,q}y) + ry\right]\overline{y}d_{\omega,q}x$$

$$= \int_c^\infty \left[-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}(pD_{\omega,q}y)\,\overline{y} + r(x)\,|y|^2\right]d_{\omega,q}x$$

$$= \int_c^\infty p\,|D_{\omega,q}y|^2 + r(x)\,|y|^2\,d_{\omega,q}x > N\int_c^\infty |y|^2\,d_{\omega,q}x = N(y,y).$$

Hence the operator T_2' is bounded from below and its closure T_2 is also bounded from below by the number N. Therefore, by Theorem 14, the half-axis $-\infty < \lambda < N$, contains no point of the continuous spectrum of the self-adjoint extension $T_{2,s}$ of T_2.

On the other hand, since the operator T_1 is regular, the spectrum of any self-adjoint extension $T_{1,s}$ of T_1 is purely discrete. Hence the half-axis $-\infty < \lambda < N$, contains no point of the continuous spectrum of $L = T_{1,s} \oplus T_{2,s}$.

By Theorem 22, every self-adjoint extension T_s of the operator $T_{\min}$ has this property. Since the number N is arbitrary, the spectrum of the operator T_s has no continuous part at all. Q.E.D.

Theorem 4.4. Let

$$\lim_{x\to\infty} r(x) = M$$

and $p(x) > 0$ $(x \in [\omega_0, \infty))$. Then the interval $(-\infty, M)$ contains no point of the continuous spectrum of any, self-adjoint extension T_s of the singular operator $T_{\min}$; on the contrary, any T_s can only have at most point-eigenvalues on this interval and these can have a point of accumulation only at $\lambda = M$.

Proof. If we decompose the operator at a point c such that

$$r(x) > M - \varepsilon \text{ for } c < x < \infty,$$

then we obtain

$$(T_2'y, y) > (M - \varepsilon)(y, y).$$

Hence, the part of the spectrum of T_2 lying in the interval $(-\infty, M - \varepsilon)$ can consist only of a finite number of eigenvalues of finite multiplicity. On the other hand, by Theorem 21, the operator T_1 is regular and bounded below. Hence its spectrum is purely discrete; and any point of accumulation of the spectrum $T_{1,s}$ can only be at $\lambda = +\infty$. Thus, from Theorem 21, we get the desired result. Q.E.D.

Now, we need the following lemma ([2]).

Lemma 4.5. If the interval $[\lambda_0 - \delta, \lambda_0 + \delta]$ contains no point of the spectrum of a self-adjoint operator L except perhaps for a finite number of eigenvalues each of finite multiplicity, and if Q is a bounded Hermitian operator satisfying the condition

$$\|Q\| < \delta,$$

then the point λ_0 does not lie in the continuous part of the spectrum of the operator $L + Q$.

Theorem 4.6. Let $p(x) \equiv 1$ and

$$\lim_{x \to \infty} |r(x)| = M.$$

Then any interval, of length greater than $2M$, of the positive half-axis contains of the continuous spectrum of any self-adjoint extension T_s of the singular operator $T_{\min}$.

Proof. Suppose, contrary to our claim, that an interval $[\lambda_0 - \delta, \lambda_0 + \delta]$ of the half-axis $\lambda > 0$ contains no point of the continuous spectrum of T_s, $\delta > M$. Then, the operator may be decomposed, this interval would contain no point of the continuous spectrum of any self-adjoint extension of T_2. If we choose the point c such that

$$|r(x)| \leq M + \varepsilon < \delta \text{ for } x > c,$$

then, by Lemma 25, λ_0 can not belong to the continuous spectrum of the self-adjoint extension of the minimal operator generated by the expression $-\frac{1}{q} D_{-\omega q^{-1}, q^{-1}} D_{\omega,q}$ and the same boundary conditions. But this is a contradiction because the continuous spectrum of the last operator covers the whole of the positive half-axis. Q.E.D.

In particular, for $M = 0$ we have the following corollary.

Corollary 4.7. Let $p(x) \equiv 1$ and

$$\overline{\lim_{x \to \infty}} |r(x)| = 0.$$

Then the whole positive half-axis is covered by the continuous spectrum of any self-adjoint extension T_s of the singular operator $T_{\min}$.

Corollary 4.8. Let $p(x) \equiv 1$ and

$$\overline{\lim_{x\to\infty}} |r(x)| = \beta < \infty, \quad \underline{\lim_{x\to\infty}} |r(x)| = \gamma > -\infty.$$

Then any interval, of length greater than $(\beta - \gamma)$, of the half-axis

$$\lambda > \frac{1}{2}(\beta + \gamma)$$

contains of the continuous spectrum of any self-adjoint extension T_s of the singular operator $T_{\min}$.

Proof. For, if $r_1(x) = r(x) - \frac{1}{2}(\beta + \gamma)$, then

$$\overline{\lim_{x\to\infty}} |r_1(x)| = \frac{1}{2}(\beta - \gamma),$$

and the result follows by replacing $r(x)$ by $r_1(x)$, i.e., by applying Theorem 21 to the operator $T_s - \frac{1}{2}(\beta + \gamma) I$. Q.E.D.

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