On non Artinian cofinite generalized local cohomology modules

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Abstract

Let R be a commutative Noetherian ring and $I \subseteq J$ be ideals of R. Let M and N be finitely generated R-modules such that $\mathrm{pd}_R(M) < \infty$. In this paper, we study the supremum of non Artinian I-cofinite modules $H^i_I(M, N)$ over a commutative Noetherian ring where $i \geq 1$ and give a bound for $\tilde{\mathrm{q}}_J(M, N)$ by using $\tilde{\mathrm{q}}_I(M, N)$. We show that $\tilde{\mathrm{q}}_J(M, N) \leq \tilde{\mathrm{q}}_I(M, N) + \mathrm{cd}_J(M, N/IN)$.

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1 Introduction

Throughout this paper, Let R denote a commutative Noetherian ring and I be an ideal of R. Let M and N be two finitely generated R-modules. The notion of generalized local cohomology was introduced by Herzog in [11]. The ith generalized local cohomology modules of M and N with respect to I is defined as

$$H^i_I(M, N) \cong \varinjlim_{n \geq 1} \mathrm{Ext}^i_R(M/I^n M, N).$$

It is clear that $H^i_I(R, N)$ is just the ordinary local cohomology module $H^i_I(N)$. Generalized local cohomology modules have been studied by several authors (see for example [4], [14], [16]).

Hartshorn in [9] defined an R-module M to be I-cofinite, if $\mathrm{Supp}(M) \subseteq \mathrm{V}(I)$ and $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated module for all $i \geq 0$.

Recall that for an R-module M, the notion $\mathrm{cd}(I, M)$, *the cohomological dimension of M with respect to I*, is defined as:

$$\mathrm{cd}(I, M) = \sup\{i \in \mathbb{N}_0 \ : \ H^i_I(M) \neq 0\}$$

and the notion $\mathrm{q}(I, M)$, which for first time was introduced by Hartshorne, is defined as:

$$\mathrm{q}(I, M) = \sup\{i \in \mathbb{N}_0 \ : \ H^i_I(M) \text{ is not Artinian}\},$$

with the usual convention that the supremum of the empty set of integers is interpreted as $-\infty$. These two notions have been studied by several authors (see [6, 7, 8, 10]).

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Amjadi and Naghipour in [3] defined for R-modules M and N, the notion $\mathrm{cd}_I(M,N)$, *the cohomological dimension of M and N with respect to I*, as:

$$\mathrm{cd}_I(M,N) = \sup\{i \in \mathbb{N}_0 \ : \ H^i_I(M,N) \neq 0\}.$$

The present author et al. in [15], introduced the notion $\tilde{\mathrm{q}}_I(M,N)$ as:

$$\tilde{\mathrm{q}}_I(M,N) = \sup\{i \in \mathbb{N}_0 \ : \ H^i_I(M,N) \text{ is not Artinian } I\text{-cofinite}\},$$

if there exist such i's and $-\infty$ otherwise.

Recall that, a subcategory $\mathcal{S}$ of the category of all R-modules is said to be a *Serre category* if an any short exact sequence of R-modules and R-homomorphism, the middle module is in $\mathcal{S}$ if and only if the two other modules are in $\mathcal{S}$.

The main aim of this paper is to prove the following result:

Theorem 1.1. Let R be a Noetherian ring, $I \subseteq J$ be ideals of R and M, N be finitely generated R-modules such that $\mathrm{pd}_R(M) < \infty$. Then

$$\tilde{\mathrm{q}}_J(M,N) \leq \tilde{\mathrm{q}}_I(M,N) + \mathrm{cd}_J(M, N/IN).$$

For any ideal I of R, we denote $\{\mathfrak{p} \in \mathrm{Spec}R : \mathfrak{p} \supseteq I\}$ by $\mathrm{V}(I)$. We refer the reader to [5] and [12] for any unexplained notion and terminology.

2 The results

The main purpose of this section is to prove Theorem 1.1. But first of all we need the following auxiliary lemmas.

