Infinite families of congruences for m-regular $[j,k]$-overpartitions in two colors

T. N. Veeranayaka[1], M. S. Mahadeva Naika[2] and Harishkumar T[3]

[1,3]Department of Mathematics, Bangalore University, Bengaluru, Karnataka, India
[2]Department of Mathematics, Central College Campus, Bengaluru City University, Bengaluru, Karnataka, India
E-mail: veernayak100@gmail.com[1], msmnaika@rediffmail.com[2], harishhaf@gmail.com[3]

Communicating editor - Emzar Khmaladze

Abstract

Let $\overline{a}^{m}_{j,k}(n)$ denote the number of overpartitions of n with two colors in which no parts are divisible by m and only parts congruent to j modulo k may be overlined. In this work, we establish many infinite families of congruences modulo powers of 2 for $\overline{a}^{3}_{1,3}(n)$, $\overline{a}^{3}_{1,4}(n)$ and congruences modulo powers of 2 and 3 for $\overline{a}^{9}_{1,4}(n)$.

2010 Mathematics Subject Classification. **11P83**. 05A15, 05A17.
Keywords. congruences, overpartitions, $[j,k]$-overpartions, color partitions.

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n with $p(0) = 1$.

In 2004, Corteel and Lovejoy [6] introduced overpartitions. An overpartition of a positive integer n is a partition in which the first occurrence of each distinct part may be overlined. For example, the overpartitions of 3 are

$$3,\ \overline{3},\ 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1,\ \overline{1}+1+1.$$

Let $\overline{p}(n)$ denote the number of overpartitions of n with $\overline{p}(0) = 1$. The generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}\,(n)\,q^n = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}, \tag{1.1}$$

where $(q;q)_\infty = (1-q)(1-q^2)(1-q^3)\cdots$.
Many authors found various Ramanujan-type identities and congruences for $\overline{p}(n)$. For more information, one can see [4, 5, 9, 10, 13, 18].

Lovejoy [14] introduced the partition function $\overline{A}_\ell(n)$ which counts the number of overpartitions of n with no parts divisible by ℓ. Shen [20] called the function $\overline{A}_\ell(n)$ as ℓ-regular overpartitions and the generating function is given by

$$\sum_{n=0}^{\infty} \overline{A}_\ell\,(n)\,q^n = \frac{f_2 f_\ell^2}{f_1^2 f_{2\ell}}, \tag{1.2}$$

where $f_\ell := \left(q^\ell;q^\ell\right)_\infty = \prod_{n=0}^{\infty}(1-q^{n\ell})$.
The arithmetic properties of ℓ-regular overpartitions have been studied by many authors, we can see [1, 17, 19, 21].

Advanced Studies: Euro-Tbilisi Mathematical Journal 15(3) (2022), pp. 53–74.
DOI: 10.32513/asetmj/19322008224
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 23 February 2020.
Accepted for publication: 25 June 2022.

Mahadeva Naika et al. [16] defined the partition function $\overline{p}_3(n)$, the number of overpartitions of n with 2-colors in which one of the color appears only in parts that are multiples of 3. For example, the 2-color overpartitions of 3 are

$$3_1,\ \overline{3}_1,\ 3_2,\ \overline{3}_2,\ 2_1+1_1,\ \overline{2}_1+1_1,\ 2_1+\overline{1}_1,\ \overline{2}_1+\overline{1}_1,\ 1_1+1_1+1_1,\ \overline{1}_1+1_1+1_1.$$

The generating function for $\overline{p}_3(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{p}_3(n)\,q^n=\frac{(-q;q)_\infty(-q^3;q^3)_\infty}{(q;q)_\infty(q^3;q^3)_\infty}. \tag{1.3}$$

They obtained many infinite families of congruences modulo powers of 2 and 3 for $\overline{p}_3(n)$. For example, for all $\alpha\geq 0$ and $n\geq 0$,

$$\overline{p}_3\left(3\cdot 4^{\alpha+2}n+10\cdot 4^{\alpha+1}\right)\equiv 0 \pmod{27}.$$

In [15], the authors defined $\overline{p}_{j,k}(n)$, the number of $[j,k]$-regular overpartitions of n in which none of the parts congruent to $j \pmod{k}$. Also, they found many infinite families of congruences modulo powers of 2 for $\overline{p}_{3,6}(n)$, $\overline{p}_{5,10}(n)$ and $\overline{p}_{9,18}(n)$.

On the motivation of the above work, in this paper, we define $\overline{a}^m_{j,k}(n)$, the number of overpartitions of n with 2-colors in which no parts are divisible by m and only parts congruent to j modulo k may be overlined. The generating function for $\overline{a}^m_{j,k}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{a}^m_{j,k}(n)\,q^n=\frac{(-q^j;q^k)_\infty(-q^{k-j};q^k)_\infty f_m f_{km}}{(-q^{mj};q^{km})_\infty(-q^{m(k-j)};q^{km})_\infty f_1 f_k}. \tag{1.4}$$

Also, we establish many infinite families of congruences modulo powers of 2 for $\overline{a}^3_{1,3}(n)$, $\overline{a}^3_{1,4}(n)$ and congruences modulo powers of 2 and 3 for $\overline{a}^9_{1,4}(n)$. For example, for all $n\geq 0$ and $\alpha,\beta\geq 0$,

$$\overline{a}^3_{1,4}\left(4\cdot 3^{4\alpha+2}\cdot 5^{4\beta+1}(5n+i)+\frac{17\cdot 3^{4\alpha+1}\cdot 5^{4\beta+1}-1}{2}\right)\equiv 0 \pmod{16},$$

where $i=0,1,3,4$.

2 Preliminary results

In this section, we record many identities which are useful in proving our main results.

Lemma 2.1. The following 2-dissections hold:

$$\frac{1}{f_1^2}=\frac{f_8^5}{f_2^5 f_{16}^2}+2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8} \tag{2.1}$$

and

$$\frac{1}{f_1^4}=\frac{f_4^{14}}{f_2^{14} f_8^4}+4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.2}$$

The identity (2.1) is the 2-dissection of $\varphi(q)$ [8, (1.9.4)]. The identity (2.2) is the 2-dissection of $\varphi(q)^2$ [8, (1.10.1)]. Also, one can see [2, p.40].

Lemma 2.2. The following 2-dissection holds:

$$\frac{f_3^2}{f_1^2}=\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}. \tag{2.3}$$

The identity (2.3) is essentially (30.10.4) in [8].

Lemma 2.3. The following 2-dissections hold:

$$\frac{f_3^3}{f_1}=\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}, \tag{2.4}$$

$$\frac{f_1^3}{f_3}=\frac{f_4^3}{f_{12}}-3q\frac{f_2^2f_{12}^3}{f_4f_6^2}, \tag{2.5}$$

$$\frac{f_1}{f_3^3}=\frac{f_2f_4^2f_{12}^2}{f_6^7}-q\frac{f_2^3f_{12}^6}{f_4^2f_6^9}, \tag{2.6}$$

$$\frac{f_3}{f_1^3}=\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}. \tag{2.7}$$

The equation (2.4) is the same as (22.1.14) in [8] (after using 22.1.6 and 22.1.7). The equation (2.5) is the same as (22.1.13) in [8] (after using 22.1.6 and 22.1.7). The equations (2.6) and (2.7) can be obtained from the equations (2.4) and (2.5) respectively by replacing q by $-q$.

Lemma 2.4. The following 2-dissections hold:

$$\frac{f_9}{f_1}=\frac{f_{12}^3f_{18}}{f_2^2f_6f_{36}}+q\frac{f_4^2f_6f_{36}}{f_2^3f_{12}} \tag{2.8}$$

and

$$\frac{f_1}{f_9}=\frac{f_2f_{12}^3}{f_4f_6f_{18}^2}-q\frac{f_4f_6f_{36}^2}{f_{12}f_{18}^3}. \tag{2.9}$$

The identity (2.8) was obtained by Xia and Yao [22]. Replacing q by $-q$ in (2.8) and using the relation $(-q;-q)_\infty=\dfrac{f_2^3}{f_1f_4}$, we obtain (2.9).

