Some integral inequalities via fractional derivatives

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Communicating editor - Managing editors' group

Abstract

In this work, we obtain new versions of the Hermite-Hadamard Inequality via generalized fractional derivatives, which differentiates our work from previous known ones, which use integral operators. Several known results from the literature are particular cases of ours, demonstrating the breadth and generality of our conclusions.

2010 Mathematics Subject Classification. **26A51**. 26D10, 26D15.
Keywords. Integral Inequalities, fractional derivatives and integrals, convex functions.

1 Introduction

With the emergence of the Differential and Integral Calculus, by Newton and Leibniz, one of the most important theoretical tools of Mathematics was consolidated, and the applications multiplied, including areas that a priori were considered "non-mathematical". The birth of the non-integer order calculus is practically contemporary, in 1675 a letter from Leibniz to L'Hopital already answered the question, will it make sense to extend the values of the order of derivation (integer) to the set of rational, irrational or complex numbers? That is, the fractional calculus is concerned with integrals and differentiation of arbitrary non integral order. This field has become in the last five decades, one of the most dynamic of Mathematics and its applications have spread to study and characterize different processes and phenomena ranging from biology to economics, through chemistry, engineering and technology, physics and many more.

As say before, one of the most developed mathematical areas today is that of Fractional Calculus, and with a diversity of applications and theoretical developments similar to Calculus with derivatives and integer integrals (see, for example, [1]), and more and more researchers have dedicated themselves to this area [3]. A more complete overview of the development of this area actually, with its overlapping with the Generalized Local Calculus, can be found at [5, 6, 29].

Comparing quantities is a fundamental tool in mathematics. The art of inequalities is found in the methods used to generate and prove them. The theory of inequalities lies in its careful interpretation and in the knowledge of its strengths and limitations.

The use of inequalities dates back to Greek Mathematics, although its development as an independent scientific discipline is linked to the contributions of Cauchy, Hilbert, Bunyakowski and many more, and its definitive consolidation came from the hand of the renowned "Inequalities" of Hardy, Littlewood and Polya of 1934.

Advanced Studies: Euro-Tbilisi Mathematical Journal 15(3) (2022), pp. 31–44.
DOI: 10.32513/asetmj/19322008222
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 27 August 2021.
Accepted for publication: 25 March 2022.

From this moment, the development of this area has been increasing and every year the number of researchers and productions related to the subject increases. Among the best known inequalities, there is the so-called Hermite-Hadamard Inequality:

Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $a, b \in I$ with $a < b$, then

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \psi(\xi)d\xi \leq \frac{\psi(a)+\psi(b)}{2}. \tag{1.1}$$

This inequality was published by Hermite [24] in 1883 and, independently, by Hadamard in 1893 [23]. It gives an estimation of the mean value of a convex function, and interpolates the image of the midpoint of the interval and the average of the images of the extremes of the interval and it is important to note that it also provides a refinement to the Jensen inequality. The above inequality has been the central axis of multiple investigations in recent years, obtained with different integral operators, if not also, under different notions of convexity see, for example [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 14, 18, 20, 21, 22, 26, 28, 30, 31, 34, 36, 38, 39] and references therein for more information and other extensions of the Hermite–Hadamard inequality.

The convex function is defined as:

Definition 1.1. A function $\psi : \Im \to \mathbb{R}$ is said to be convex function, if the following inequality holds:

$$\psi(\xi\tau_1 + (1-\xi)\tau_2) \leq \xi\psi(\tau_1) + (1-\xi)\psi(\tau_2).$$

For all $\tau_1, \tau_2 \in \Im$ and $\xi \in [0, 1]$. If the above inequality holds in reverse order then the function ψ is said to be concave function.

In Mathematical Sciences, the notion of convex function plays a very prominent role, due to its multiple applications and its theoretical overlaps with various mathematical areas. Readers interested in this notion, can consult [35], where a panorama, practically complete, of these topic is presented. Authors have established many identities and inequalities for convex functions such as Ostrowski, Hardy and Gagliardo Nirenberg inequality.

To facilitate the reading of the work, we present some necessary definitions of Fractional Calculus.

Definition 1.2. Let $\digamma \in L_1[a, b]$ and $\Re^{\varrho}_{a+}\digamma$ and $\Re^{\varrho}_{b-}\digamma$ be the left-sided and right-sided Riemann Liouville fractional integral of order $\varrho > 0$ defined by

$$\Re^{\varrho}_{a+}\digamma(\varsigma) = \frac{1}{\Gamma(\varrho)}\int_a^{\varsigma} \digamma(\zeta)(\varsigma-\zeta)^{\varrho-1}d\zeta \qquad (\varrho > a),$$

and

$$\Re^{\varrho}_{b-}\digamma(\varsigma) = \frac{1}{\Gamma(\varrho)}\int_{\varsigma}^{b} \digamma(\zeta)(\zeta-\varsigma)^{\varrho-1}d\zeta \qquad (\varrho < b).$$

respectively, where $\Gamma(\varrho) = \int_0^\infty e^{-\wp} - \wp^{\varrho-1}d\wp$ is the usual gamma function.

