Vertical rescaled Cheeger-Gromoll metric and harmonicity on the cotangent bundle

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Communicating editor - Managing editors' group

Abstract

In this paper, we introduce the harmonicity on the cotangent bundle equipped with vertical rescaled Cheeger-Gromoll metric which rescales the vertical part by a non-zero differentiable function f. We establish a necessary and sufficient condition under which a covector field is harmonic with respect to this metric. We also construct some examples of harmonic vector fields. Finally, we study the harmonicity of a map between a Riemannian manifold and a cotangent bundle of another Riemannian manifold and vice versa.

2010 Mathematics Subject Classification. **53A45**. 53C20, 58E20, 53C25.
Keywords. cotangent bundles, horizontal lift and vertical lift, vertical rescaled Cheeger-Gromoll metric, harmonic maps.

1 Introduction

In this field, one of the first works which deal with the cotangent bundle of a manifold as a Riemannian manifold is that of Patterson and Walker [10], who constructed a pseudo-Riemannian metric on the cotangent bundle, using a torsion-free linear connection on the base manifold. They called the pseudo-Riemannian metric as Riemann extension metric. A generalization of this metric had been given by Sekizawa [13]. He obtained the class of natural Riemann extensions which is a 2-parameter family of metrics. This metric had been intensively studied by many authors. In [19], Zayatuev introduced the rescaled Sasaki metric which rescale the horizontal part of the well known Sasaki metric on the tangent bundle over a Riemannian manifold (for Sasaki metric, see [12]). Wang and Wang called this metric the rescaled Sasaki metric and studied geodesics and some curvature properties for the rescaled Sasaki metric [14]. Inspired by the works on the tangent bundle, Salimov and Ağca [1, 11] searched the geometry of Sasaki and Cheeger-Gromoll type metrics on the cotangent bundle of a Riemannian manifold. Also, the second author and Altunbas [5] study the paraholomorphy of the rescaled Sasaki metric on the cotangent bundle. This is done by using some compatible paracomplex structures on the cotangent bundle. They also investigate some curvature properties of the cotangent bundle and construct a locally decomposable golden Riemannian structure on it. In addition, we refer to [4, 6, 7] for the rescaled Rienannian metrics on bundles.

The main idea in this note consists in the deformation (in the vertical part) of the Cheeger-Gromoll metric [1]. Firstly, we introduce the vertical rescaled Cheeger-Gromoll metric on the cotangent bundle T^*M over a Riemannian manifold (M, g) and we investigate the Levi-Civita connection of this metric (Theorem 3.4). Secondly, we study the harmonicity on the cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric, then we establish necessary and

Advanced Studies: Euro-Tbilisi Mathematical Journal 15(3) (2022), pp. 11–29.
DOI: 10.32513/asetmj/19322008221
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 27 August 2021.
Accepted for publication: 25 March 2022.

sufficient conditions under which a covector field is harmonic with respect to this metric (Theorem 4.4, Corollary 4.5 and Theorem 4.6). Next, we also construct some examples of harmonic covector fields. Finally, we study the harmonicity of the map $\sigma : (M, g) \longrightarrow (T^*N, \tilde{h})$ (Theorem 4.16 and Theorem 4.17) and the map $\Phi : (T^*M, \tilde{g}) \longrightarrow (N, h)$ (Theorem 4.20 and Theorem 4.22).

2 Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \rightarrow M$ the natural projection. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U),\, x^i,\, x^{\bar{i}} = p_i)_{i=\overline{1,m}, \bar{i}=m+i}$ on T^*M, where p_i is the component of covector p in each cotangent space T^*_xM, $x \in U$ with respect to the natural coframe dx^i. Let $C^\infty(M)$ (resp. $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M(resp. T^*M) and $\Im^r_s(M)$ (resp. $\Im^r_s(T^*M)$) be the module over $C^\infty(M)$ (resp. $C^\infty(T^*M)$) of C^∞ tensor fields of type (r, s).

Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. The Levi Civita connection ∇ defines a direct sum decomposition

$$TT^*M = VT^*M \oplus HT^*M. \tag{1}$$

of the tangent bundle to T^*M into vertical distribution $VT^*M = Ker(d\pi)$ and the horizontal distribution HT^*M.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be a local expressions in $U \subset M$ of a vector and covector field $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, respectively. Then the horizontal and the vertical lifts of X and ω are defined, respectively by

$$X^H = X^i \frac{\partial}{\partial x^i} + p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial p_i}, \tag{2}$$

$$\omega^V = \omega_i \frac{\partial}{\partial p_i} \tag{3}$$

with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i}\}$, where Γ^h_{ij} are components of the Levi-Civita connection ∇ on M. (see [15] for more details).

From(1), (2) and (3) we have

$$d\pi(\omega^V) = 0\ ,\ \ d\pi(X^H) = X \circ \pi.$$

If $\mathcal{P}$ be a local covector field constant on each fiber T^*_xM i.e ($\mathcal{P}_x = p \in T^*_xM$), the vertical lift $\mathcal{P}^V$ is called the canonical vertical vector field or Liouville vector field on T^*M.

Lemma 2.1. [15] Let (M, g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M of M satisfies the following

1. $[\omega^V, \theta^V] = 0$,
2. $[X^H, \theta^V] = (\nabla_X \theta)^V$,
3. $[X^H, Y^H] = [X, Y]^H - (pR(X, Y))^V$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$.

Let (M, g) be a Riemannian manifold, we define the map by

$$\begin{array}{rcl} \Im_1^0(M) & \to & \Im_0^1(M) \\ \omega & \mapsto & \tilde{\omega} \end{array}$$

for all $X \in \Im_0^1(M)$, $g(\tilde{\omega}, X) = \omega(X)$. Locally for all $\omega = \omega_i dx^i \in \Im_1^0(M)$, we have $\tilde{\omega} = g^{ij}\omega_i \frac{\partial}{\partial x^j}$, where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).

For each $x \in M$, the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $T_x^* M$ by $g^{-1}(\omega, \theta) = g(\tilde{\omega}, \tilde{\theta}) = g^{ij}\omega_i\theta_j$. In this case, we have $\tilde{\omega} = g^{-1} \circ \omega$. If ∇ be the Levi-Civita connection of (M, g) we have

$$\nabla_X \tilde{\omega} = \widetilde{\nabla_X \omega},$$

$$X g^{-1}(\omega, \theta) = g^{-1}(\nabla_X \omega, \theta) + g^{-1}(\omega, \nabla_X \theta),$$

for all $X \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$.