Lemma 2.5. The following 3-dissections hold:

$$\frac{f_1^2}{f_2}=\frac{f_9^2}{f_{18}}-2q\frac{f_3f_{18}^2}{f_6f_9}, \tag{2.10}$$

$$\frac{f_2^2}{f_1}=\frac{f_6f_9^2}{f_3f_{18}}+q\frac{f_{18}^2}{f_9} \tag{2.11}$$

and

$$\frac{f_2}{f_1^2}=\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_6^3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^3}{f_3^6}. \tag{2.12}$$

Lemma (2.5) was proved by Hirschhorn and Sellers [11].

Lemma 2.6. The following 3-dissection holds:

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}. \tag{2.13}$$

The equation (2.13) is the same as (14.8.5) in [8]. See also [2, p.345].

Lemma 2.7. The following 3-dissections hold:

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}, \tag{2.14}$$

$$\frac{1}{f_1 f_2} = \frac{f_9^9}{f_3^6 f_6^2 f_{18}^3} + q \frac{f_9^6}{f_3^5 f_6^3} + 3q^2 \frac{f_9^3 f_{18}^3}{f_3^4 f_6^4} - 2q^3 \frac{f_{18}^6}{f_3^3 f_6^5} + 4q^4 \frac{f_{18}^9}{f_3^2 f_6^6 f_9^3}. \tag{2.15}$$

For a proof of (2.14), we can see [12]. The identity (2.15) was obtained by Chan [3].

Lemma 2.8. We require the 5-dissection formula,

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2 R(q^5)), \tag{2.16}$$

where

$$R(q) = \frac{f(-q, -q^4)}{f(-q^2, -q^3)}.$$

The identity (2.16) is the same as (8.1.1) in [8].

Lemma 2.9. We require the 7-dissection formula,

$$f_1 = f_{49} \left(\frac{B(q^7)}{C(q^7)} - q \frac{A(q^7)}{B(q^7)} - q^2 + q^5 \frac{C(q^7)}{A(q^7)} \right), \tag{2.17}$$

where $A(q) = f(-q^3, -q^4)$, $B(q) = f(-q^2, -q^5)$ and $C(q) = f(-q, -q^6)$.

Lemma 2.9 is an exercise in [8], see [8, (10.5.1)]. Also, one can see [2, p.303, Entry 17(v)].

Lemma 2.10. [7, Theorem 2.2] For any prime $p \geq 5$,

$$f_1 = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq (\pm p - 1)/6}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}} \right) + (-1)^{\frac{\pm p - 1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}. \tag{2.18}$$

Furthermore, for $-(p-1)/2 \leq k \leq (p-1)/2$ and $k \neq (\pm p - 1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

3 Congruences for $\overline{a}^{3}_{1,3}(n)$

Theorem 3.1. For all $n \geq 0$, we have

$$\overline{a}^{3}_{1,3}(12n+11) \equiv 0 \pmod{32}, \tag{3.1}$$

$$\overline{a}^{3}_{1,3}(12n+8) \equiv 0 \pmod{8}. \tag{3.2}$$

Proof. Setting $j = 1, k = 3$ and $m = 3$ in (1.4), we find that

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,3}(n)\, q^n = \frac{(-q;q^3)_\infty(-q^2;q^3)_\infty f_3 f_9}{(-q^3;q^9)_\infty(-q^6;q^9)_\infty f_1 f_3}, \tag{3.3}$$

which gives

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,3}(n)\, q^n = \frac{f_2 f_3^2 f_{18}}{f_1^2 f_6^2}. \tag{3.4}$$

Employing (2.12) in (3.4), we obtain

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,3}(3n)\, q^n = \frac{f_2^2 f_3^6}{f_1^6 f_6^2}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,3}(3n+1)\, q^n = 2\frac{f_2 f_3^3 f_6}{f_1^5} \tag{3.6}$$

and

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,3}(3n+2)\, q^n = 4\frac{f_6^4}{f_1^4}. \tag{3.7}$$

From the binomial theorem, it is easy to see that for any positive integers k and m,

$$f_k^{2m} \equiv f_{2k}^{m} \pmod{2}, \tag{3.8}$$

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod{4}, \tag{3.9}$$

$$f_k^{8m} \equiv f_{2k}^{4m} \pmod{8}. \tag{3.10}$$

Using (2.2) in (3.7) along with (3.8), we find that

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,3}(6n+2)\, q^n \equiv 4\frac{f_{12}}{f_2} \pmod{8} \tag{3.11}$$

and

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,3}(6n+5)\, q^n \equiv 16 f_2^5 f_{12} \pmod{32}. \tag{3.12}$$

Collecting the coefficients of q^{2n+1} from both sides of the equations (3.11) and (3.12), we obtain (3.1) and (3.2) respectively. Q.E.D.

Theorem 3.2. For all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(36 \cdot 5^{2\beta} n + \frac{39 \cdot 5^{2\beta} - 1}{2}\right) q^n \equiv 4 f_1^{13} \pmod{8}, \tag{3.13}$$

$$\overline{a}_{1,3}^{3}\left(36 \cdot 5^{2\beta+1}(5n+h) + \frac{51 \cdot 5^{2\beta+1} - 1}{2}\right) \equiv 0 \pmod{8}, \tag{3.14}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(4 \cdot 3^{2\alpha+3} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + \frac{3^{2\alpha+3} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1}{2}\right) q^n \equiv 4 f_1^{3} \pmod{8}, \tag{3.15}$$

$$\begin{aligned} &\overline{a}_{1,3}^{3}\left(4 \cdot 3^{2\alpha+4} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + \frac{3^{2\alpha+3} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1}{2}\right) \\ &\equiv \begin{cases} 4 \pmod{8} & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 \pmod{8} & \text{otherwise,} \end{cases} \end{aligned} \tag{3.16}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(4 \cdot 3^{2\alpha+4} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + \frac{3^{2\alpha+5} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1}{2}\right) q^n \equiv 4 f_3^{3} \pmod{8}, \tag{3.17}$$

$$\overline{a}_{1,3}^{3}\left(4 \cdot 3^{2\alpha+4} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + \frac{17 \cdot 3^{2\alpha+3} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1}{2}\right) \equiv 0 \pmod{8}, \tag{3.18}$$

$$\overline{a}_{1,3}^{3}\left(4 \cdot 3^{2\alpha+4} \cdot 5^{2\beta} \cdot 7^{2\gamma}(3n+k) + \frac{3^{2\alpha+5} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1}{2}\right) \equiv 0 \pmod{8}, \tag{3.19}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(4 \cdot 3^{2\alpha+3} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + \frac{3^{2\alpha+3} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} - 1}{2}\right) q^n \equiv 4 f_5^{3} \pmod{8}, \tag{3.20}$$

$$\overline{a}_{1,3}^{3}\left(4 \cdot 3^{2\alpha+3} \cdot 5^{2\beta} \cdot 7^{2\gamma}(5n+m) + \frac{3^{2\alpha+3} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1}{2}\right) \equiv 0 \pmod{8}, \tag{3.21}$$

$$\overline{a}_{1,3}^{3}\left(4 \cdot 3^{2\alpha+3} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}(5n+i) + \frac{3^{2\alpha+3} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} - 1}{2}\right) \equiv 0 \pmod{8}, \tag{3.22}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(4 \cdot 3^{2\alpha+3} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + \frac{3^{2\alpha+3} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} - 1}{2}\right) q^n \equiv 4 f_7^{3} \pmod{8}, \tag{3.23}$$

$$\overline{a}_{1,3}^{3}\left(4 \cdot 3^{2\alpha+3} \cdot 5^{2\beta} \cdot 7^{2\gamma+1}(7n+j) + \frac{3^{2\alpha+3} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} - 1}{2}\right) \equiv 0 \pmod{8}, \tag{3.24}$$

where $h = 0, 1, 3, 4$, $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4, 5, 6$, $k = 1, 2$ and $m = 2, 4$.