Definition 1.3. Let $\left(\tau_1^+, \tau_2^-\right)$ where $-\infty \leq \tau_1^+ < \tau_2^- \leq +\infty$ be the finite or infinite real interval and $\theta > 0$. Let $\digamma(\zeta)$ be positive monotone function and it is increasing on the interval $(\tau_1^+, \tau_2^-]$

then, the left hand fractional derivative of Caputo in the F-Hilfer sense of order $\theta > 0$ is defined as:

$$ {}^{c}\breve{D}^{\theta,F}_{\tau_1^+} G(\lambda) = I^{n-\theta,F}_{\tau_1^+} \left(\frac{1}{F'(\lambda)} \frac{d}{d\lambda} \right)^n G(\lambda). \tag{1.2} $$

the right hand fractional derivative of Caputo in the F-Hilfer sense of order $\theta > 0$ is defined as:

$$ {}^{c}\breve{D}^{\theta,F}_{\tau_2^-} G(\lambda) = I^{n-\theta,F}_{\tau_2^-} \left(-\frac{1}{F'(\lambda)} \frac{d}{d\lambda} \right)^n G(\lambda). \tag{1.3} $$

where $n = [\theta] + 1$ for $\theta \notin \mathbb{N}, n = \theta$ for $\theta \in \mathbb{N}$ and if $\theta \notin \mathbb{N}$ then

$$ {}^{c}\breve{D}^{\theta,F}_{\tau_1^+} G(\lambda) = \frac{1}{\Gamma(n-\theta)} \int_{\tau_1^+}^{\lambda} F'(\zeta)(F(\lambda) - F(\zeta))^{n-\theta-1} G^{[n]}_{F}(\zeta) d\zeta. \tag{1.4} $$

$$ {}^{c}\breve{D}^{\theta,F}_{\tau_2^-} G(\lambda) = \frac{(-1)^n}{\Gamma(n-\theta)} \int_{\lambda}^{\tau_2^-} F'(\zeta)(F(\zeta) - F(\lambda))^{n-\theta-1} G^{[n]}_{F}(\zeta) d\zeta. \tag{1.5} $$

Definition 1.4. Let (τ_1, τ_2) where $-\infty \leq \tau_1 < \tau_2 \leq +\infty$ be the finite or infinite real interval and $\theta > 0$. Let $F(\zeta)$ be positive monotone function and it is increasing on the interval $(\tau_1, \tau_2]$ then, the left-sided and right-sided fractional derivative of Caputo in the F-Hilfer sense of order $\theta > 0$ is defined as:

$$ \breve{D}^{\theta,F}_{0^+} G(\lambda) = \frac{1}{\Gamma(n-\theta)} \int_{0}^{\lambda} F'(\zeta)(F(\lambda) - F(\zeta))^{n-\theta-1} G(\zeta) d\zeta \tag{1.6} $$

this is also known as generalized Caputo fractional derivative.

Remark 1.5. You can see about (1.4), (1.5) and (1.6) in [1].

Remark 1.6. If we take $F(\zeta) = \zeta$ in definition represented by (1.4), (1.5) and(1.6), we obtain the classical Caputo derivative.

Remark 1.7. Under the above conditions, if $F(\zeta) = \ln \zeta$, we obtain Caputo Hadamard fractional derivative.

Remark 1.8. If we put $F(\zeta) = \frac{\zeta^\theta}{\theta}$ in definition represented by (1.4) and (1.5), then the both left handed and right handed generalized fractional derivative is recaptured.

Remark 1.9. If $F(\zeta) = \frac{(\zeta - a)^\theta}{\theta}$ in (1.6) then we obtain the fractional conformable derivative.

It should be noted that most of the results published on the Hermite-Hadamard Inequality (1.1), are formulated in terms of integral operators, some attempts can be consulted in [13, 15, 16, 17, 19, 25, 27, 32, 37, 40].

The main objective of the present work is to establish new integral inequalities, using the Caputo right and left fractional derivatives in the Hilfer sense, via convex functions. The most novel thing is that we use fractional derivatives to establish integral inequalities, something that generally does not appear in known works.

2 Main results

In this section we obtain new integral inequalities, within the framework of the generalized operators of the definition (1.4), (1.5) and (1.6).