Lemma 2.1. Let R be a Noetherian ring, I and J be ideals of R such that $I \subseteq J$. Let M be finitely generated and N be an arbitrary module. Then

(a) $\Gamma_I(M,N) \cong \mathrm{Hom}_R(M, \Gamma_I(N))$,

(b) $\Gamma_J(M,N) \cong \Gamma_J(M, \Gamma_I(N))$.

Proof. The proof is straightforward and is left to the reader. Q.E.D.

Lemma 2.2. Let R be a Noetherian ring, I and J be ideals of R and M, N be finitely generated R-modules such that $H^j_J(M, H^i_I(N))$ is Artinian and J-cofinite, for each $i \geq 0$ and each $j \geq 0$. Then $H^i_J(M,N)$ is also Artinian and J-cofinite, for each $i \geq 0$.

Proof. We use inuction on i. If $i = 0$, then $\Gamma_J(M, \Gamma_I(N))$ is Artinian and J-cofinite so in view of lemma 2.1, $\Gamma_J(M,N)$ is Artinian and J-cofinite. Now assume that $i > 0$ and the claim holds for $i-1$. Let $\bar{N} = N/\Gamma_I(N)$. Suppose that $E(\bar{N})$ is an injective hull of $\bar{N}$. Let

$$0 \longrightarrow \bar{N} \longrightarrow E(\bar{N}) \longrightarrow L \longrightarrow 0, \tag{2.1}$$

is an exact sequence such that $L = E(\bar{N})/\bar{N}$. We apply $\Gamma_I(-)$ to the exact sequence (2.1). We get the exact sequence

$$0 \longrightarrow \Gamma_I(\bar{N}) \longrightarrow \Gamma_I(E(\bar{N})) \longrightarrow \Gamma_I(L) \longrightarrow$$
$$H_I^1(\bar{N}) \longrightarrow H_I^1(E(\bar{N})) \longrightarrow H_I^1(L) \longrightarrow \cdots$$
$$\longrightarrow H_I^i(\bar{N}) \longrightarrow H_I^i(E(\bar{N})) \longrightarrow H_I^i(L) \longrightarrow H_I^{i+1}(\bar{N}).$$

Since $\Gamma_I(\bar{N}) = 0$, it follows that $\Gamma_I(E(\bar{N})) = 0$. we can deduce from long exact sequence

$$H_I^i(L) \cong H_I^{i+1}(\bar{N}). \tag{2.2}$$

Now we apply $\Gamma_J(M, -)$ to the exact sequence (2.1). We get the exact sequence

$$0 \longrightarrow \Gamma_J(M, \bar{N}) \longrightarrow \Gamma_J(M, E(\bar{N})) \longrightarrow \Gamma_J(M, L) \longrightarrow$$
$$H_J^1(M, \bar{N}) \longrightarrow H_J^1(M, E(\bar{N})) \longrightarrow H_J^1(M, L) \longrightarrow \cdots$$
$$\longrightarrow H_J^i(M, \bar{N}) \longrightarrow H_J^i(M, E(\bar{N})) \longrightarrow H_J^i(M, L) \longrightarrow H_J^{i+1}(M, \bar{N}).$$

By using lemma 2.1 and $\Gamma_I(\bar{N}) = 0$ we see that $\Gamma_J(M, \bar{N}) = 0$. Hence $\Gamma_J(M, E(\bar{N})) = 0$ so from the exact sequence, we can deduce that

$$H_J^i(M, L) \cong H_J^{i+1}(M, \bar{N}). \tag{2.3}$$

Since $H_I^i(N) \cong H_I^i(\bar{N})$, it follows the assumption that $H_J^j(M, H_I^i(\bar{N}))$ is Artinian and J-cofinite, for each $i \geq 0$ and $j \geq 0$. In view of the relation (2.2), we get $H_J^j(M, H_I^{i-1}(L))$ is Artinian and J-cofinite, for each $i \geq 0$ and $j \geq 0$. then by induction hypothesis, $H_J^i(M, L)$ is Artinian and J-cofinite, for each $i \geq 0$. Then by using the relation (2.3), we can deduce that $H_J^{i+1}(M, \bar{N})$ is Artinian and J-cofinite, for each $i \geq 0$. The short exact sequence

$$0 \longrightarrow \Gamma_I(N) \longrightarrow N \longrightarrow \bar{N} \longrightarrow 0$$

induces the exact sequenc

$$\cdots \longrightarrow H_J^{i+1}(M, \Gamma_I(N)) \longrightarrow H_J^{i+1}(M, N) \longrightarrow H_J^{i+1}(M, \bar{N}) \longrightarrow \cdots .$$