Proof. Invoking (3.9) in (3.6), we have

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(3n+1) q^n \equiv 2 \frac{f_3^3 f_6}{f_1 f_2} \pmod{8}. \tag{3.25}$$

Using (2.15) in (3.25), we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(9n+1)\, q^n \equiv 2\frac{f_3 f_6}{f_1^3 f_2} + 4q\frac{f_6^6}{f_2^4} \pmod{8}, \tag{3.26}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(9n+4)\, q^n \equiv 2\frac{f_3^6}{f_1^2 f_2^2} \pmod{8} \tag{3.27}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(9n+7)\, q^n \equiv 2\frac{f_3^3 f_6^3}{f_1 f_2^3} \pmod{4}. \tag{3.28}$$

Utilizing (2.7) in (3.26), we find that

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(18n+1)\, q^n \equiv 2\frac{f_2^2}{f_1^2} \pmod{8}. \tag{3.29}$$

Substituting (2.1) in (3.29), we have

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(36n+1)\, q^n \equiv 2\frac{f_4}{f_1^3} \pmod{8} \tag{3.30}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(36n+19)\, q^n \equiv 4 f_1^{13} \pmod{8}. \tag{3.31}$$

The equation (3.31) is $\beta = 0$ case of (3.13). Suppose that the equation (3.13) is true for $\beta \geq 0$. Employing (2.16) in (3.13) and then collecting the coefficients of q^{5n+3} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(36 \cdot 5^{2\beta+1} n + \frac{51 \cdot 5^{2\beta+1} - 1}{2}\right) q^n \equiv 4q^2 f_5^{13} \pmod{8}, \tag{3.32}$$

which implies (3.14) and

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(36 \cdot 5^{2\beta+2} n + \frac{39 \cdot 5^{2\beta+2} - 1}{2}\right) q^n \equiv 4 f_1^{13} \pmod{8}, \tag{3.33}$$

which proves that the equation (3.13) is true for $\beta + 1$. By induction, the congruence (3.13) holds for all integer $\beta \geq 0$.

Employing (2.4) in (3.27), we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(18n+13)\, q^n \equiv 4 f_6^3 \pmod{8}, \tag{3.34}$$

which implies

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\,(108n+13)\,q^n \equiv 4f_1^3 \pmod 8, \tag{3.35}$$

which is $\alpha=\beta=\gamma=0$ case of the congruence (3.15). Suppose that the congruence (3.15) is true for $\alpha \geq 0$ with $\beta=\gamma=0$. The equation (3.15) with $\beta=\gamma=0$, we have

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(4\cdot 3^{2\alpha+3}n+\frac{3^{2\alpha+3}-1}{2}\right)q^n \equiv 4f_1^3 \pmod 8. \tag{3.36}$$

Using (2.13) in (3.36), we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(4\cdot 3^{2\alpha+4}n+\frac{3^{2\alpha+5}-1}{2}\right)q^n \equiv 4f_3^3 \pmod 8, \tag{3.37}$$

which implies

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(4\cdot 3^{2\alpha+5}n+\frac{3^{2\alpha+5}-1}{2}\right)q^n \equiv 4f_1^3 \pmod 8, \tag{3.38}$$

which implies that the congruence (3.15) is true for $\alpha+1$. By induction, the congruence (3.15) holds for all $\alpha \geq 0$ with $\beta=\gamma=0$. Suppose that the congruence (3.15) is true for $\alpha,\beta \geq 0$ with $\gamma=0$. Employing (2.16) in (3.15) with $\gamma=0$, we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(4\cdot 3^{2\alpha+3}\cdot 5^{2\beta+1}n+\frac{3^{2\alpha+3}\cdot 5^{2\beta+2}-1}{2}\right)q^n \equiv 4f_5^3 \pmod 8, \tag{3.39}$$

which yields

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(4\cdot 3^{2\alpha+3}\cdot 5^{2\beta+2}n+\frac{3^{2\alpha+3}\cdot 5^{2\beta+2}-1}{2}\right)q^n \equiv 4f_1^3 \pmod 8, \tag{3.40}$$

which implies that the congruence (3.15) is true for $\beta+1$. By induction, the congruence (3.15) holds for all $\alpha,\beta \geq 0$ with $\gamma=0$. Suppose that the congruence (3.15) is true for $\alpha,\beta,\gamma \geq 0$. Utilizing (2.17) in (3.15), we arrive at

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(4\cdot 3^{2\alpha+3}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+\frac{3^{2\alpha+3}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1}{2}\right)q^n \equiv 4f_7^3 \pmod 8, \tag{3.41}$$

which yields

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(4\cdot 3^{2\alpha+3}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+\frac{3^{2\alpha+3}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1}{2}\right)q^n \equiv 4f_1^3 \pmod 8, \tag{3.42}$$

which implies that the congruence (3.15) is true for $\gamma+1$. By induction, the congruence (3.15) holds for all integers $\alpha,\beta,\gamma \geq 0$.

Using (2.13) in (3.15) and then collecting the coefficients of q^{3n}, q^{3n+1} and q^{3n+2} from both sides of the resultant equation, we obtain (3.16), (3.17) and (3.18) respectively.

The equation (3.17) implies (3.19).

From the congruence (3.15) along with (2.16), we arrive at (3.20) and (3.21).

From the equation (3.20), we obtain (3.22).

Utilizing (2.17) in (3.15) and then collecting the coefficients of q^{7n+6} from the resultant equation, we get (3.23).

The congruence (3.23) implies (3.24). Q.E.D.

Theorem 3.3. For any prime $p \geq 5$ with $\left(\frac{-6}{p}\right) = -1$, $1 \leq j \leq p-1$ and for all $n \geq 0$ and $\alpha \geq 0$, we have

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(12 \cdot p^{2\alpha} n + \frac{7 \cdot p^{2\alpha} - 1}{2}\right) q^n \equiv 2 f_1 f_6 \pmod{4}, \tag{3.43}$$

$$\overline{a}_{1,3}^{3}\left(12 \cdot p^{2\alpha+1}(pn+j) + \frac{7 \cdot p^{2\alpha+2} - 1}{2}\right) \equiv 0 \pmod{4}. \tag{3.44}$$

Proof. Invoking (3.9) in (3.5), we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(3n)\, q^n \equiv \frac{f_3^2}{f_1^2} \pmod{4}. \tag{3.45}$$

Utilizing (2.3) in (3.45) and then extracting the terms involving q^{2n+1} from both sides of the resultant equation, we have

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(6n+3)\, q^n \equiv 2 f_2 f_{12} \pmod{4}, \tag{3.46}$$

which implies

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(12n+3)\, q^n \equiv 2 f_1 f_6 \pmod{4}, \tag{3.47}$$

which is $\alpha = 0$ case of the congruence (3.43). Suppose that the congruence (3.43) is true for all $\alpha \geq 0$. Employing (2.18) in (3.43), we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(12 \cdot p^{2\alpha+1} n + \frac{7 \cdot p^{2\alpha+2} - 1}{2}\right) q^n \equiv 2 f_p f_{6p} \pmod{4}, \tag{3.48}$$

which implies (3.44) and

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(12 \cdot p^{2\alpha+2} n + \frac{7 \cdot p^{2\alpha+2} - 1}{2}\right) q^n \equiv 2 f_1 f_6 \pmod{4}, \tag{3.49}$$

which implies that the congruence (3.43) is true for $\alpha + 1$. By induction, the congruence (3.43) holds for all integer $\alpha \geq 0$. Q.E.D.