3 Vertical rescaled Cheeger-Gromoll metric

Definition 3.1. Let (M, g) be a Riemannian manifold and $f : M \to]0, +\infty[$ be a strictly positive smooth function on M. On the cotangent bundle T^*M of (M, g), we define a vertical rescaled Cheeger-Gromoll metric noted g^f by

$$\begin{array}{rcl} g^f(X^H, Y^H) & = & g(X, Y)^V = g(X, Y) \circ \pi, \\ g^f(X^H, \theta^V) & = & 0, \\ g^f(\omega^V, \theta^V) & = & \dfrac{f}{\alpha}(g^{-1}(\omega, \theta) + g^{-1}(\omega, p) g^{-1}(\theta, p)) \end{array} \tag{4}$$

for all $X, Y \in \Im_0^1(M)$, $\omega, \theta \in \Im_1^0(M)$, where $\alpha = 1 + \|p\|^2$ and $\|p\| = \sqrt{g^{-1}(p, p)}$ is the norm of p with respect to the metric g.

Note that, if $f = 1$, then g^f is the Cheeger-Gromoll metric [1].

Lemma 3.2. [16] Let (M^m, g) be a Riemannian manifold and $\rho : \mathbb{R} \to \mathbb{R}$ a smooth function, we have the followings

1. $X^H(\rho(r^2)) = 0$,
2. $\omega^V(\rho(r^2)) = 2\rho'(r^2) g^{-1}(\omega, p)$,
3. $X^H(g^{-1}(\theta, p)) = g^{-1}(\nabla_X \theta, p)$,
4. $\omega^V(g^{-1}(\theta, p)) = g^{-1}(\omega, \theta)$

for all $X \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$, $r^2 = g^{-1}(p, p)$.

Lemma 3.3. Let (M,g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric, we have the followings

$$\begin{aligned} (1)\ \ X^H g^f(\theta^V,\eta^V) &= \frac{1}{f}X(f)g^f(\theta^V,\eta^V) + g^f((\nabla_X\theta)^V,\eta^V) + g^f(\theta^V,(\nabla_X\eta)^V),\\ (2)\ \ \omega^V g^f(\theta^V,\eta^V) &= \frac{-2}{\alpha}g^{-1}(\omega,p)g^f(\theta^V,\eta^V) + \frac{1}{\alpha}g^{-1}(\omega,\theta)g^f(\eta^V,\mathcal{P}^V)\\ &\quad +\frac{1}{\alpha}g^{-1}(\omega,\eta)g^f(\theta^V,\mathcal{P}^V) \end{aligned}$$

for all $X \in \Im_0^1(M)$ and $\omega,\theta,\eta \in \Im_1^0(M)$, where $\mathcal{P}^V$ is the canonical vertical vector field on T^*M.

Proof. From Lemma 3.2, we have

$$\begin{aligned} (1)\ \ X^H g^f(\theta^V,\eta^V) &= X^H\big(\frac{f}{\alpha}(g^{-1}(\theta,\eta) + g^{-1}(\theta,p)g^{-1}(\eta,p))\big)\\ &= X(f)\frac{1}{\alpha}(g^{-1}(\theta,\eta) + g^{-1}(\theta,p)g^{-1}(\eta,p))\\ &\quad +\frac{f}{\alpha}(g^{-1}(\nabla_X\theta,\eta) + g^{-1}(\theta,\nabla_X\eta)\\ &\quad +g^{-1}(\nabla_X\theta,p)g^{-1}(\eta,p)) + g^{-1}(\theta,p)g^{-1}(\nabla_X\eta,p)\\ &= \frac{1}{f}X(f)g^f(\theta^V,\eta^V) + g^f((\nabla_X\theta)^V,\eta^V) + g^f(\theta^V,(\nabla_X\eta)^V). \end{aligned}$$

$$\begin{aligned} (2)\ \ \omega^V g^f(\theta^V,\eta^V) &= \omega^V\big(\frac{f}{\alpha}(g^{-1}(\theta,\eta) + g^{-1}(\theta,p)g^{-1}(\eta,p))\big)\\ &= \omega^V(\frac{f}{\alpha})(g^{-1}(\theta,\eta) + g^{-1}(\theta,p)g^{-1}(\eta,p))\\ &\quad +\frac{f}{\alpha}\omega^V(g^{-1}(\theta,\eta) + g^{-1}(\theta,p)g^{-1}(\eta,p))\\ &= -\frac{2f}{\alpha^2}g^{-1}(\omega,p)(g^{-1}(\theta,\eta) + g^{-1}(\theta,p)g^{-1}(\eta,p))\\ &\quad +\frac{f}{\alpha}(g^{-1}(\omega,\theta)g^{-1}(\eta,p) + g^{-1}(\theta,p))g^{-1}(\omega,\eta))\\ &= -\frac{2}{\alpha}g^{-1}(\omega,p)g^f(\theta^V,\eta^V) + \frac{1}{\alpha}g^{-1}(\omega,\theta)g^f(\eta^V,\mathcal{P}^V)\\ &\quad +\frac{1}{\alpha}g^{-1}(\omega,\eta)g^f(\theta^V,\mathcal{P}^V). \end{aligned}$$

∎

Theorem 3.4. Let (M,g) be a Riemannian manifold and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. If ∇ (resp ∇^f) denote the Levi-Civita connection of (M,g) (resp (T^*M,g^f)), we have

$$\begin{aligned}
(1)\ (\nabla^f_{X^H}Y^H)_\xi &= (\nabla_X Y)^H_\xi + \frac{1}{2}(pR_x(X,Y))^V,\\
(2)\ (\nabla^f_{X^H}\theta^V)_\xi &= (\nabla_X \theta)^V_\xi + \frac{1}{2f(x)}X_x(f)\theta^V_\xi + \frac{f(x)}{2\alpha}(R_x(\tilde{p},\tilde{\theta})X)^H,\\
(3)\ (\nabla^f_{\omega^V}Y^H)_\xi &= \frac{1}{2f(x)}Y_x(f)\omega^V_x + \frac{f(x)}{2\alpha}(R_x(\tilde{p},\tilde{\omega})Y)^H,\\
(4)\ (\nabla^f_{\omega^V}\theta^V)_\xi &= -\frac{1}{2f(x)}g^f_\xi(\omega^V,\theta^V)(grad\, f)^H_\xi\\
&\quad -\frac{1}{\alpha f(x)}\big[g^f_\xi(\omega^V,\mathcal{P}^V)\theta^V_\xi + g^f_\xi(\theta^V,\mathcal{P}^V)\omega^V_\xi\big]\\
&\quad +\big[\frac{\alpha+1}{\alpha f(x)}g^f_\xi(\omega^V,\theta^V) - \frac{1}{\alpha f^2(x)}g^f_\xi(\omega^V,\mathcal{P}^V)g^f_\xi(\theta^V,\mathcal{P}^V)\big]\mathcal{P}^V_\xi
\end{aligned}$$

for all $\xi=(x,p)\in T^*M$, $X,Y\in\Im^1_0(M)$ and $\omega,\theta\in\Im^0_1(M)$, where R denotes the Riemannian curvature tensor of (M,g).