Since $H_J^{i+1}(M, \Gamma_I(N))$ and $H_J^{i+1}(M, \bar{N})$ are Artinian and J-cofinite and by using [13, Corollary 4.4], we can deduce that $H_J^{i+1}(M, N)$ is Artinian and J-cofinite for each $i \geq 0$. Q.E.D.

We need the following lemma in the proof of Lemma 2.4.

Lemma 2.3. (See [2, Theorem 2.9]) Let R be a Noetherian ring and I be an ideal of R. Let $\mathcal{S}$ be a Serre subcategory of the category of R-module. Then for each R-module N, the following conditions are equivalent.

(a) $\operatorname{Ext}_R^i(R/I, N)$ is in $\mathcal{S}$ for all i,

(b) $\operatorname{Ext}_R^i(M, N)$ is in $\mathcal{S}$ for all i and each finitely generated module M such that $\operatorname{Supp}(M) \subset V(I)$,

(c) $H^i_I(M,N)$ is in $\mathcal{S}$ for each finitely generated R-module M and for all i.

Q.E.D.

Lemma 2.4. Let R be a Noetherian ring, $I \subseteq J$ be ideals of a Noetherian ring R and M, N be finitely generated R-modules. If $\tilde{\mathrm{q}}_J(M,N) \geq 0$, then $\tilde{\mathrm{q}}_I(M,N) \geq 0$.

Proof. Set $t = \tilde{\mathrm{q}}_J(M,N) \geq 0$. From the definition we have R-module $H^t_J(M,N)$ is not Artinian and J-cofinite. Therefore, lemma 2.3 implies that R-module $\mathrm{Ext}^i_R(R/J,N)$ is not Artinian and J-cofinite for each $i \geq 0$. Since $\mathrm{Supp}(R/J) \subseteq \mathrm{Supp}(R/I) = V(I)$, it follows from lemma 2.3 that $\mathrm{Ext}^j_R(R/I,N)$ is not Artinian and I-cofinite for some $j \geq 0$ Also, using lemma 2.3 it follows that $H^i_I(M,N)$ is not Artinian and I-cofinite for some $i \geq 0$. Hence $\tilde{\mathrm{q}}_I(M,N) \geq 0$. Q.E.D.

Lemma 2.5. Let R be a Noetherian ring and $I \subseteq J$ be ideals of R. Let M and N be finitely generated R-modules. If $\tilde{\mathrm{q}}_J(M,N) \geq 0$, then $\tilde{\mathrm{q}}_J(M, \oplus_{i=0}^{\mathrm{cd}(I,N)} H^i_I(N)) \geq 0$ such that

$$\tilde{\mathrm{q}}_J(M, \oplus_{i=0}^{\mathrm{cd}(I,N)} H^i_I(N)) = \mathrm{Sup}\{\tilde{\mathrm{q}}_J(M, H^i_I(N)) : i \in \mathbb{N}_0\}.$$

Proof. Suppose the contrary and look for a contradiction.

Set $\tilde{\mathrm{q}}_J(M, \oplus_{i=0}^{\mathrm{cd}(I,N)} H^i_I(N)) = -\infty$. From the definition we have $H^j_J(M, H^i_I(N))$ is Artinian and J-cofinite. Then, in view of Lemma 2.1, $H^i_J(M,N)$ is Artinian and J-cofinite, for each $i \geq 0$ consequently $\tilde{\mathrm{q}}_J(M,N) = -\infty$, which is a contradiction. Q.E.D.