Theorem 3.4. For all $n \geq 0$ and $\beta, \gamma \geq 0$, we have

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(36 \cdot 7^{2\gamma} n + \frac{15 \cdot 7^{2\gamma} - 1}{2}\right) q^n \equiv 2 f_1^5 \pmod 4, \tag{3.50}$$

$$\overline{a}_{1,3}^{3}\left(36 \cdot 7^{2\gamma+1}(7n+i) + \frac{33 \cdot 7^{2\gamma+1} - 1}{2}\right) \equiv 0 \pmod 4, \tag{3.51}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(36 \cdot 5^{2\beta} n + \frac{33 \cdot 5^{2\beta} - 1}{2}\right) q^n \equiv 2 f_2 f_3^3 \pmod 4, \tag{3.52}$$

$$\overline{a}_{1,3}^{3}\left(36 \cdot 5^{2\beta+1}(5n+j) + \frac{21 \cdot 5^{2\beta+1} - 1}{2}\right) \equiv 0 \pmod 4, \tag{3.53}$$

$$\overline{a}_{1,3}^{3}(36n+1) \equiv \begin{cases} 2 \pmod 4 & \text{if } n = k(3k+1)/2 \text{ for some } k \in \mathbb{Z}, \\ 0 \pmod 4 & \text{otherwise,} \end{cases} \tag{3.54}$$

where $i = 0, 2, 3, 4, 5, 6$ and $j = 0, 1, 3, 4$.

Proof. Using (2.4) in (3.28), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(18n+7) q^n \equiv 2 \frac{f_2 f_3^3}{f_1} \pmod 4 \tag{3.55}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(18n+16) q^n \equiv 2 \frac{f_3^3 f_6^3}{f_1 f_2^2} \pmod 4. \tag{3.56}$$

Employing (2.4) in (3.55), we have

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}(36n+7) q^n \equiv 2 f_1^5 \pmod 4, \tag{3.57}$$

which is $\gamma = 0$ case of (3.50). Suppose that the congruence (3.50) is true for $\gamma \geq 0$. Employing (2.17) in (3.50) and then collecting the coefficients of q^{7n+3} from both sides, we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(36 \cdot 7^{2\gamma+1} n + \frac{33 \cdot 7^{2\gamma+1} - 1}{2}\right) q^n \equiv 2 q f_7^5 \pmod 4, \tag{3.58}$$

which implies (3.51) and

$$\sum_{n=0}^{\infty} \overline{a}_{1,3}^{3}\left(36 \cdot 7^{2\gamma+2} n + \frac{15 \cdot 7^{2\gamma+2} - 1}{2}\right) q^n \equiv 2 f_1^5 \pmod 4, \tag{3.59}$$

which proves that the congruence (3.50) is true for $\gamma + 1$. By induction, the congruence (3.50) holds for all integer $\gamma \geq 0$.

Substituting (2.4) in (3.56), we get

$$\sum_{n=0}^{\infty}\overline{a}_{1,3}^{3}\,(36n+16)\,q^{n}\equiv 2f_2f_3^3 \pmod{4}, \tag{3.60}$$

which is $\beta=0$ case of (3.52). Suppose that the congruence (3.52) holds for $\beta\geq 0$. Utilizing (2.16) in (3.52) and then comparing the coefficients of q^{5n+1} on both sides, we obtain

$$\sum_{n=0}^{\infty}\overline{a}_{1,3}^{3}\left(36\cdot 5^{2\beta+1}n+\frac{21\cdot 5^{2\beta+1}-1}{2}\right)q^{n}\equiv 2q^2f_{10}f_{15}^3 \pmod{4}, \tag{3.61}$$

which implies (3.53) and

$$\sum_{n=0}^{\infty}\overline{a}_{1,3}^{3}\left(36\cdot 5^{2\beta+2}n+\frac{33\cdot 5^{2\beta+2}-1}{2}\right)q^{n}\equiv 2f_2f_3^3 \pmod{4}, \tag{3.62}$$

which proves that the congruence (3.52) holds for $\beta+1$. By induction, the congruence (3.52) is true for all integer $\beta\geq 0$.

From the equation (3.30), we obtain (3.54). Q.E.D.

4 Congruences for $\overline{a}_{1,4}^{3}(n)$

Theorem 4.1. For all $n\geq 0$ and $\alpha,\beta\geq 0$, we have

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha}n+\frac{7\cdot 3^{4\alpha}-1}{2}\right)q^{n}\equiv 4\frac{f_2f_3f_4^3f_6}{f_1^2} \pmod{16}, \tag{4.1}$$

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+2}\cdot 5^{4\beta}n+\frac{13\cdot 3^{4\alpha+1}\cdot 5^{4\beta}-1}{2}\right)q^{n}\equiv 4f_1f_4^3 \pmod{16}, \tag{4.2}$$

$$\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+2}\cdot 5^{4\beta+1}(5n+i)+\frac{17\cdot 3^{4\alpha+1}\cdot 5^{4\beta+1}-1}{2}\right)\equiv 0 \pmod{16}, \tag{4.3}$$

$$\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+2}\cdot 5^{4\beta+3}(5n+i)+\frac{17\cdot 3^{4\alpha+1}\cdot 5^{4\beta+3}-1}{2}\right)\equiv 0 \pmod{16}, \tag{4.4}$$

where $i=0,1,3,4$.

Proof. Setting $j=1,k=4$ and $m=3$ in (1.4), we find that

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}\,(n)\,q^{n}=\frac{(-q;q^4)_\infty(-q^3;q^4)_\infty f_3f_{12}}{(-q^3;q^{12})_\infty(-q^9;q^{12})_\infty f_1f_4}, \tag{4.5}$$

which gives

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}\,(n)\,q^{n}=\frac{f_2^2f_3^2f_{12}^2}{f_1^2f_4^2f_6^2}. \tag{4.6}$$

Substituting (2.3) in (4.6) and then collecting the coefficients of q^{2n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{3}\left(2n+1\right)q^{n} = 2\frac{f_4 f_6 f_{12}}{f_1^2 f_2}. \tag{4.7}$$

Using (2.1) in (4.7), we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{3}\left(4n+1\right)q^{n} = 2\frac{f_2 f_3 f_4^5 f_6}{f_1^6 f_8^2} \tag{4.8}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{3}\left(4n+3\right)q^{n} = 4\frac{f_2^3 f_3 f_6 f_8^2}{f_1^6 f_4}. \tag{4.9}$$

Invoking (3.9) in (4.9), we find that

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{3}\left(4n+3\right)q^{n} \equiv 4\frac{f_2 f_3 f_4^3 f_6}{f_1^2} \pmod{16}. \tag{4.10}$$

which is $\alpha = 0$ case of (4.1). Suppose that the congruence (4.1) is true for $\alpha \geq 0$. Employing (2.12) and (2.13) in (4.1), we obtain

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+1}n + \frac{7\cdot 3^{4\alpha}-1}{2}\right)q^{n} \equiv 4\frac{f_1 f_2 f_3^2 f_6^3 f_8}{f_4 f_{24}} \pmod{16}, \tag{4.11}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+1}n + \frac{5\cdot 3^{4\alpha+1}-1}{2}\right)q^{n} \equiv 8f_2^3 f_3^3 + 4q\frac{f_1 f_2 f_3^2 f_{12}^3}{f_6} \pmod{16} \tag{4.12}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+1}n + \frac{23\cdot 3^{4\alpha}-1}{2}\right)q^{n} \equiv 8qf_2 f_3^{15} \pmod{16}. \tag{4.13}$$

Using (2.13) and (2.14) in (4.12), we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+2}n + \frac{5\cdot 3^{4\alpha+1}-1}{2}\right)q^{n} \equiv 8f_1^5 + 8qf_1^{11}f_6^3 \pmod{16}, \tag{4.14}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+2}n + \frac{13\cdot 3^{4\alpha+1}-1}{2}\right)q^{n} \equiv 4f_1 f_4^3 \pmod{16} \tag{4.15}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+2}n + \frac{7\cdot 3^{4\alpha+2}-1}{2}\right)q^{n} \equiv 8f_1^3 f_6^3 + 12\frac{f_1^2 f_3 f_4^3 f_6}{f_2} \pmod{16}. \tag{4.16}$$

Substituting (2.10) and (2.13) in (4.16) and then collecting the coefficients of q^{3n+1} from both sides of the resultant equation, we arrive at

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+3}n + \frac{5\cdot 3^{4\alpha+3}-1}{2}\right)q^{n} \equiv 12q\frac{f_1 f_2 f_3^2 f_{12}^3}{f_6} \pmod{16}. \tag{4.17}$$

Employing (2.14) in (4.17) and then comparing the terms involving q^{3n+2} on both sides, we obtain

$$\sum_{n=0}^{\infty} \overline{a}^3_{1,4}\left(4\cdot 3^{4\alpha+4}n+\frac{7\cdot 3^{4\alpha+4}-1}{2}\right)q^n \equiv 4\frac{f_2f_3f_4^3f_6}{f_1^2} \pmod{16}, \tag{4.18}$$

which implies that the congruence (4.1) is true for $\alpha+1$. By induction, the congruence (4.1) holds for all integer $\alpha \geq 0$.