Proof. The proof of Theorem 3.4 follows directly from Koszul formula and Lemma 3.3. ∎

4 Vertical rescaled Cheeger-Gromoll metric and Harmonicity.

Consider a smooth map $\phi:(M^m,g)\to(N^n,h)$ between two Riemannian manifolds, then the second fundamental form of ϕ is defined by

$$(\nabla d\phi)(X,Y)=\nabla^\phi_X d\phi(Y)-d\phi(\nabla_X Y). \tag{5}$$

Here ∇ is the Levi-Civita connection on M and ∇^ϕ is the pull-back connection on the pull-back bundle $\phi^{-1}TN$, and

$$\tau(\phi)=trace_g\nabla d\phi \tag{6}$$

is the tension field of ϕ.

The energy functional of ϕ is defined by

$$E(\phi)=\int_K e(\phi)dv_g \tag{7}$$

such that K is any compact of M, where

$$e(\phi)=\frac{1}{2}trace_g h(d\phi,d\phi) \tag{8}$$

is the energy density of ϕ.

A map is called harmonic if it is a critical point of the energy functional E. For any smooth variation $\{\phi_t\}_{t\in I}$ of ϕ with $\phi_0=\phi$ and $V=\frac{d}{dt}\phi_t\Big|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0}=-\int_K h(\tau(\phi),V)dv_g. \tag{9}$$

Then ϕ is harmonic if and only if $\tau(\phi)=0$.

One can refer to [2, 3, 8] for background on harmonic maps.

4.1 Harmonic sections $\omega : (M,g) \longrightarrow (T^*M, g^f)$

Lemma 4.1. [16, 17] Let (M,g) be a Riemannian manifold. If $\omega \in \Im_1^0(M)$ is a covector field (1-form) on M and $\xi = (x,p) \in T^*M$ such that $\omega_x = p$, then we have

$$d_x\omega(X_x) = X_\xi^H + (\nabla_X \omega)_\xi^V,$$

where $X \in \Im_0^1(M)$ and ∇ denote the Levi-Civita connection of (M,g).

Lemma 4.2. Let (M^m,g) be a Riemannian $m-$dimensional manifold and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. If $\omega \in \Im_1^0(M)$, then the energy density associated to ω is given by

$$e(\omega) = \frac{m}{2} + \frac{f}{2\alpha} trace_g \big[g^{-1}(\nabla_*\omega, \nabla_*\omega) + g^{-1}(\nabla_*\omega, \omega)^2 \big].$$

Proof. Let $\xi = (x,p) \in T^*M$, $\omega \in \Im_1^0(M)$, $\omega_x = p$ and $(E_1, \cdots, E_m)$ be a local orthonormal frame on M, then:

$$\begin{aligned} e(\omega)_x &= \frac{1}{2} trace_g g^f(d\omega, d\omega)_\xi \\ &= \frac{1}{2}\sum_{i=1}^m g^f(d\omega(E_i), d\omega(E_i))_\xi. \end{aligned}$$

Using Lemma 4.1, we obtain

$$\begin{aligned} e(\omega) &= \frac{1}{2}\sum_{i=1}^m g^f(E_i^H + (\nabla_{E_i}\omega)^V, E_i^H + (\nabla_{E_i}\omega)^V) \\ &= \frac{1}{2}\sum_{i=1}^m \big\{ g^f(E_i^H, E_i^H) + g^f((\nabla_{E_i}\omega)^V, (\nabla_{E_i}\omega)^V) \big\} \\ &= \frac{1}{2}\sum_{i=1}^m \big\{ g(E_i, E_i) + \frac{f}{2\alpha}\big[g^{-1}(\nabla_{E_i}\omega, \nabla_{E_i}\omega) + g^{-1}(\nabla_{E_i}\omega, \omega)^2 \big] \big\} \\ &= \frac{m}{2} + \frac{f}{2\alpha} trace_g \big[g^{-1}(\nabla_*\omega, \nabla_*\omega) + g^{-1}(\nabla_*\omega, \omega)^2 \big]. \end{aligned}$$

∎

Theorem 4.3. Let (M^m,g) be a Riemannian $m-$dimensional manifold and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. If $\omega \in \Im_1^0(M)$, then the tension field associated to ω is given by

$$\tau(\omega) = \big[trace_g A(\omega) \big]^H + \big[trace_g B(\omega) \big]^V,$$

where $A(\omega)$ and $B(\omega)$ are bilinear maps defined by

$$\begin{aligned} A(\omega) &= \frac{f}{\alpha} R(\tilde{\omega}, \widetilde{\nabla_*\omega}) * - \frac{1}{2\alpha}\big[g^{-1}(\nabla_*\omega, \nabla_*\omega) + g^{-1}(\nabla_*\omega, \omega)^2 \big] grad\, f, \\ B(\omega) &= \nabla_*^2\omega + \big[\frac{1}{f} df(*) - \frac{2}{\alpha} g^{-1}(\nabla_*\omega, \omega) \big] \nabla_*\omega \\ &\quad + \big[\frac{\alpha+1}{\alpha^2} g^{-1}(\nabla_*\omega, \nabla_*\omega) + \frac{1}{\alpha^2} g^{-1}(\nabla_*\omega, \omega)^2 \big] \omega, \end{aligned}$$

$\nabla^2_*\omega = \nabla_*\nabla_*\omega - \nabla_{\nabla_* *}\omega$ and $\alpha = 1 + \|\omega\|^2 = 1 + g^{-1}(\omega, \omega)$.