Theorem 2.1. Suppose $K_1^{(n)}$ and $K_3^{(n)}$ be continuous, positive and integrable functions which are defined on the interval $[0,+\infty)$ such that $K_1^{(n)} \leq K_3^{(n)}$. Under the condition that $K_1^{(n)}$ is increasing and $\frac{K_1^{(n)}}{K_3^{(n)}}$ is decreasing over the interval $[0,+\infty)$, then for any convex function $\hat{H}$ that satisfies the condition $\hat{H}(0)=0$, we have the following inequality:

$$\frac{{}^{c}\check{D}^{\theta,F}_{\tau_1^+}\left[K_1(\lambda)\right]}{{}^{c}\check{D}^{\theta,F}_{\tau_1^+}\left[K_3(\lambda)\right]} \geq \frac{{}^{c}\check{D}^{\theta,F}_{\tau_1^+}\left[\hat{H}(K_1(\lambda))\right]}{{}^{c}\check{D}^{\theta,F}_{\tau_1^+}\left[\hat{H}(K_3(\lambda))\right]}. \tag{2.1}$$

Proof. Using the convexity of $\hat{H}$ and using the assumption $\hat{H}(0)=0$, the function $\frac{\hat{H}(K_1^{(n)}(x))}{x}$ is increasing. As the function $K_1^{(n)}$ is increasing then the function $\frac{\hat{H}(K_1^{(n)}(x))}{K_1^{(n)}(x)}$ is also increasing. It is obvious that the function $\frac{K_1^{(n)}}{K_3^{(n)}}$ is decreasing, then for all $u,v\in[0,+\infty)$, we have

$$\left(\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)} - \frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\right)\left(\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)} - \frac{K_1^{(n)}(u)}{K_3^{(n)}(u)}\right) \geq 0. \tag{2.2}$$

From (2.2), we have

$$\begin{aligned} & \frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)} + \frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\frac{K_1^{(n)}(u)}{K_3^{(n)}(u)} \\ \geq\ & \frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)} + \frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}\frac{K_1^{(n)}(u)}{K_3^{(n)}(u)}. \end{aligned} \tag{2.3}$$

On multiplying (2.3) by $K_3^{(n)}(u)K_3^{(n)}(v)$, we obtain

$$\begin{aligned} & \frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}K_3^{(n)}(u)K_1^{(n)}(v) + \frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}K_3^{(n)}(v)K_1^{(n)}(u) \\ \geq\ & \hat{H}(K_1^{(n)}(v))K_3^{(n)}(u) + \hat{H}(K_1^{(n)}(u))K_3^{(n)}(v). \end{aligned} \tag{2.4}$$

Multiplying (2.4) by $\frac{1}{\Gamma(n-\theta)}F'(v)(F(\lambda)-F(v))^{n-\theta-1}$ then after integrating over $[\tau_1^+,\lambda]$ with respect to v, we obtain

$$\begin{aligned} & \frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}K_3^{(n)}(u)\ {}^{c}\check{D}^{\theta,F}_{\tau_1^+}\left[K_1(\lambda)\right] + K_1^{(n)}(u)\ {}^{c}\check{D}^{\theta,F}_{\tau_1^+}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right] \\ \geq\ & K_3^{(n)}(u)\ {}^{c}\check{D}^{\theta,F}_{\tau_1^+}\left[\hat{H}(K_1(\lambda))\right] + \hat{H}(K_1^{(n)}(u))\ {}^{c}\check{D}^{\theta,F}_{\tau_1^+}\left[K_3(\lambda)\right]. \end{aligned} \tag{2.5}$$

Similarly, multiplying the inequality (2.5) by $\frac{1}{\Gamma(n-\theta)}\digamma^{'}(u)(\digamma(\lambda)-\digamma(u))^{n-\theta-1}$ and integrating over $[\tau_1^+,\lambda]$ with respect to u, we obtain

$$\begin{aligned} &{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right]\,{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[K_1(\lambda)\right]+{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[K_1(\lambda)\right]\,{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right] \\ &\geq {}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[\hat{H}(K_1(\lambda))\right]\,{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[K_3(\lambda)\right]+{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[\hat{H}(K_1(\lambda))\right]\,{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[K_3(\lambda)\right]. \end{aligned} \tag{2.6}$$

We have from (2.6)

$$\frac{{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[K_1(\lambda)\right]}{{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[K_3(\lambda)\right]}\geq\frac{{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_1(\lambda)\right]}{{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right]}. \tag{2.7a}$$

Since $K_1^{(n)}\leq K_3^{(n)}$ and from the properties of $\hat{H}$, it is easy to obtain

$$\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\leq\frac{\hat{H}(K_3^{(n)}(v))}{K_3^{(n)}(v)}, v\in[0,+\infty). \tag{2.8}$$

Now from (2.8), we can easily obtain

$${}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_1(\lambda)\right]\leq{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[\hat{H}(K_1(\lambda))\right], \tag{2.9}$$

and

$${}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right]\leq{}^c\check{D}^{\theta,\digamma}_{\tau_1^+}\left[\hat{H}(K_3(\lambda))\right]. \tag{2.10}$$

Utilizing (2.9) and (2.10) in (2.7a), we obtain the required result. Q.E.D.