The equation (4.15) is $\beta = 0$ case of (4.2). Suppose that the congruence (4.2) is true for $\alpha, \beta \geq 0$. Using (2.16) in (4.2), we obtain

$$\sum_{n=0}^{\infty} \overline{a}^3_{1,4}\left(4\cdot 3^{4\alpha+2}\cdot 5^{4\beta+1}n+\frac{17\cdot 3^{4\alpha+1}\cdot 5^{4\beta+1}-1}{2}\right)q^n \equiv 12q^2f_5f_{20}^3 \pmod{16}, \tag{4.19}$$

which implies (4.3) and

$$\sum_{n=0}^{\infty} \overline{a}^3_{1,4}\left(4\cdot 3^{4\alpha+2}\cdot 5^{4\beta+2}n+\frac{13\cdot 3^{4\alpha+1}\cdot 5^{4\beta+2}-1}{2}\right)q^n \equiv 12f_1f_4^3 \pmod{16}. \tag{4.20}$$

Again, using (2.16) in (4.20), we get

$$\sum_{n=0}^{\infty} \overline{a}^3_{1,4}\left(4\cdot 3^{4\alpha+2}\cdot 5^{4\beta+3}n+\frac{17\cdot 3^{4\alpha+1}\cdot 5^{4\beta+3}-1}{2}\right)q^n \equiv 4q^2f_5f_{20}^3 \pmod{16}, \tag{4.21}$$

which implies (4.4) and

$$\sum_{n=0}^{\infty} \overline{a}^3_{1,4}\left(4\cdot 3^{4\alpha+2}\cdot 5^{4\beta+4}n+\frac{13\cdot 3^{4\alpha+1}\cdot 5^{4\beta+4}-1}{2}\right)q^n \equiv 4f_1f_4^3 \pmod{16}, \tag{4.22}$$

which implies that the congruence (4.2) is true for $\beta+1$. Hence, by mathematical induction, the congruence (4.2) holds for all integers $\alpha, \beta \geq 0$. Q.E.D.

Theorem 4.2. For all $n \geq 0$ and $\alpha, \beta \geq 0$, we have

$$\overline{a}^3_{1,4}\left(4\cdot 3^{4\alpha+1}n+\frac{23\cdot 3^{4\alpha}-1}{2}\right) \equiv 0 \pmod{8}, \tag{4.23}$$

$$\overline{a}^3_{1,4}\left(4\cdot 3^{4\alpha+2}n+\frac{5\cdot 3^{4\alpha+1}-1}{2}\right) \equiv 0 \pmod{8}, \tag{4.24}$$

$$\sum_{n=0}^{\infty} \overline{a}^3_{1,4}\left(4\cdot 3^{4\alpha}n+\frac{3^{4\alpha+1}-1}{2}\right)q^n \equiv 2\frac{f_1^2f_3f_4f_6}{f_2^3} \pmod{8}, \tag{4.25}$$

$$\sum_{n=0}^{\infty} \overline{a}^3_{1,4}\left(4\cdot 3^{4\alpha+1}\cdot 5^{2\beta}n+\frac{7\cdot 3^{4\alpha}\cdot 5^{2\beta}-1}{2}\right)q^n \equiv 4f_1^7 \pmod{8}, \tag{4.26}$$

$$\overline{a}^3_{1,4}\left(4\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}(5n+i)+\frac{11\cdot 3^{4\alpha}\cdot 5^{2\beta+1}-1}{2}\right) \equiv 0 \pmod{8}, \tag{4.27}$$

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+1}\cdot 5^{2\beta}n+\frac{11\cdot 3^{4\alpha}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4f_2f_3^3 \pmod 8, \tag{4.28}$$

$$\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}(5n+j)+\frac{7\cdot 3^{4\alpha}\cdot 5^{2\beta+1}-1}{2}\right)\equiv 0 \pmod 8, \tag{4.29}$$

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+1}\cdot 5^{2\beta}n+\frac{19\cdot 3^{4\alpha}\cdot 5^{2\beta}-1}{2}\right)q^n\equiv 4f_1f_6^3 \pmod 8, \tag{4.30}$$

$$\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}(5n+k)+\frac{23\cdot 3^{4\alpha}\cdot 5^{2\beta+1}-1}{2}\right)\equiv 0 \pmod 8, \tag{4.31}$$

where $i=0,2,3,4$, $j=0,1,3,4$ and $k=0,1,2,4$.

Proof. From the equations (4.13) and (4.14), we obtain (4.23) and (4.24) respectively.

Invoking (3.9) in (4.8), we find that

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}(4n+1)q^n\equiv 2\frac{f_1^2f_3f_4f_6}{f_2^3} \pmod 8, \tag{4.32}$$

which is $\alpha=0$ case of (4.25). Suppose that the congruence (4.25) is true for $\alpha\geq 0$. Employing (2.10) and (2.12) in (4.25), we have

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+1}n+\frac{3^{4\alpha+1}-1}{2}\right)q^n\equiv 2\frac{f_1f_2f_3^2f_6}{f_{12}} \pmod 8, \tag{4.33}$$

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+1}n+\frac{11\cdot 3^{4\alpha}-1}{2}\right)q^n\equiv 4f_2f_3^3 \pmod 8 \tag{4.34}$$

and

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+1}n+\frac{19\cdot 3^{4\alpha}-1}{2}\right)q^n\equiv 4f_1f_6^3 \pmod 8. \tag{4.35}$$

Using (2.14) in (4.33) and then extracting the terms involving q^{3n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+2}n+\frac{3^{4\alpha+3}-1}{2}\right)q^n\equiv 6\frac{f_1^2f_3f_4f_6}{f_2^3} \pmod 8. \tag{4.36}$$

Using (2.10) and (2.12) in (4.36) and then comparing the coefficients of q^{3n} on both sides, we obtain

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+3}n+\frac{3^{4\alpha+3}-1}{2}\right)q^n\equiv 6\frac{f_1f_2f_3^2f_6}{f_{12}} \pmod 8. \tag{4.37}$$

Utilizing (2.14) in (4.37) and then collecting the terms involving q^{3n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{3}\left(4\cdot 3^{4\alpha+4}n+\frac{3^{4\alpha+5}-1}{2}\right)q^n\equiv 2\frac{f_1^2f_3f_4f_6}{f_2^3} \pmod 8, \tag{4.38}$$

which implies that the congruence (4.25) is true for $\alpha+1$. By induction, the congruence (4.25) holds for all integer $\alpha \geq 0$.

The equation (4.11) becomes

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,4}\left(4\cdot 3^{4\alpha+1}n+\frac{7\cdot 3^{4\alpha}-1}{2}\right)q^n \equiv 4f_1^7 \pmod 8, \tag{4.39}$$

which is $\beta=0$ case of (4.26). Suppose that the congruence (4.26) is true for $\beta\geq 0$. Employing (2.16) in (4.26) and then extracting the coefficients of q^{5n+2} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,4}\left(4\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}n+\frac{11\cdot 3^{4\alpha}\cdot 5^{2\beta+1}-1}{2}\right)q^n \equiv 4qf_5^7 \pmod 8, \tag{4.40}$$

which implies (4.27) and

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,4}\left(4\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}n+\frac{7\cdot 3^{4\alpha}\cdot 5^{2\beta+2}-1}{2}\right)q^n \equiv 4f_1^7 \pmod 8, \tag{4.41}$$

which implies that the congruence (4.26) is true for $\beta+1$. Hence, by induction, the congruence (4.26) holds for all integers $\alpha,\beta\geq 0$.