Proof. Let $\xi = (x, p) \in T^*M$, $\omega \in \Im^0_1(M)$, $\omega_x = p$ and $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M such that $(\nabla_{E_i} E_i)_x = 0$, then

$$\begin{aligned}
\tau(\omega)_x &= trace_g(\nabla d\omega)_x \\
&= \sum_{i=1}^{m} \{\nabla^\omega_{E_i} d\omega(E_i) - d\omega(\nabla_{E_i} E_i)\}_x \\
&= \sum_{i=1}^{m} \{\nabla^f_{d\omega(E_i)} d\omega(E_i)\}_\xi,
\end{aligned}$$

where ∇^ω is the pull-back connection. Using Lemma 4.1, we obtain

$$\begin{aligned}
\tau(\omega)_x &= \sum_{i=1}^{m} \{\nabla^f_{(E_i^H + (\nabla_{E_i}\omega)^V)} (E_i^H + (\nabla_{E_i}\omega)^V)\}_\xi \\
&= \sum_{i=1}^{m} \{\nabla^f_{E_i^H} E_i^H + {}^f\nabla_{E_i^H} (\nabla_{E_i}\omega)^V + \nabla^f_{(\nabla_{E_i}\omega)^V} (E_i)^H \\
&\quad + \nabla^f_{(\nabla_{E_i}\omega)^V} (\nabla_{E_i}\omega)^V\}_\xi.
\end{aligned}$$

Using Theorem 3.4, we obtain

$$\begin{aligned}
\tau(\omega) &= \sum_{i=1}^{m} \Big[(\nabla_{E_i} E_i)^H + \frac{1}{2}(\omega R(E_i, E_i))^V + (\nabla_{E_i}\nabla_{E_i}\omega)^V \\
&\quad + \frac{1}{2f} E_i(f)(\nabla_{E_i}\omega)^V + \frac{f}{2\alpha}(R(\tilde{\omega}, \widetilde{\nabla_{E_i}\omega})E_i)^H + \frac{1}{2f} E_i(f)(\nabla_{E_i}\omega)^V \\
&\quad + \frac{f}{2\alpha}(R(\tilde{\omega}, \widetilde{\nabla_{E_i}\omega})E_i)^H - \frac{1}{2f} g^f((\nabla_{E_i}\omega)^V, (\nabla_{E_i}\omega)^V)(grad\, f)^H \\
&\quad - \frac{1}{\alpha f}\big[g^f((\nabla_{E_i}\omega)^V, \omega^V)(\nabla_{E_i}\omega)^V + g^f((\nabla_{E_i}\omega)^V, \omega^V)(\nabla_{E_i}\omega)^V \big] \\
&\quad + \frac{\alpha + 1}{\alpha f} g^f((\nabla_{E_i}\omega)^V, (\nabla_{E_i}\omega)^V)\omega^V \\
&\quad - \frac{1}{\alpha f^2} g^f((\nabla_{E_i}\omega)^V, \omega^V) g^f((\nabla_{E_i}\omega)^V, \omega^V)\omega^V \Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\Big[\frac{f}{\alpha}(R(\tilde{\omega},\widetilde{\nabla_{E_i}\omega})E_i)^H - \frac{1}{2f}g^f((\nabla_{E_i}\omega)^V,(\nabla_{E_i}\omega)^V)(grad\, f)^H \\
&\quad +(\nabla_{E_i}\nabla_{E_i}\omega)^V + \frac{1}{f}E_i(f)(\nabla_{E_i}\omega)^V - \frac{2}{\alpha f}g^f((\nabla_{E_i}\omega)^V,\omega^V)(\nabla_{E_i}\omega)^V \\
&\quad +\big[\frac{\alpha+1}{\alpha f}g^f((\nabla_{E_i}\omega)^V,(\nabla_{E_i}\omega)^V) - \frac{1}{\alpha f^2}g^f((\nabla_{E_i}\omega)^V,\omega^V)^2\big]\omega^V\Big] \\
&= \sum_{i=1}^{m}\Big[\frac{f}{\alpha}(R(\tilde{\omega},\widetilde{\nabla_{E_i}\omega})E_i)^H \\
&\quad -\frac{1}{2\alpha}\big[g^{-1}(\nabla_{E_i}\omega,\nabla_{E_i}\omega)+g^{-1}(\nabla_{E_i}\omega,\omega)^2\big](grad\, f)^H \\
&\quad +(\nabla_{E_i}\nabla_{E_i}\omega)^V + \big[\frac{1}{f}df(E_i) - \frac{2}{\alpha}g^{-1}(\nabla_{E_i}\omega,\omega)\big](\nabla_{E_i}\omega)^V \\
&\quad +\big[\frac{\alpha+1}{\alpha^2}g^{-1}(\nabla_{E_i}\omega,\nabla_{E_i}\omega) + \frac{1}{\alpha^2}g^{-1}(\nabla_{E_i}\omega,\omega)^2\big]\omega^V\Big] \\
&= \Big[trace_g\big(\frac{f}{\alpha}R(\tilde{\omega},\widetilde{\nabla_*\omega})* - \frac{1}{2\alpha}\big[g^{-1}(\nabla_*\omega,\nabla_*\omega)+g^{-1}(\nabla_*\omega,\omega)^2\big]grad\, f\big)\Big]^H \\
&\quad +\Big[trace_g\big(\nabla_*^2\omega + \big[\frac{1}{f}df(*) - \frac{2}{\alpha}g^{-1}(\nabla_*\omega,\omega)\big]\nabla_*\omega \\
&\quad +\big[\frac{\alpha+1}{\alpha^2}g^{-1}(\nabla_*\omega,\nabla_*\omega) + \frac{1}{\alpha^2}g^{-1}(\nabla_*\omega,\omega)^2\big]\omega\big)\Big]^V.
\end{aligned}
$$

∎

Theorem 4.4. Let (M^m, g) be a Riemannian m-dimensional manifold and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. If $\omega \in \Im_1^0(M)$, then ω is harmonic covector field if and only if the following conditions are verified

$$
\begin{aligned}
&trace_g\big(\frac{f}{\alpha}R(\tilde{\omega},\widetilde{\nabla_*\omega})* - \frac{1}{2\alpha}\big[g^{-1}(\nabla_*\omega,\nabla_*\omega)+ \\
&\qquad g^{-1}(\nabla_*\omega,\omega)^2\big]grad\, f\big) = 0
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
&trace_g\big(\nabla_*^2\omega + \big[\frac{1}{f}df(*) - \frac{2}{\alpha}g^{-1}(\nabla_*\omega,\omega)\big]\nabla_*\omega + \\
&\big[\frac{\alpha+1}{\alpha^2}g^{-1}(\nabla_*\omega,\nabla_*\omega) + \frac{1}{\alpha^2}g^{-1}(\nabla_*\omega,\omega)^2\big]\omega\big) \\
&= 0,
\end{aligned} \tag{11}
$$

where $\alpha = 1 + \|\omega\|^2 = 1 + g^{-1}(\omega,\omega)$.