Theorem 2.2. Suppose $K_1^{(n)}$ and $K_3^{(n)}$ be continuous, positive and integrable functions which are defined on the interval $[0,+\infty)$, such that $K_1^{(n)}\leq K_3^{(n)}$. Under the condition that $K_1^{(n)}$ is increasing and $\frac{K_1^{(n)}}{K_3^{(n)}}$ is decreasing over the interval $[0,+\infty)$ then, for any convex function $\hat{H}$ that satisfies the condition $\hat{H}(0)=0$, then we have the following inequality:

$$\frac{{}^c\check{D}^{\theta,\digamma}_{\tau_2^-}\left[K_1(\lambda)\right]}{{}^c\check{D}^{\theta,\digamma}_{\tau_2^-}\left[K_3(\lambda)\right]}\geq\frac{{}^c\check{D}^{\theta,\digamma}_{\tau_2^-}\left[\hat{H}(K_1(\lambda))\right]}{{}^c\check{D}^{\theta,\digamma}_{\tau_2^-}\left[\hat{H}(K_3(\lambda))\right]}. \tag{2.11}$$

Proof. Using the convexity of $\hat{H}$ and using the assumption $\hat{H}(0)=0$, the function $\frac{\hat{H}(K_1^{(n)}(x))}{x}$ is increasing. As the function $K_1^{(n)}$ is increasing then the function $\frac{\hat{H}(K_1^{(n)}(x))}{K_1^{(n)}(x)}$ is also increasing. It is obvious that the function $\frac{K_1^{(n)}}{K_3^{(n)}}$ is decreasing, then for all $u,v\in[0,+\infty)$, we have

$$\left(\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}-\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\right)\left(\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)}-\frac{K_1^{(n)}(u)}{K_3^{(n)}(u)}\right)\geq 0. \tag{2.12}$$

From (2.12), we have

$$\begin{aligned} & \frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)} \frac{K_1^{(n)}(v)}{K_3^{(n)}(v)} + \frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)} \frac{K_1^{(n)}(u)}{K_3^{(n)}(u)} \\ \geq \quad & \frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)} \frac{K_1^{(n)}(v)}{K_3^{(n)}(v)} + \frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)} \frac{K_1^{(n)}(u)}{K_3^{(n)}(u)}. \end{aligned} \tag{2.13}$$

On multiplying (2.13) by $K_3^{(n)}(u)K_3^{(n)}(v)$, we obtain

$$\begin{aligned} & \frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)} K_3^{(n)}(u)K_1^{(n)}(v) + \frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)} K_3^{(n)}(v)K_1^{(n)}(u) \\ \geq \quad & \hat{H}(K_1^{(n)}(v))K_3^{(n)}(u) + \hat{H}(K_1^{(n)}(u))K_3^{(n)}(v). \end{aligned} \tag{2.14}$$

Multiplying (2.4) by $\frac{(-1)^n}{\Gamma(n-\theta)}F'(v)(F(v) - F(\lambda))^{n-\theta-1}$ then after integrating over $[\lambda, \tau_2^-]$ with respect to v, we obtain

$$\begin{aligned} & \frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)} K_3^{(n)}(u)\ {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[K_1(\lambda)\right] + K_1^{(n)}(u)\ {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right] \\ \geq \quad & K_3^{(n)}(u)\ {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[\hat{H}(K_1(\lambda))\right] + \hat{H}(K_1^{(n)}(u))\ {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[K_3(\lambda)\right]. \end{aligned} \tag{2.15}$$

Similarly, on multiplying (2.15) by $\frac{(-1)^n}{\Gamma(n-\theta)}F'(u)(F(u) - F(\lambda))^{n-\theta-1}$ and integrating over $[\lambda, \tau_2^-]$ with respect to u, we obtain

$$\begin{aligned} & {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right]\ {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[K_1(\lambda)\right] + {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[K_1(\lambda)\right]\ {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right] \\ & \quad \geq {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[\hat{H}(K_1(\lambda))\right]\ {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[K_3(\lambda)\right] + {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[\hat{H}(K_1(\lambda))\right]\ {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[K_3(\lambda)\right]. \end{aligned} \tag{2.16}$$

We have from (2.16)

$$\frac{{}^c\check{D}^{\theta,F}_{\tau_2^-}\left[K_1(\lambda)\right]}{{}^c\check{D}^{\theta,F}_{\tau_2^-}\left[K_3(\lambda)\right]} \geq \frac{{}^c\check{D}^{\theta,F}_{\tau_2^-}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_1(\lambda)\right]}{{}^c\check{D}^{\theta,F}_{\tau_2^-}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right]}. \tag{2.17}$$