Lemma 2.6. (See [1, Proposition 3.2]) Let R be a Notherian ring, I be ideal of R and M, N be finitely generated R-modules such that $\mathrm{pd}_R(M) < \infty$ and $\mathrm{Supp}\, L \subseteq \mathrm{Supp}\, N$. Then $\tilde{\mathrm{q}}_I(M,L) \leq \tilde{\mathrm{q}}_I(M,N)$.

Q.E.D.

The following proposition plays an important role in the proof of Theorem 2.8.

Proposition 2.7. Let R be a Notherian ring, $I \subseteq J$ be ideals of R and M, N be finitely generated R-modules such that $\mathrm{pd}_R(M) < \infty$. Then

$$\tilde{\mathrm{q}}_J(M,N) \leq \tilde{\mathrm{q}}_I(M,N) + \tilde{\mathrm{q}}_J(M, \oplus_{i=0}^{\mathrm{cd}(I,N)} H^i_I(N)).$$

Proof. Set $t := \tilde{\mathrm{q}}_J(M,N)$. We prove the result by using the induction on t. If $t = 0$, then the assertion is hold. Suppose that $t > 0$ and that the case $t-1$ is settled. Then it follows from lemma 2.4 that $\tilde{\mathrm{q}}_I(M,N) > 0$. From the exact sequence

$$0 \longrightarrow \Gamma_I(N) \longrightarrow N \longrightarrow N/\Gamma_I(N) \longrightarrow 0$$

we deduce that

$$\begin{aligned}\tilde{\mathrm{q}}_J(M,N) &= \sup\{\tilde{\mathrm{q}}_J(M,\Gamma_I(N)), \tilde{\mathrm{q}}_J(M, N/\Gamma_I(N))\} \\ &= \tilde{\mathrm{q}}_J(M, N/\Gamma_I(N)). \qquad (2.4)\end{aligned}$$

We set $L = N/\Gamma_I(N)$. The exact sequence

$$0 \longrightarrow L \longrightarrow E_R(L) \longrightarrow E_R(L)/L \longrightarrow 0 \tag{2.5}$$

and relation (2.4) yields

$$\begin{aligned}\tilde{\mathrm{q}}_J(M, E_R(L)/L) &= \tilde{\mathrm{q}}_J(M, L) - 1\\ &= \tilde{\mathrm{q}}_J(M, N) - 1\\ &= t - 1.\end{aligned}$$

By induction hypothesis, it follows that

$$\tilde{\mathrm{q}}_J(M, E_R(L)/L) \leq \tilde{\mathrm{q}}_I(M, E_R(L)/L) + \sup\{\tilde{\mathrm{q}}_J(M, H^i_I(E_R(L)/L)) : i \in \mathbb{N}_0\}. \tag{2.6}$$

By the exact sequence (2.5) we get the following exact sequence.

$$H^i_I(L) \longrightarrow H^i_I(E_R(L)) \longrightarrow H^i_I(E_R(L)/L) \longrightarrow H^{i+1}_I(L) \longrightarrow$$
$$H^{i+1}_I(E_R(L)) \longrightarrow H^{i+1}_I(E_R(L)/L).$$

Therefore, $H^i_I(E_R(L)/L) \cong H^{i+1}_I(L)$. Then

$$\tilde{\mathrm{q}}_J(M, H^i_I(E_R(L)/L)) = \tilde{\mathrm{q}}_J(M, H^{i+1}_I(L)). \tag{2.7}$$

Hence by relations (2.6) and (2.7) we have

$$t - 1 \leq \tilde{\mathrm{q}}_I(M, E_R(L)/L) + \sup\{\tilde{\mathrm{q}}_J(M, H^i_I(L)) : i \in \mathbb{N}_0\},$$

Since $H^i_I(L) \cong H^i_I(N)$, it follows that

$$t \leq 1 + \tilde{\mathrm{q}}_I(M, E_R(L)/L) + \sup\{\tilde{\mathrm{q}}_J(M, H^i_I(N)) : i \in \mathbb{N}_0\}. \tag{2.8}$$