Proof. The proof of Theorem 4.4 follows directly of Theorem 4.3. ∎

Corollary 4.5. Let (M^m, g) be a Riemannian $m-$dimensional manifold and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. If $\omega \in \Im^1_0(M)$ is a parallel covector field (i.e $\nabla_*\omega = 0$) then ω is harmonic.

Theorem 4.6. Let (M^m, g) be a Riemannian compact m-dimensional manifold and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. If $\omega \in \Im^0_1(M)$, then ω is harmonic covector field if and only if ω is parallel.

Proof. If ω is parallel from Corollary 4.5, we deduce that ω is harmonic covector field.
Inversely, let φ_t be a compactly supported variation of ω defined by

$$\begin{array}{rcl} \mathbb{R}\times M & \longrightarrow & T^*_xM \\ (t,x) & \longmapsto & \varphi_t(x) = (1+t)\omega_x. \end{array}$$

From lemma 4.2 we have

$$e(\varphi_t) = \frac{m}{2} + \frac{(1+t)^2}{2}\frac{f}{\alpha}trace_g g^{-1}(\nabla_*\omega, \nabla_*\omega) + \frac{(1+t)^4}{2}\frac{f}{\alpha}trace_g g^{-1}(\nabla_*\omega, \omega)^2,$$

$$\begin{aligned} E(\varphi_t) &= \frac{m}{2}Vol(M) + \frac{(1+t)^2}{2}\int_M \frac{f}{\alpha}trace_g g^{-1}(\nabla_*\omega, \nabla_*\omega)dv_g \\ &\quad + \frac{(1+t)^4}{2}\int_M \frac{f}{\alpha}trace_g g^{-1}(\nabla_*\omega, \omega)^2 dv_g. \end{aligned}$$

If ω is a critical point of the energy functional, then we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}E(\varphi_t)|_{t=0} \\ &= \frac{\partial}{\partial t}\Big[\frac{m}{2}Vol(M)\Big]_{t=0} + \frac{\partial}{\partial t}\Big[\frac{(1+t)^2}{2}\int_M \frac{f}{\alpha}trace_g g^{-1}(\nabla_*\omega, \nabla_*\omega)dv_g\Big]_{t=0} \\ &\quad + \frac{\partial}{\partial t}\Big[\frac{(1+t)^4}{2}\int_M \frac{f}{\alpha}trace_g g^{-1}(\nabla_*\omega, \omega)^2 dv_g\Big]_{t=0} \\ &= \int_M \frac{f}{\alpha}trace_g g^{-1}(\nabla_*\omega, \nabla_*\omega)dv_g + 2\int_M \frac{f}{\alpha}trace_g g^{-1}(\nabla_*\omega, \omega)^2 dv_g \\ &= \int_M \frac{f}{\alpha}trace_g\big(g^{-1}(\nabla_*\omega, \nabla_*\omega) + 2g^{-1}(\nabla_*\omega, \omega)^2\big)dv_g \end{aligned}$$

which gives

$$g^{-1}(\nabla_*\omega, \nabla_*\omega) + 2g^{-1}(\nabla_*\omega, \omega)^2 = 0,$$

hence $\nabla_*\omega = 0$. ∎

Example 4.7. Let $\mathbb{R}^3$ be equipped with the Riemannian metric in cylindrical coordinates defined by

$$g = dr^2 + r^2 d\theta^2 + dt^2.$$

The non-null Christoffel symbols of the Riemannian connection are

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},\ \Gamma^1_{22} = -r,$$

then we have

$$\nabla_{\frac{\partial}{\partial r}} dr = 0,\ \nabla_{\frac{\partial}{\partial r}} d\theta = -\frac{1}{r} d\theta,\ \nabla_{\frac{\partial}{\partial r}} dt = 0,\ \nabla_{\frac{\partial}{\partial \theta}} dr = r d\theta,\ \nabla_{\frac{\partial}{\partial \theta}} d\theta = -\frac{1}{r} dr,$$
$$\nabla_{\frac{\partial}{\partial \theta}} dt = 0,\ \nabla_{\frac{\partial}{\partial t}} dr = 0,\ \nabla_{\frac{\partial}{\partial t}} d\theta = 0,\ \nabla_{\frac{\partial}{\partial t}} dt = 0.$$

The covector field $\omega = \cos\theta dr - r\sin\theta d\theta + dt$ is harmonic because ω is parallel, indeed,

$$\nabla_{\frac{\partial}{\partial r}} \omega = \cos\theta \nabla_{\frac{\partial}{\partial r}} dr - \sin\theta d\theta - r\sin\theta \nabla_{\frac{\partial}{\partial r}} d\theta + \nabla_{\frac{\partial}{\partial r}} dt = 0,$$
$$\nabla_{\frac{\partial}{\partial \theta}} \omega = -\sin\theta dr + \cos\theta \nabla_{\frac{\partial}{\partial \theta}} dr - r\cos\theta d\theta - r\sin\theta \nabla_{\frac{\partial}{\partial \theta}} d\theta + \nabla_{\frac{\partial}{\partial \theta}} dt = 0,$$
$$\nabla_{\frac{\partial}{\partial t}} \omega = \cos\theta \nabla_{\frac{\partial}{\partial t}} dr - r\sin\theta \nabla_{\frac{\partial}{\partial t}} d\theta + \nabla_{\frac{\partial}{\partial t}} dt = 0,$$

i.e., $\nabla_* \omega = 0$, then ω is harmonic.