Since $K_1^{(n)} \leq K_3^{(n)}$ and from the properties of $\hat{H}$ it is easy to obtain that

$$\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)} \leq \frac{\hat{H}(K_3^{(n)}(v))}{K_3^{(n)}(v)}, v \in [\lambda, \tau_2^-] \tag{2.18}$$

Now from (2.18), we can easily obtain

$${}^c\check{D}^{\theta,F}_{\tau_2^-}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_1(\lambda)\right] \leq {}^c\check{D}^{\theta,F}_{\tau_2^-}\left[\hat{H}(K_1(\lambda))\right], \tag{2.19}$$

and

$$ {}^{c}\check{D}^{\theta,F}_{\tau_2^-}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right] \leq {}^{c}\check{D}^{\theta,F}_{\tau_2^-}\left[\hat{H}(K_3(\lambda))\right]. \tag{2.20} $$

Utilizing (2.19) and (2.20) in (2.17), we obtain the required result. Q.E.D.

Theorem 2.3. Suppose $K_1^{(n)}$ and $K_3^{(n)}$ be continuous, positive and integrable functions which are defined on the interval $[0,+\infty)$, such that $K_1^{(n)} \leq K_3^{(n)}$. Under the condition that $K_1^{(n)}$ is increasing and $\frac{K_1^{(n)}}{K_3^{(n)}}$ is decreasing over the interval $[0,+\infty)$ then, for any convex function $\hat{H}$ that satisfies the condition $\hat{H}(0)=0$, then we have the following inequality:

$$ \frac{\check{D}^{\theta,F}_{0^+} K_1(\lambda)}{\check{D}^{\theta,F}_{0^+} K_3(\lambda)} \geq \frac{\check{D}^{\theta,F}_{0^+}\left[\hat{H}(K_1(\lambda))\right]}{\check{D}^{\theta,F}_{0^+}\left[\hat{H}(K_3(\lambda))\right]} \tag{2.21} $$

Proof. Using the convexity of $\hat{H}$ and using the assumption $\hat{H}(0)=0$, the function $\frac{\hat{H}(K_1^{(n)}(x))}{x}$ is increasing. As the function $K_1^{(n)}$ is increasing then the function $\frac{\hat{H}(K_1^{(n)}(x))}{K_1^{(n)}(x)}$ is also increasing. It is obvious that the function $\frac{K_1^{(n)}}{K_3^{(n)}}$ is decreasing, then for all $u,v\in[0,+\infty)$, we have

$$ \left(\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}-\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\right)\left(\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)}-\frac{K_1^{(n)}(u)}{K_3^{(n)}(u)}\right)\geq 0. \tag{2.22} $$

From (2.22), we have

$$ \begin{aligned} &\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)}+\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\frac{K_1^{(n)}(u)}{K_3^{(n)}(u)} \\ \geq\ &\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)}+\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}\frac{K_1^{(n)}(u)}{K_3^{(n)}(u)}. \end{aligned} \tag{2.23} $$

On multiplying (2.23) by $K_3^{(n)}(u)K_3^{(n)}(v)$, we obtain

$$ \begin{aligned} &\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}K_3^{(n)}(u)K_1^{(n)}(v)+\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}K_3^{(n)}(v)K_1^{(n)}(u) \\ \geq\ &\hat{H}(K_1^{(n)}(v))K_3^{(n)}(u)+\hat{H}(K_1^{(n)}(u))K_3^{(n)}(v). \end{aligned} \tag{2.24} $$

Multiplying (2.24) by $\frac{1}{\Gamma(n-\theta)}F'(v)(F(\lambda)-F(v))^{n-\theta-1}$ then after integrating over $[0^+,\lambda]$ with respect to v, we obtain

$$ \begin{aligned} &\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}K_3^{(n)}(u)\ \check{D}^{\theta,F}_{0^+}\left[K_1(\lambda)\right]+K_1^{(n)}(u)\ \check{D}^{\theta,F}_{0^+}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right] \\ \geq\ &K_3^{(n)}(u)\check{D}^{\theta,F}_{0^+}\left[\hat{H}(K_1(\lambda))\right]+\hat{H}(K_1^{(n)}(u))\ \check{D}^{\theta,F}_{0^+}\left[K_3(\lambda)\right]. \end{aligned} \tag{2.25} $$

Similarly, multiplying the inequality (2.25) by $\frac{1}{\Gamma(n-\theta)}F'(u)(F(\lambda)-F(u))^{n-\theta-1}$ and integrating over $[0^+,\lambda]$ with respect to u, we obtain

$$\check{D}_{0^+}^{\theta,F}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right]\ \check{D}_{0^+}^{\theta,F}\left[K_1(\lambda)\right]+\ \check{D}_{0^+}^{\theta,F}\left[K_1(\lambda)\right]\ \check{D}_{0^+}^{\theta,F}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right] \tag{2.26}$$
$$\geq \check{D}_{0^+}^{\theta,F}\left[\hat{H}(K_1(\lambda))\right]\ \check{D}_{0^+}^{\theta,F}\left[K_3(\lambda)\right]+\ \check{D}_{0^+}^{\theta,F}\left[\hat{H}(K_1(\lambda))\right]\ \check{D}_{0^+}^{\theta,F}\left[K_3(\lambda)\right].$$