The congruence (4.34) is $\beta=0$ case of (4.28). Suppose that the congruence (4.28) is true for $\beta\geq 0$. Substituting (2.16) in (4.28) and then collecting the coefficients of q^{5n+1} from both sides of the resultant equation, we arrive at

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,4}\left(4\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}n+\frac{7\cdot 3^{4\alpha}\cdot 5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^2f_{10}f_{15}^3 \pmod 8, \tag{4.42}$$

which implies (4.29) and

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,4}\left(4\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}n+\frac{11\cdot 3^{4\alpha}\cdot 5^{2\beta+2}-1}{2}\right)q^n \equiv 4f_2f_3^3 \pmod 8, \tag{4.43}$$

which implies that the congruence (4.28) is true for $\beta+1$. Hence, by induction, the congruence (4.28) holds for all integers $\alpha,\beta\geq 0$.

The congruence (4.35) is $\beta=0$ case of (4.30). Suppose that the congruence (4.30) is true for $\beta\geq 0$. Using (2.16) in (4.30) and then comparing the coefficients of q^{5n+4}, we obtain

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,4}\left(4\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}n+\frac{23\cdot 3^{4\alpha}\cdot 5^{2\beta+1}-1}{2}\right)q^n \equiv 4q^3f_5f_{30}^3 \pmod 8, \tag{4.44}$$

which implies (4.31) and

$$\sum_{n=0}^{\infty} \overline{a}^{3}_{1,4}\left(4\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}n+\frac{19\cdot 3^{4\alpha}\cdot 5^{2\beta+2}-1}{2}\right)q^n \equiv 4f_1f_6^3 \pmod 8, \tag{4.45}$$

which implies that the congruence (4.30) is true for $\beta+1$. Hence, by induction, the congruence (4.30) holds for all integers $\alpha,\beta\geq 0$.

Q.E.D.

5 Congruences for $\overline{a}^{9}_{1,4}(n)$

Theorem 5.1. For all $n \geq 0$ and $\alpha, \beta \geq 0$, we have

$$\overline{a}^{9}_{1,4}\left(24n+23\right) \equiv 0 \pmod{16}, \tag{5.1}$$

$$\overline{a}^{9}_{1,4}\left(8 \cdot 3^{4\alpha+2}n + 23 \cdot 3^{4\alpha+1} - 2\right) \equiv 0 \pmod{16}, \tag{5.2}$$

$$\sum_{n=0}^{\infty} \overline{a}^{9}_{1,4}\left(4 \cdot 3^{4\alpha+1}n + 3^{4\alpha+2} - 2\right) q^n \equiv 4f_3^6 + 8f_1^9 f_3^3 \pmod{16}, \tag{5.3}$$

$$\sum_{n=0}^{\infty} \overline{a}^{9}_{1,4}\left(8 \cdot 3^{4\alpha+2} \cdot 5^{2\beta}n + 7 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} - 2\right) q^n \equiv 8f_1^7 \pmod{16}, \tag{5.4}$$

$$\overline{a}^{9}_{1,4}\left(8 \cdot 3^{4\alpha+2} \cdot 5^{2\beta+1}(5n+i) + 11 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} - 2\right) \equiv 0 \pmod{16}, \tag{5.5}$$

$$\sum_{n=0}^{\infty} \overline{a}^{9}_{1,4}\left(8 \cdot 3^{4\alpha+2} \cdot 5^{2\beta}n + 11 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} - 2\right) q^n \equiv 8f_2 f_3^3 \pmod{16}, \tag{5.6}$$

$$\overline{a}^{9}_{1,4}\left(8 \cdot 3^{4\alpha+2} \cdot 5^{2\beta+1}(5n+k) + 7 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} - 2\right) \equiv 0 \pmod{16}, \tag{5.7}$$

$$\sum_{n=0}^{\infty} \overline{a}^{9}_{1,4}\left(8 \cdot 3^{4\alpha+2} \cdot 5^{2\beta}n + 19 \cdot 3^{4\alpha+1} \cdot 5^{2\beta} - 2\right) q^n \equiv 8f_1 f_6^3 \pmod{16}, \tag{5.8}$$

$$\overline{a}^{9}_{1,4}\left(8 \cdot 3^{4\alpha+2} \cdot 5^{2\beta+1}(5n+j) + 23 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} - 2\right) \equiv 0 \pmod{16}, \tag{5.9}$$

where $i = 0, 2, 3, 4$, $j = 0, 1, 2, 4$ and $k = 0, 1, 3, 4$.

Proof. Setting $j = 1, k = 4$ and $m = 9$ in (1.4), we see that

$$\sum_{n=0}^{\infty} \overline{a}^{9}_{1,4}\left(n\right) q^n = \frac{(-q;q^4)_\infty(-q^3;q^4)_\infty f_9 f_{36}}{(-q^9;q^{36})_\infty(-q^{27};q^{36})_\infty f_1 f_4}, \tag{5.10}$$

which gives

$$\sum_{n=0}^{\infty} \overline{a}^{9}_{1,4}\left(n\right) q^n = \frac{f_2^2 f_9^2 f_{36}^2}{f_1^2 f_4^2 f_{18}^2}. \tag{5.11}$$

Employing (2.8) in (5.11) and then collecting the coefficients of q^{2n+1} from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} \overline{a}^{9}_{1,4}\left(2n+1\right) q^n = 2\frac{f_6^2 f_{18}^2}{f_1^3 f_9}. \tag{5.12}$$

Using (2.2) and (2.9) in (5.12), obtain

$$\sum_{n=0}^{\infty} \overline{a}^{9}_{1,4}\left(4n+1\right) q^n = 2\frac{f_2^{13} f_3 f_6^3}{f_1^{13} f_4^4} - 8q\frac{f_2^3 f_3^3 f_4^4 f_{18}^2}{f_1^{10} f_6 f_9} \tag{5.13}$$

and

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{9}\left(4n+3\right)q^{n}=8\frac{f_2f_3f_4^4f_6^3}{f_1^9}-2\frac{f_2^{15}f_3^3f_{18}^2}{f_1^{14}f_4^4f_6f_9}. \tag{5.14}$$

Invoking (3.8) and (3.10) in (5.14), we find that

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{9}\left(4n+3\right)q^{n}\equiv 8f_1^9f_3^7-2\frac{f_1^2f_3^3f_{18}^2}{f_2f_6f_9} \pmod{16}. \tag{5.15}$$

Substituting (2.10) and (2.13) in (5.15), we arrive at

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{9}\left(12n+3\right)q^{n}\equiv 8f_2^5+14\frac{f_1^3f_3f_6}{f_2}+8qf_1^7f_3^9 \pmod{16}, \tag{5.16}$$

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{9}\left(12n+7\right)q^{n}\equiv 4f_3^6+8f_1^9f_3^3 \pmod{16} \tag{5.17}$$

and

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{9}\left(12n+11\right)q^{n}\equiv 8f_6^3f_8 \pmod{16}. \tag{5.18}$$

Extracting the coefficients of q^{2n+1} from both sides of the equation (5.18), we obtain (5.1).

The equation (5.17) is $\alpha=0$ case of (5.3). Suppose that the congruence (5.3) is true for $\alpha\geq 0$. Employing (2.13) in (5.3), we obtain

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{9}\left(4\cdot 3^{4\alpha+2}n+3^{4\alpha+2}-2\right)q^{n}\equiv 12f_1^6+8qf_1^3f_3^9 \pmod{16}, \tag{5.19}$$

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{9}\left(4\cdot 3^{4\alpha+2}n+7\cdot 3^{4\alpha+1}-2\right)q^{n}\equiv 8\frac{f_2^3f_3^3}{f_1} \pmod{16} \tag{5.20}$$

and

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{9}\left(4\cdot 3^{4\alpha+2}n+11\cdot 3^{4\alpha+1}-2\right)q^{n}\equiv 8f_4f_6^3 \pmod{16}. \tag{5.21}$$

Collecting the coefficients of q^{2n+1} from both sides of the equation (5.21), we obtain (5.2).