Example 4.8. Let $\mathbb{R}$ (Riemannian manifold) be equipped with the metric

$$g_{\mathbb{R}} = e^x dx^2.$$

The Christoffel symbols of the Levi-Civita connection are given by

$$\Gamma^1_{11} = \frac{1}{2} g^{11} (\frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{11}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_1}) = \frac{1}{2}.$$

The covector field $\omega = f(x)dx$, $f \in \mathcal{C}^\infty(\mathbb{R})$ is harmonic if ω is parallel,

$$\begin{aligned} \nabla_* \omega = 0 \quad &\Leftrightarrow \quad f'(x) - \frac{1}{2} f(x) = 0 \\ &\Leftrightarrow \quad f(x) = k\exp(\frac{x}{2})\ ,\ k \in \mathbb{R} \\ &\Leftrightarrow \quad \omega = k\exp(\frac{x}{2})dx\ ,\ k \in \mathbb{R}. \end{aligned}$$

Remark 4.9. In general, using Corollary 4.5 and Theorem 4.6, we can construct many examples for harmonic covector fields.

Theorem 4.10. Let $(\mathbb{R}^m, g_0)$ be the real Euclidean space and $(T^*\mathbb{R}^m, g_0^f)$ its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric.
If $\omega = (\omega_1, \cdots, \omega_m) \in \Im^0_1(\mathbb{R}^m)$, then ω is a harmonic covector field if and only if the following conditions are verified

$$\omega = constant \ \ or \ \ f = constant,$$

$$\sum_{i=1}^m \big[\frac{\partial^2 \omega_k}{\partial (x^i)^2} + \frac{1}{f}\frac{\partial f}{\partial x^i}\frac{\partial \omega_k}{\partial x^i}\big] - \frac{2}{1+\|\omega\|^2} \sum_{i,j=1}^m (\frac{\partial \omega_j}{\partial x^i})\omega_j(\frac{\partial \omega_k}{\partial x^i})$$
$$+ \sum_{i,j=1}^m (\frac{\partial \omega_j}{\partial x^i})^2 \big[\frac{2+\|\omega\|^2}{(1+\|\omega\|^2)^2} + \frac{1}{(1+\|\omega\|^2)^2}\omega_j^2\big]\omega_k = 0 \qquad (12)$$

for all $k = \overline{1, m}$. where $\|\omega\|^2 = g^{-1}(\omega, \omega)$.

Proof. Let $\{\frac{\partial}{\partial x^i}\}_{i=\overline{1,m}}$ be a canonical frame on $\mathbb{R}^m$. Using Theorem 4.4, we have $\tau(\omega)=0$ equivalent the following conditions (10) and (11) are verified

$$
\begin{aligned}
(10)\quad \Leftrightarrow\quad & trace_g\Big(-\frac{1}{2\alpha}\big[g^{-1}(\nabla_*\omega,\nabla_*\omega)+g^{-1}(\nabla_*\omega,\omega)^2\big]grad\, f\Big)=0\\
\Leftrightarrow\quad & \sum_{i=1}^m g^{-1}\big[(\nabla_{\frac{\partial}{\partial x^i}}\omega,\nabla_{\frac{\partial}{\partial x^i}}\omega)+g^{-1}(\nabla_{\frac{\partial}{\partial x^i}}\omega,\omega)^2\big]=0\ \ or\ \ grad\, f=0\\
\Leftrightarrow\quad & \sum_{i,j=1}^m\big[(\frac{\partial\omega_j}{\partial x^i})^2+(\frac{\partial\omega_j}{\partial x^i})^2\omega_j^2\big]=0\ \ or\ \ f=constant\\
\Leftrightarrow\quad & \sum_{i,j=1}^m(\frac{\partial\omega_j}{\partial x^i})^2\big[1+\omega_j^2\big]=0\ \ or\ \ f=constant\\
\Leftrightarrow\quad & \frac{\partial\omega_j}{\partial x^i}=0\ ,forall\ i,j=\overline{1,m}\ \ or\ \ f=constant\\
\Leftrightarrow\quad & \omega=constant\ \ or\ \ f=constant.
\end{aligned}
$$

$$
\begin{aligned}
(11)\quad \Leftrightarrow\quad & trace_g\big(\nabla_*^2\omega+\big[\frac{1}{f}df(*)-\frac{2}{\alpha}g^{-1}(\nabla_*\omega,\omega)\big]\nabla_*\omega\\
& +\big[\frac{\alpha+1}{\alpha^2}g^{-1}(\nabla_*\omega,\nabla_*\omega)+\frac{1}{\alpha^2}g^{-1}(\nabla_*\omega,\omega)^2\big]\omega\big)=0\\
\Leftrightarrow\quad & \sum_{i=1}^m\Big\{\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^i}}\omega+\big[\frac{1}{f}df(\frac{\partial}{\partial x^i})-\frac{2}{\alpha}g^{-1}(\nabla_{\frac{\partial}{\partial x^i}}\omega,\omega)\big]\nabla_{\frac{\partial}{\partial x^i}}\omega\\
& +\big[\frac{\alpha+1}{\alpha^2}g^{-1}(\nabla_{\frac{\partial}{\partial x^i}}\omega,\nabla_{\frac{\partial}{\partial x^i}}\omega)+\frac{1}{\alpha^2}g^{-1}(\nabla_{\frac{\partial}{\partial x^i}}\omega,\omega)^2\big]\omega\Big\}=0\\
\Leftrightarrow\quad & \sum_{i=1}^m\Big\{\sum_{k=1}^m\Big(\frac{\partial^2\omega_k}{\partial(x^i)^2}dx^k\Big)+\big[\frac{1}{f}\frac{\partial f}{\partial x^i}-\frac{2}{\alpha}\sum_{j=1}^m(\frac{\partial\omega_j}{\partial x^i})\omega_j\big]\sum_{k=1}^m\frac{\partial\omega_k}{\partial x^i}dx^k\\
& +\big[\frac{\alpha+1}{\alpha^2}\sum_{j=1}^m(\frac{\partial\omega_j}{\partial x^i})^2+\frac{1}{\alpha^2}\sum_{j=1}^m(\frac{\partial\omega_j}{\partial x^i})^2\omega_j^2\big]\sum_{k=1}^m\omega_k dx^k\Big\}=0\\
\Leftrightarrow\quad & \sum_{i=1}^m\big[\frac{\partial^2\omega_k}{\partial(x^i)^2}+\frac{1}{f}\frac{\partial f}{\partial x^i}\frac{\partial\omega_k}{\partial x^i}\big]-\frac{2}{\alpha}\sum_{i,j=1}^m(\frac{\partial\omega_j}{\partial x^i})\omega_j(\frac{\partial\omega_k}{\partial x^i})\\
& +\sum_{i,j=1}^m(\frac{\partial\omega_j}{\partial x^i})^2\big[\frac{\alpha+1}{\alpha^2}+\frac{1}{\alpha^2}\omega_j^2\big]\omega_k=0\\
\Leftrightarrow\quad & \sum_{i=1}^m\big[\frac{\partial^2\omega_k}{\partial(x^i)^2}+\frac{1}{f}\frac{\partial f}{\partial x^i}\frac{\partial\omega_k}{\partial x^i}\big]-\frac{2}{1+\|\omega\|^2}\sum_{i,j=1}^m(\frac{\partial\omega_j}{\partial x^i})\omega_j(\frac{\partial\omega_k}{\partial x^i})\\
& +\sum_{i,j=1}^m(\frac{\partial\omega_j}{\partial x^i})^2\big[\frac{2+\|\omega\|^2}{(1+\|\omega\|^2)^2}+\frac{1}{(1+\|\omega\|^2)^2}\omega_j^2\big]\omega_k=0
\end{aligned}
$$