We have from (2.26)

$$\frac{\check{D}_{0^+}^{\theta,F}\left[K_1(\lambda)\right]}{\check{D}_{0^+}^{\theta,F}\left[K_3(\lambda)\right]}\geq\frac{\check{D}_{0^+}^{\theta,F}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_1(\lambda)\right]}{\check{D}_{0^+}^{\theta,F}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right]}. \tag{2.27}$$

Since $K_1^{(n)}\leq K_3^{(n)}$ and from the properties of $\hat{H}$, it is easy to obtain

$$\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\leq\frac{\hat{H}(K_3^{(n)}(v))}{K_3^{(n)}(v)}, v\in[0,+\infty). \tag{2.28}$$

Now from (2.28), we can easily obtain

$$\check{D}_{0^+}^{\theta,F}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_1(\lambda)\right]\leq\ \check{D}_{0^+}^{\theta,F}\left[\hat{H}(K_1(\lambda))\right], \tag{2.29}$$

and

$$\check{D}_{0^+}^{\theta,F}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right]\leq\ \check{D}_{0^+}^{\theta,F}\left[\hat{H}(K_3(\lambda))\right]. \tag{2.30}$$

Utilizing (2.29) and (2.30) in (2.27), we obtain the required result. Q.E.D.

Theorem 2.4. Suppose $K_1^{(n)}$ and $K_3^{(n)}$ be continuous, positive and integrable functions which are defined on the interval $[0,+\infty)$, such that $K_1^{(n)}\leq K_3^{(n)}$. Under the condition that $K_1^{(n)}$ is increasing and $\frac{K_1^{(n)}}{K_3^{(n)}}$ is decreasing over the interval $[0,+\infty)$ then, for any convex function $\hat{H}$ that satisfies the condition $\hat{H}(0)=0$, then we have the following inequality:

$${}^{c}\check{D}_{\tau_1^+}^{\theta,F}\left[K_1(\lambda)\right]\ {}^{c}\check{D}_{\tau_2^-}^{\theta,F}\left[\hat{H}(K_3(\lambda))\right]+\ {}^{c}\check{D}_{\tau_2^-}^{\theta,F}\left[K_1(\lambda)\right]\ {}^{c}\check{D}_{\tau_1^+}^{\theta,F}\left[\hat{H}(K_3(\lambda))\right] \tag{2.31}$$
$$\geq\ {}^{c}\check{D}_{\tau_1^+}^{\theta,F}\left[\hat{H}(K_1(\lambda))\right]\ {}^{c}\check{D}_{\tau_2^-}^{\theta,F}\left[K_3(\lambda)\right]+\ {}^{c}\check{D}_{\tau_2^-}^{\theta,F}\left[\hat{H}(K_1(\lambda))\right]\ {}^{c}\check{D}_{\tau_2^+}^{\theta,F}\left[K_3(\lambda)\right].$$

Proof. Using the convexity of $\hat{H}$ and using the assumption $\hat{H}(0)=0$, the function $\frac{\hat{H}(K_1^{(n)}(x))}{x}$ is increasing. As the function $K_1^{(n)}$ is increasing then the function $\frac{\hat{H}(K_1^{(n)}(x))}{K_1^{(n)}(x)}$ is also increasing. It is obvious that the function $\frac{K_1^{(n)}}{K_3^{(n)}}$ is decreasing, then for all $u,v\in[0,+\infty)$, we have

$$\left(\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}-\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\right)\left(\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)}-\frac{K_1^{(n)}(u)}{K_3^{(n)}(u)}\right)\geq 0. \tag{2.32}$$

From (2.32), we have

$$\begin{aligned}&\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)}+\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\frac{K_1^{(n)}(u)}{K_3^{(n)}(u)} \\ \geq\ &\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)}+\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}\frac{K_1^{(n)}(u)}{K_3^{(n)}(u)}.\end{aligned} \tag{2.33}$$

On multiplying (2.33) by $K_3^{(n)}(u)K_3^{(n)}(v)$, we obtain

$$\begin{aligned}&\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}K_3^{(n)}(u)K_1^{(n)}(v)+\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}K_3^{(n)}(v)K_1^{(n)}(u) \\ \geq\ &\hat{H}(K_1^{(n)}(v))K_3^{(n)}(u)+\hat{H}(K_1^{(n)}(u))K_3^{(n)}(v).\end{aligned} \tag{2.34}$$