Utilizing (2.13) in (5.19) and then collecting the coefficients of q^{3n+2} from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{9}\left(4\cdot 3^{4\alpha+3}n+3^{4\alpha+4}-2\right)q^{n}\equiv 8f_1^9f_3^3+12f_3^6 \pmod{16}. \tag{5.22}$$

Using (2.13) in (5.22), we get

$$\sum_{n=0}^{\infty}\overline{a}_{1,4}^{9}\left(4\cdot 3^{4\alpha+4}n+3^{4\alpha+4}-2\right)q^{n}\equiv 4f_1^6+8qf_1^3f_3^9 \pmod{16}. \tag{5.23}$$

Again, using (2.13) in (5.23) and then extracting the coefficients of q^{3n+2} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(4\cdot 3^{4\alpha+5}n+3^{4\alpha+6}-2\right)q^n \equiv 4f_3^6+8f_1^9f_3^3 \pmod{16}, \tag{5.24}$$

which implies that the congruence (5.3) is true for $\alpha+1$. By induction, the congruence (5.3) holds for all integer $\alpha \geq 0$.

Using (2.4) in (5.20), we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(8\cdot 3^{4\alpha+2}n+7\cdot 3^{4\alpha+1}-2\right)q^n \equiv 8f_1^7 \pmod{16} \tag{5.25}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(8\cdot 3^{4\alpha+2}n+19\cdot 3^{4\alpha+1}-2\right)q^n \equiv 8f_1f_6^3 \pmod{16}. \tag{5.26}$$

The equation (5.21) implies

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(8\cdot 3^{4\alpha+2}n+11\cdot 3^{4\alpha+1}-2\right)q^n \equiv 8f_2f_3^3 \pmod{16}. \tag{5.27}$$

The rest of the proofs of the identities (5.4)-(5.9) are similar to the proofs of the identities (4.26)-(4.31). So, we omit the details. Q.E.D.

Theorem 5.2. For all $n \geq 0$ and $\alpha, \beta, \gamma \geq 0$, we have

$$\overline{a}_{1,4}^{9}(24n+17) \equiv 0 \pmod{4}, \tag{5.28}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(24\cdot 7^{2\gamma}n+5\cdot 7^{2\gamma}-2\right)q^n \equiv 2f_1f_4 \pmod{4}, \tag{5.29}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(24\cdot 7^{2\gamma+1}n+11\cdot 7^{2\gamma+1}-2\right)q^n \equiv 2qf_7f_{28} \pmod{4}, \tag{5.30}$$

$$\overline{a}_{1,4}^{9}\left(24\cdot 7^{2\gamma+1}(7n+i)+11\cdot 7^{2\gamma+1}-2\right) \equiv 0 \pmod{4}, \tag{5.31}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}-2\right)q^n \equiv 2f_1^3 \pmod{4}, \tag{5.32}$$

$$\begin{aligned}&\overline{a}_{1,4}^{9}\left(8\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}-2\right)\\ &\equiv \begin{cases} 2 \pmod{4} & \text{if } n=k(3k+1)/2 \text{ for some } k\in\mathbb{Z},\\ 0 \pmod{4} & \text{otherwise,}\end{cases}\end{aligned} \tag{5.33}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(8\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+3^{2\alpha+3}\cdot 5^{2\beta}\cdot 7^{2\gamma}-2\right)q^n \equiv 2f_3^3 \pmod{4}, \tag{5.34}$$

$$\overline{a}^{9}_{1,4}\left(8\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+17\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}-2\right)\equiv 0 \pmod 4, \tag{5.35}$$

$$\overline{a}^{9}_{1,4}\left(8\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}(3n+j)+3^{2\alpha+3}\cdot 5^{2\beta}\cdot 7^{2\gamma}-2\right)\equiv 0 \pmod 4, \tag{5.36}$$

$$\sum_{n=0}^{\infty}\overline{a}^{9}_{1,4}\left(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+3^{2\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-2\right)q^n\equiv 2f_5^3 \pmod 4, \tag{5.37}$$

$$\overline{a}^{9}_{1,4}\left(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}(5n+k)+3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}-2\right)\equiv 0 \pmod 4, \tag{5.38}$$

$$\overline{a}^{9}_{1,4}\left(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}(5n+m)+3^{2\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-2\right)\equiv 0 \pmod 4, \tag{5.39}$$

$$\sum_{n=0}^{\infty}\overline{a}^{9}_{1,4}\left(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-2\right)q^n\equiv 2f_7^3 \pmod 4, \tag{5.40}$$

$$\overline{a}^{9}_{1,4}\left(8\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}(7n+r)+3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-2\right)\equiv 0 \pmod 4, \tag{5.41}$$

$$\sum_{n=0}^{\infty}\overline{a}^{9}_{1,4}\left(24\cdot 5^{2\beta}n+11\cdot 5^{2\beta}-2\right)q^n\equiv 2f_2f_3^3 \pmod 4, \tag{5.42}$$

$$\overline{a}^{9}_{1,4}\left(24\cdot 5^{2\beta+1}(5n+s)+7\cdot 5^{2\beta+1}-2\right)\equiv 0 \pmod 4, \tag{5.43}$$

where $i=0,2,3,4,5,6$, $j=1,2$, $k=2,4$, $m=1,2,3,4$, $r=1,2,3,4,5,6$ and $s=0,1,3,4$.

Proof. The equation (5.16) becomes

$$\sum_{n=0}^{\infty}\overline{a}^{9}_{1,4}(12n+3)q^n\equiv 2\frac{f_2f_3^3}{f_1} \pmod 4. \tag{5.44}$$

Substituting (2.4) in (5.44) and then collecting the coefficients of q^{2n} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty}\overline{a}^{9}_{1,4}(24n+3)q^n\equiv 2f_1f_4 \pmod 4, \tag{5.45}$$

which is $\gamma=0$ case of (5.29). Suppose that the congruence (5.29) is true for $\gamma\geq 0$. Employing (2.17) in (5.29) and then collecting the coefficients of q^{7n+3}, we arrive at

$$\sum_{n=0}^{\infty}\overline{a}^{9}_{1,4}\left(24\cdot 7^{2\gamma+1}n+11\cdot 7^{2\gamma+1}-2\right)q^n\equiv 2qf_7f_{28} \pmod 4, \tag{5.46}$$

which implies

$$\sum_{n=0}^{\infty}\overline{a}^{9}_{1,4}\left(24\cdot 7^{2\gamma+2}n+5\cdot 7^{2\gamma+2}-2\right)q^n\equiv 2f_1f_4 \pmod 4, \tag{5.47}$$

which implies that the congruence (5.29) is true for $\gamma+1$. Hence, by induction, the congruence (5.29) holds for all integer $\gamma\geq 0$.

Utilizing (2.17) in (5.29) and then extracting the coefficients of q^{7n+3} from both sides of the resultant equation, we obtain (5.30).

From the equation (5.30), we arrive at (5.31).

The equation (5.13) becomes

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(4n+1\right)q^{n} \equiv 2\frac{f_3 f_6^3}{f_1^3} \pmod 4. \tag{5.48}$$

Employing (2.7) in (5.48) and then comparing the coefficients of q^{2n} on both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(8n+1\right)q^{n} \equiv 2f_1^3 f_6 \pmod 4. \tag{5.49}$$

Using (2.13) in (5.49), we arrive at (5.28),

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(24n+1\right)q^{n} \equiv 2f_1^3 \pmod 4 \tag{5.50}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(24n+9\right)q^{n} \equiv 2f_2 f_3^3 \pmod 4. \tag{5.51}$$

The equation (5.50) is $\alpha = \beta = \gamma = 0$ case of (5.32). The rest of the proofs of the identities (5.32)-(5.41) are similar to the proofs of the identities (3.15)-(3.24). So, we omit the details.