for all $k=\overline{1,m}$. ∎

From Theorem 4.10 we deduce

Corollary 4.11. Let $(\mathbb{R}^m, g_0)$ be the real Euclidean space and $(T^*\mathbb{R}^m, g_0^f)$ its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. If $f \neq$ *constant*, then $\omega = (\omega_1, \cdots, \omega_m) \in \Im_1^0(\mathbb{R}^m)$ is harmonic if and only if ω is constant.

Corollary 4.12. Let $(\mathbb{R}^m, g_0)$ be the real Euclidean space and $(T^*\mathbb{R}^m, g_0^f)$ its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. If f is a constant function then $\omega = (\omega_1, \cdots, \omega_m) \in \Im_1^0(\mathbb{R}^m)$ is a harmonic covector field if and only if ω verifies the following system of equations

$$\sum_{i=1}^{m}(\frac{\partial^2 \omega_k}{\partial (x^i)^2}) - \frac{2}{1+\|\omega\|^2}\sum_{i,j=1}^{m}(\frac{\partial \omega_j}{\partial x^i})\omega_j(\frac{\partial \omega_k}{\partial x^i})$$
$$+\sum_{i,j=1}^{m}(\frac{\partial \omega_j}{\partial x^i})^2\big[\frac{2+\|\omega\|^2}{(1+\|\omega\|^2)^2} + \frac{1}{(1+\|\omega\|^2)^2}\omega_j^2\big]\omega_k = 0 \tag{13}$$

for all $k = \overline{1, m}$. where $\|\omega\|^2 = g^{-1}(\omega, \omega)$.

Remark 4.13. Using Theorem 4.10, we can construct many examples of non trivial harmonic vector fields.

Example 4.14. If $\mathbb{R}^m$ is endowed with the canonical metric and $T^*\mathbb{R}^m$ its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric such as f is a constant function. From Corollary 4.12, we deduce that
If $\omega = (y(x), 0, \cdots, 0) \in \Im_1^0(\mathbb{R}^m)$ is a harmonic covector field if and only if the function y is solution of differential equation $y'' = 0$, i.e., $y(x) = ax + b$, where $x, a, b \in \mathbb{R}$.

4.2 Harmonicity of the map $\sigma : (M, g) \longrightarrow (T^*N, h^f)$

Lemma 4.15. [17, 18] Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds and let $\sigma : M \longrightarrow T^*N$ be a smooth map such that $\varphi = \pi_N \circ \sigma$, where $\pi_N : T^*N \to N$ is the canonical projection, then

$$d\sigma(X) = (d\varphi(X))^H + (\nabla_X^{\varphi}\sigma)^V$$

for all $X \in \Im_0^1(M)$, where ∇^{φ} is the pull-back connection.

Theorem 4.16. Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds, $f : N \to]0, +\infty[$ be a strictly positive smooth function on N and (T^*N, h^f) the cotangent bundle of N equipped with the vertical rescaled Cheeger-Gromoll metric. If $\sigma : M \longrightarrow T^*N$ is a smooth map such that $\varphi = \pi_N \circ \sigma$, then the tension field of σ is given by

$$\tau(\sigma) = \big[\tau(\varphi) + trace_g A(\sigma)\big]^H + \big[trace_g B(\sigma)\big]^V,$$

where $A(\sigma)$ and $B(\sigma)$ are a bilinear maps defined by

$$\begin{aligned}
A(\sigma) &= \frac{f}{\alpha}R^N(\tilde{\sigma}, \widetilde{\nabla_*^{\varphi}\sigma})d\varphi(*) - \frac{1}{2\alpha}\big[h^{-1}(\nabla_*^{\varphi}\sigma, \nabla_*^{\varphi}\sigma) + h^{-1}(\nabla_*^{\varphi}\sigma, \sigma)^2\big]grad\, f,\\
B(\sigma) &= (\nabla_*^{\varphi})^2\sigma + \big[\frac{1}{f}df(d\varphi(*)) - \frac{2}{\alpha}h^{-1}(\nabla_*^{\varphi}\sigma, \sigma)\big]\nabla_*^{\varphi}\sigma\\
&\quad + \big[\frac{1+\alpha}{\alpha^2}h^{-1}(\nabla_*^{\varphi}\sigma, \nabla_*^{\varphi}\sigma) + \frac{1}{\alpha^2}h^{-1}(\nabla_*^{\varphi}\sigma, \sigma)^2\big]\sigma,
\end{aligned}$$

$(\nabla^{\varphi}_{*})^2\sigma = \nabla^{\varphi}_{*}\nabla^{\varphi}_{*}\sigma - \nabla^{\varphi}_{\nabla_{*}*}\sigma$ and $\alpha = 1 + \|\sigma\|^2 = 1 + h^{-1}(\sigma,\sigma)$.