Multiplying (2.34) by $\frac{1}{\Gamma(n-\theta)}F'(v)(F(\lambda)-F(v))^{n-\theta-1}$ then after integrating over $[\tau_1^+,\lambda]$ with respect to v, we obtain

$$\begin{aligned}&\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}K_3^{(n)}(u)\ {}^{c}\check{D}_{\tau_1^+}^{\theta,F}\left[K_1(\lambda)\right]+K_1^{(n)}(u)\ {}^{c}\check{D}_{\tau_1^+}^{\theta,F}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right] \\ \geq\ &K_3^{(n)}(u)\ {}^{c}\check{D}_{\tau_1^+}^{\theta,F}\left[\hat{H}(K_1(\lambda))\right]+\hat{H}(K_1^{(n)}(u))\ {}^{c}\check{D}_{\tau_1^+}^{\theta,F}\left[K_3(\lambda)\right].\end{aligned} \tag{2.35}$$

Similarly, multiplying the (2.35) by $\frac{(-1)^n}{\Gamma(n-\theta)}F'(u)(F(u)-F(\lambda))^{n-\theta-1}$ and integrating over $[\lambda,\tau_2^-]$ with respect to u, we obtain

$$\begin{aligned}&{}^{c}\check{D}_{\tau_2^-}^{\theta,F}\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\ {}^{c}\check{D}_{\tau_1^+}^{\theta,F}\left[K_1(\lambda)\right]+\ {}^{c}\check{D}_{\tau_2^-}^{\theta,F}\left[K_1(\lambda)\right]\ {}^{c}\check{D}_{\tau_1^+}^{\theta,F}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right] \\ &\geq \check{D}_{\tau_1^+}^{\theta,F}\left[\hat{H}(K_1(\lambda))\right]\ {}^{c}\check{D}_{\tau_2^-}^{\theta,F}\left[K_3(\lambda)\right]+\ {}^{c}\check{D}_{\tau_2^-}^{\theta,F}\left[\hat{H}(K_1(\lambda))\right]\ {}^{c}\check{D}_{\tau_1^+}^{\theta,F}\left[K_3(\lambda)\right].\end{aligned} \tag{2.36}$$

Since $K_1^{(n)}\leq K_3^{(n)}$ and from the properties of $\hat{H}$, it is easy to obtain that

$$\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\leq\frac{\hat{H}(K_3^{(n)}(v))}{K_3^{(n)}(v)},v\in[0,+\infty). \tag{2.37}$$

Now from (2.28), we can easily obtain

$${}^{c}\check{D}_{\tau_1^+}^{\theta,F}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right]\leq\ {}^{c}\check{D}_{\tau_1^+}^{\theta,F}\left[\hat{H}(K_3(\lambda))\right], \tag{2.38}$$

and

$${}^{c}\check{D}_{\tau_2^-}^{\theta,F}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right]\leq\ {}^{c}\check{D}_{\tau_2^-}^{\theta,F}\left[\hat{H}(K_3(\lambda))\right]. \tag{2.39}$$

Utilizing (2.38) and (2.39) in (2.36), we obtain the required result. Q.E.D.

Theorem 2.5. Suppose $K_1^{(n)}, K_2^{(n)}$ and $K_3^{(n)}$ be the positive continuous and integrable functions on the interval $[0,+\infty)$ and $K_1^{(n)} \leq K_3^{(n)}$ on $[0,+\infty)$. If $K_1^{(n)}$ and $K_2^{(n)}$ are increasing and $\frac{K_1^{(n)}}{K_3^{(n)}}$ is decreasing over $[0,+\infty)$,then for a convex function $\hat{H}$ such that $\hat{H}(0)=0$ then we have the following inequality:

$$\begin{aligned} &\check{D}_{0^+}^{\theta,\omega}\left[\hat{H}(K_3(\lambda))\right]\check{D}_{0^+}^{\theta,F}\left[K_1(\lambda)\right]+\check{D}_{0^+}^{\theta,\omega}\left[K_1(\lambda)\right]\check{D}_{0^+}^{\theta,F}\left[\hat{H}(K_3(\lambda))\right] \\ \geq\ &\check{D}_{0^+}^{\theta,F}\left[\hat{H}(K_1(\lambda))\right]\check{D}_{0^+}^{\theta,\omega}\left[K_3(\lambda)\right]+\check{D}_{0^+}^{\theta,\omega}\left[\hat{H}(K_1(\lambda))\right]\check{D}_{0^+}^{\theta,F}\left[K_3(\lambda)\right], \end{aligned} \tag{2.40}$$

where $\lambda \in [0,1]$.