The equation (5.51) is $\beta = 0$ case of (5.42). The rest of the proofs of the identities (5.42) and (5.43) are similar to the proofs of the identities (3.52) and (3.53). So, we omit the details. Q.E.D.

Theorem 5.3. For all $n \geq 0$ and $\alpha, \beta \geq 0$, we have

$$\overline{a}_{1,4}^{9}\left(6n+3\right) \equiv 0 \pmod 3, \tag{5.52}$$

$$\overline{a}_{1,4}^{9}\left(6n+5\right) \equiv 0 \pmod 3, \tag{5.53}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(2\cdot 3^{\alpha+1}n + 3^{\alpha+1} - 2\right)q^{n} \equiv 2\frac{f_2^2 f_6^2}{f_1 f_3} \pmod 3, \tag{5.54}$$

$$\overline{a}_{1,4}^{9}\left(2\cdot 3^{\alpha+2}n + 5\cdot 3^{\alpha+1} - 2\right) \equiv 0 \pmod 3, \tag{5.55}$$

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(2\cdot 3^{\alpha+2}\cdot 5^{2\beta}n + 3^{\alpha+1}\cdot 5^{2\beta} - 2\right)q^{n} \equiv 2f_1 f_3 \pmod 3, \tag{5.56}$$

$$\overline{a}_{1,4}^{9}\left(2\cdot 3^{\alpha+2}\cdot 5^{2\beta+1}(5n+i) + 3^{\alpha+1}\cdot 5^{2\beta+1} - 2\right) \equiv 0 \pmod 3, \tag{5.57}$$

where $i = 1, 2, 3, 4$.

Proof. From the binomial theorem, it is easy to see that for any positive integers ℓ and k,

$$f_\ell^{3k} \equiv f_{3\ell}^{k} \pmod 3. \tag{5.58}$$

Invoking (5.58) in (5.12), we find that

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(2n+1\right)q^{n} \equiv 2\frac{f_6^2 f_{18}^2}{f_3 f_9} \pmod 3. \tag{5.59}$$

Equating the coefficients of q^{3n+1} and q^{3n+2} on both sides of the above equation, we obtain (5.52) and (5.53) respectively.

The equation (5.59) implies

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\,(6n+1)\,q^n \equiv 2\frac{f_2^2 f_6^2}{f_1 f_3} \pmod 3, \tag{5.60}$$

which is $\alpha = 0$ case of (5.54). Suppose that the congruence (5.54) is true for $\alpha \geq 0$. Employing (2.11) in (5.54), we get (5.55),

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(2\cdot 3^{\alpha+2}n + 3^{\alpha+1} - 2\right) q^n \equiv 2 f_1 f_3 \pmod 3 \tag{5.61}$$

and

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(2\cdot 3^{\alpha+2}n + 3^{\alpha+2} - 2\right) q^n \equiv 2\frac{f_2^2 f_6^2}{f_1 f_3} \pmod 3, \tag{5.62}$$

which implies that the congruence (5.54) is true for $\alpha+1$. By induction, the congruence (5.54) holds for all integer $\alpha \geq 0$.

The equation (5.61) is $\beta = 0$ case of (5.56). Suppose that the congruence (5.56) is true for $\beta \geq 0$. Using (2.16) in (5.56), we get

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(2\cdot 3^{\alpha+2}\cdot 5^{2\beta+1}n + 3^{\alpha+1}\cdot 5^{2\beta+2} - 2\right) q^n \equiv 2 f_5 f_{15} \pmod 3, \tag{5.63}$$

which implies (5.57) and

$$\sum_{n=0}^{\infty} \overline{a}_{1,4}^{9}\left(2\cdot 3^{\alpha+2}\cdot 5^{2\beta+2}n + 3^{\alpha+1}\cdot 5^{2\beta+2} - 2\right) q^n \equiv 2 f_1 f_3 \pmod 3, \tag{5.64}$$

which implies that the congruence (5.56) is true for $\beta+1$. Hence, by induction, the congruence (5.56) holds for all integers $\alpha, \beta \geq 0$. Q.E.D.

Acknowledgments.

The authors would like to thank the editor/referee for his/her useful comments.

References

[1] C. Adiga, M. S. Mahadeva Naika, D. Ranganatha and C. Shivashankar, *Congruences modulo 8 for $(2,k)$-regular overpartitions for odd $k > 1$*, Arab. J. Math., 7 (2018), 61-75.

[2] B. C. Berndt, *Ramanujan's Notebooks Part III*, Springer-Verlag, New York, 1991.

[3] H. C. Chan, *Ramanujan's cubic continued fraction and an analog of his "most beautiful identity"*, Int. J. Number Theory, 6 (2010), 673-680.

[4] W. Y. C. Chen and E. X. W. Xia, *Proof of conjecture of Hirschhorn and Sellers on overpartitions*, Acta Arith., 163 (1), (2014), 59-69.

[5] S. Chern and M. G. Dostidar, *Some congruences modulo* 5 *and* 25 *for overpartitions*, Ramanujan J., 47 (2), (2018), 435-445.

[6] S. Corteel and J. Lovejoy, *Overpartitions*, Trans. Amer. Math. Soc., 356 (2004), 1623-1635.

[7] S. P. Cui and N. S. S. Gu, *Arithmetic properties of ℓ-regular partitions*, Adv. Appl. Math., 51 (2013), 507-523.

[8] M. D. Hirschhorn, The Power of q, *Springer International Publishing,* Switzerland, 2017.

[9] M. D. Hirschhorn and J. A. Sellers, *An infinite family of overpartition congruences modulo* 12, Integers, 5 (1), (2005), #20.

[10] M. D. Hirschhorn and J. A. Sellers, *Arithmetic relations for overpartitions*, J. Comb. Math. Comb. Comput., 53 (2005), 65-73.

[11] M. D. Hirschhorn and J. A. Sellers, *Arithmetic properties of partitions with odd distinct*, Ramanujan J., 22 (2010), 273-284.

[12] M. D. Hirschhorn and J. A. Sellers, *A congruence modulo* 3 *for partitions into distinct non-multiples of four*, J. Integer Seq., 17 (2014), Article 14.9.6.

[13] B. Kim, *The overpartition function modulo* 128, Integers, 38 (2008), #8.

[14] J. Lovejoy, *Gordon's theorem for overpartitions*, J. Comb. Theory, Series A, 103 (2003), 393-401.

[15] M. S. Mahadeva Naika, Harishkumar T and T. N. Veeranayaka, *On some congruences for (j, k)-regular overpartitions*, Gulf J. Math., 10 (1), (2021), 43-68.

[16] M. S. Mahadeva Naika, S. Shivaprasada Nayaka and C. Shivashankar, *Infinite families of congruences for* 2*-color overpartitions*, Tamsui Oxford J. Inf. Math. Sci., 30 (1), (2016), 61-79.

[17] M. S. Mahadeva Naika and C. Shivashankar, *Arithmetic properties of ℓ-regular overpartitions pairs*, Turkish J. Math., 41 (2017), 756-774.

[18] K. Mahlburg, *The overpartition function modulo small powers of* 2, Discrete Math., 286 (3), (2004), 263-267.

[19] N. Saikia and C. Boruah, *Congruences for ℓ-regular overpartition for $\ell \in \{5, 6, 8\}$*, Ind. J. Pure Appl. Math., 48 (2), (2017), 295-308.

[20] E. Y. Y. Shen, *Arithmetic properties of ℓ-regular overpartitions,* Int. J. Number Theory, 12 (3), (2016), 841-852.

[21] H. S. Sumanth Bharadwaj, B. Hemanthkumar and M. S. Mahadeva Naika, *On* 3*- and* 9*-regular overpartitions modulo powers of* 3, Colloquium Math., 154 (1), (2018), 121-130.

[22] E. X. W. Xia and O. X. M. Yao, *Some modular relations for the Göllnitz Gordon functions by an even-odd method,* J. Math. Anal. Appl., 387 (2012), 126-138.