On the other hand, since $\tilde{\mathrm{q}}_J(M, E_R(L)/L) = t-1 \geq 0$, it follows from lemma 2.4 that $\tilde{\mathrm{q}}_I(M, E_R(L)/L) \geq 0$. Consesequently by the exact sequence (2.5) we have

$$\tilde{\mathrm{q}}_I(M, E_R(L)/L) = \tilde{\mathrm{q}}_I(M, L) - 1 \geq 0. \tag{2.9}$$

Thus by relations (2.8), (2.9) and Lemma 2.6, we have

$$\begin{aligned}\tilde{\mathrm{q}}_J(M, N) &= t\\ &\leq 1 + \tilde{\mathrm{q}}_I(M, E_R(L)/L) + \sup\{\tilde{\mathrm{q}}_J(M, H^i_I(N)) : i \in \mathbb{N}_0\}\\ &= \tilde{\mathrm{q}}_I(M, L) + \sup\{\tilde{\mathrm{q}}_J(M, H^i_I(N)) : i \in \mathbb{N}_0\}\\ &\leq \tilde{\mathrm{q}}_I(M, N) + \sup\{\tilde{\mathrm{q}}_J(M, H^i_I(N)) : i \in \mathbb{N}_0\}.\end{aligned}$$

Q.E.D.

Now we are ready to state and prove the main result.

Theorem 2.8. Let R be a Noetherian ring, $I \subseteq J$ be ideals of R and M, N be finitely generated R-modules such that $\mathrm{pd}_R(M) < \infty$. Then

$$\tilde{\mathrm{q}}_J(M,N) \leq \tilde{\mathrm{q}}_I(M,N) + \mathrm{cd}_J(M, N/IN).$$

Proof. Assume that $\tilde{\mathrm{q}}_J(M,N) \geq 0$. Then by using Proposition 2.7, it follows that

$$\tilde{\mathrm{q}}_J(M,N) \leq \tilde{\mathrm{q}}_I(M,N) + \tilde{\mathrm{q}}_J(M, \oplus_{i\geq 0}^{\mathrm{cd}(I,N)} H_I^i(N)).$$

Set $k := \tilde{\mathrm{q}}_J(M, \oplus_{i\geq 0}^{\mathrm{cd}(I,N)} H_I^i(N))$. Since

$$\tilde{\mathrm{q}}_J(M, \oplus_{i\geq 0} H_I^i(N)) \leq \mathrm{cd}_J(M, \oplus_{i\geq 0} H_I^i(N)),$$

it follows that $\mathrm{cd}_J(M, \oplus_{i\geq 0} H_I^i(N)) \geq k$ which implies that

$$H_J^k(M, \oplus_{i\geq 0}^{\mathrm{cd}(I,N)} H_I^i(N)) \neq 0.$$

Therefore, there exists a finitely generated submodule L of the R-module $\oplus_{i\geq 0}^{\mathrm{cd}(I,N)} H_I^i(N)$, such that $H_J^k(M, L) \neq 0$ concequently

$$k \leq \mathrm{cd}_J(M, L). \tag{2.10}$$

Since

$$\begin{aligned} \mathrm{Supp}\, L &\subseteq \mathrm{Supp}(\oplus_{i\geq 0}^{\mathrm{cd}(I,N)} H_I^i(N)) \\ &\subseteq \mathrm{Supp}(N/IN), \end{aligned}$$

it follows from [3, Theorem B] that

$$\mathrm{cd}_J(M, L) \leq \mathrm{cd}_J(M, N/IN). \tag{2.11}$$

Then by relations (2.10) and (2.11) we have

$$\begin{aligned} \tilde{\mathrm{q}}_J(M,N) &\leq \tilde{\mathrm{q}}_I(M,N) + k \\ &\leq \tilde{\mathrm{q}}_I(M,N) + \mathrm{cd}_J(M, L) \\ &\leq \tilde{\mathrm{q}}_I(M,N) + \mathrm{cd}_J(M, N/IN). \end{aligned}$$

So, the assertion holds. Q.E.D.

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