Proof. Using the convexity of $\hat{H}$ and using the assumption $\hat{H}(0)=0$, the function $\frac{\hat{H}(K_1^{(n)}(x))}{x}$ is increasing. As the function $K_1^{(n)}$ is increasing then the function $\frac{\hat{H}(K_1^{(n)}(x))}{K_1^{(n)}(x)}$ is also increasing. It is obvious that the function $\frac{K_1^{(n)}}{K_3^{(n)}}$ is decreasing, then for all $u,v \in [0,+\infty)$, we have

$$\left(\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}-\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\right)\left(\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)}-\frac{K_1^{(n)}(u)}{K_3^{(n)}(u)}\right)\geq 0. \tag{2.41}$$

From (2.41), we have

$$\begin{aligned} &\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)}+\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\frac{K_1^{(n)}(u)}{K_3^{(n)}(u)} \\ \geq\ &\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}\frac{K_1^{(n)}(v)}{K_3^{(n)}(v)}+\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}\frac{K_1^{(n)}(u)}{K_3^{(n)}(u)}. \end{aligned} \tag{2.42}$$

On multiplying (2.42) by $K_3^{(n)}(u)K_3^{(n)}(v)$, we obtain

$$\begin{aligned} &\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}K_3^{(n)}(u)K_1^{(n)}(v)+\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}K_3^{(n)}(v)K_1^{(n)}(u) \\ \geq\ &\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)}K_1^{(n)}(v)K_3^{(n)}(u)+\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}K_1^{(n)}(u)K_3^{(n)}(v). \end{aligned} \tag{2.43}$$

Multiplying (2.43) by $\frac{1}{\Gamma(n-\theta)}F'(v)(F(\lambda)-F(v))^{n-\theta-1}$ then after integrating over $[0,\lambda]$ with respect to v, we obtain

$$\begin{aligned} &\frac{\hat{H}(K_1^{(n)}(u))}{K_1^{(n)}(u)}K_3^{(n)}(u)\ {}^{c}\check{D}_{0^+}^{\theta,F}\left[K_1(\lambda)\right]+{}^{c}\check{D}_{0^+}^{\theta,F}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right]K_1^{(n)}(u) \\ \geq\ &{}^{c}\check{D}_{0^+}^{\theta,F}\left[\hat{H}(K_1(\lambda))\right]K_3^{(n)}(u)+\hat{H}(K_1^{(n)}(u))\ {}^{c}\check{D}_{0^+}^{\theta,F}\left[K_3(\lambda)\right]. \end{aligned} \tag{2.44}$$

Similarly, multiplying (2.15) by $\frac{1}{\Gamma(n-\theta)}\omega^{'}(u)(\omega(\lambda)-\omega(u))^{n-\theta-1}$ and integrating over $[0^{+},\lambda]$ with respect to u, we obtain

$$\begin{aligned} &{}^{c}\check{D}_{0^+}^{\theta,\omega}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right]\ {}^{c}\check{D}_{0^+}^{\theta,F}\left[K_1(\lambda)\right]+\ {}^{c}\check{D}_{0^+}^{\theta,\omega}\left[K_1(\lambda)\right]\ {}^{c}\check{D}_{0^+}^{\theta,F}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right] \\ &\geq\ {}^{c}\check{D}_{0^+}^{\theta,F}\left[\hat{H}(K_1(\lambda))\right]\ {}^{c}\check{D}_{0^+}^{\theta,\omega}\left[K_3(\lambda)\right]+\ {}^{c}\check{D}_{0^+}^{F,\omega}\left[\hat{H}(K_1(\lambda))\right]\ {}^{c}\check{D}_{0^+}^{F,\varrho}\left[K_3(\lambda)\right]. \end{aligned} \tag{2.45}$$

Since $K_1^{(n)} \leq K_3^{(n)}$ and the properties of $\hat{H}$ it is easy to obtain that

$$\frac{\hat{H}(K_1^{(n)}(v))}{K_1^{(n)}(v)} \leq \frac{\hat{H}(K_3^{(n)}(v))}{K_3^{(n)}(v)}, v \in [0,+\infty). \tag{2.46}$$

Now from (2.46), we can easily obtain

$${}^{c}\check{D}_{0^+}^{\theta,\omega}\left[\frac{\hat{H}(K_1(\lambda))}{K_1(\lambda)}K_3(\lambda)\right] \leq\ {}^{c}\check{D}_{0^+}^{\theta,\omega}\left[\hat{H}(K_3(\lambda))\right]. \tag{2.47}$$

Utilizing (2.47) in (2.45), we obtain the required result. Q.E.D.

Remark 2.6. If we replace $\omega = F$ then we obtain 2.31 inequality.

3 Conclusion

The main objective of the present work was to establish new integral inequalities by using the Caputo left and right fractional derivatives in the Hilfer sense for convex functions. The results obtained allow us to expand the work in different directions. The same method can be used for other well-known inequalities such as Gruss and Chebishev.

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