Proof. Let $x \in M$ and $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M such that $(\nabla_{E_i}E_i)_x = 0$ and $\sigma(x) = (\varphi(x), q) \in T^*N$. Using Lemma 4.15, we obtain

$$\begin{aligned}
\tau(\sigma)_x &= trace_g(\nabla d\sigma)_x \\
&= \sum_{i=1}^{m}\{\nabla^{\sigma}_{E_i}d\sigma(E_i) - d\sigma(\nabla_{E_i}E_i)\}_x \\
&= \sum_{i=1}^{m}\{\nabla^{T^*N}_{d\sigma(E_i)}d\sigma(E_i)\}_{(\varphi(x),q)} \\
&= \sum_{i=1}^{m}\{\nabla^{T^*N}_{(d\varphi(E_i))^H}(d\varphi(E_i))^H + \nabla^{T^*N}_{(d\varphi(E_i))^H}(\nabla^{\varphi}_{E_i}\sigma)^V \\
&\quad + \nabla^{T^*N}_{(\nabla^{\varphi}_{E_i}\sigma)^V}(d\varphi(E_i))^H + \nabla^{T^*N}_{(\nabla^{\varphi}_{E_i}\sigma)^V}(\nabla^{\varphi}_{E_i}\sigma)^V\}_{(\varphi(x),q)}.
\end{aligned}$$

From Theorem 3.4, we get

$$\begin{aligned}
\tau(\sigma) = \sum_{i=1}^{m}\Big[&(\nabla^N_{d\varphi(E_i)}d\varphi(E_i))^H + \frac{1}{2}(\sigma R^N(d\varphi(E_i), d\varphi(E_i)))^V \\
&+ \frac{f}{2\alpha}(R^N(\tilde{\sigma}, \widetilde{\nabla^{\varphi}\sigma})d\varphi(E_i))^H + (\nabla^N_{d\varphi(E_i)}\nabla^{\varphi}_{E_i}\sigma)^V \\
&+ \frac{1}{2f}d\varphi(E_i)(f)(\nabla^{\varphi}_{E_i}\sigma)^V + \frac{f}{2\alpha}(R^N(\tilde{\sigma}, \widetilde{\nabla^{\varphi}\sigma})d\varphi(E_i))^H \\
&+ \frac{1}{2f}d\varphi(E_i)(f)(\nabla^{\varphi}_{E_i}\sigma)^V - \frac{1}{2f}h^f((\nabla^{\varphi}_{E_i}\sigma)^V, (\nabla^{\varphi}_{E_i}\sigma)^V)(grad\, f)^H \\
&- \frac{1}{\alpha f^2}h^f((\nabla^{\varphi}_{E_i}\sigma)^V, \sigma^V)^2\sigma^V + \frac{(\alpha+1)}{\alpha f}h^f((\nabla^{\varphi}_{E_i}\sigma)^V, (\nabla^{\varphi}_{E_i}\sigma)^V)\sigma^V \\
&- \frac{2}{\alpha f}h^f((\nabla^{\varphi}_{E_i}\sigma)^V, \sigma^V)(\nabla^{\varphi}_{E_i}\sigma)^V\Big]
\end{aligned}$$

$$
\begin{aligned}
= & \sum_{i=1}^{m}\Big[(\nabla^{\varphi}_{E_i}d\varphi(E_i))^H+\frac{f}{\alpha}(R^N(\tilde{\sigma},\widetilde{\nabla^{\varphi}\sigma})d\varphi(E_i))^H+\frac{1}{f}d\varphi(E_i)(f)(\nabla^{\varphi}_{E_i}\sigma)^V\\
&+(\nabla^{\varphi}_{E_i}\nabla^{\varphi}_{E_i}\sigma)^V-\frac{1}{2\alpha}\big[h^{-1}(\nabla^{\varphi}_{E_i}\sigma,\nabla^{\varphi}_{E_i}\sigma)+h^{-1}(\nabla^{\varphi}_{E_i}\sigma,\sigma)^2\big](grad\, f)^H\\
&-\frac{1}{\alpha}h^{-1}(\nabla^{\varphi}_{E_i}\sigma,\sigma)^2\sigma^V+\frac{\alpha+1}{\alpha^2}\big[h^{-1}(\nabla^{\varphi}_{E_i}\sigma,\nabla^{\varphi}_{E_i}\sigma)+h^{-1}(\nabla^{\varphi}_{E_i}\sigma,\sigma)^2\big]\sigma^V\\
&-\frac{2}{\alpha}h^{-1}(\nabla^{\varphi}_{E_i}\sigma,\sigma)(\nabla^{\varphi}_{E_i}\sigma)^V\Big]\\
= & \sum_{i=1}^{m}\Big[(\nabla^{\varphi}_{E_i}d\varphi(E_i))^H+\frac{f}{\alpha}(R^N(\tilde{\sigma},\widetilde{\nabla^{\varphi}\sigma})d\varphi(E_i))^H\\
&-\frac{1}{2\alpha}\big[h^{-1}(\nabla^{\varphi}_{E_i}\sigma,\nabla^{\varphi}_{E_i}\sigma)+h^{-1}(\nabla^{\varphi}_{E_i}\sigma,\sigma)^2\big](grad\, f)^H\\
&+(\nabla^{\varphi}_{E_i}\nabla^{\varphi}_{E_i}\sigma)^V+\big[\frac{1}{f}df(d\varphi(*))-\frac{2}{\alpha}h^{-1}(\nabla^{\varphi}_{E_i}\sigma,\sigma)\big](\nabla^{\varphi}_{E_i}\sigma)^V\\
&+\big[\frac{\alpha+1}{\alpha^2}h^{-1}(\nabla^{\varphi}_{E_i}\sigma,\nabla^{\varphi}_{E_i}\sigma)+\frac{1}{\alpha^2}h^{-1}(\nabla^{\varphi}_{E_i}\sigma,\sigma)^2\big]\sigma^V\Big]
\end{aligned}
$$

$$
\begin{aligned}
= & \Big[\tau(\varphi)+trace_g\Big(\frac{f}{\alpha}R^N(\tilde{\sigma},\widetilde{\nabla^{\varphi}_{*}\sigma})d\varphi(*)\\
&-\frac{1}{2\alpha}\big[h^{-1}(\nabla^{\varphi}_{*}\sigma,\nabla^{\varphi}_{*}\sigma)+h^{-1}(\nabla^{\varphi}_{*}\sigma,\sigma)^2\big]grad\, f\Big)\Big]^H\\
&+\Big[trace_g\Big((\nabla^{\varphi}_{*})^2\sigma+\big[\frac{1}{f}df(d\varphi(*))-\frac{2}{\alpha}h^{-1}(\nabla^{\varphi}_{*}\sigma,\sigma)\big]\nabla^{\varphi}_{*}\sigma\\
&+\big[\frac{1+\alpha}{\alpha^2}h^{-1}(\nabla^{\varphi}_{*}\sigma,\nabla^{\varphi}_{*}\sigma)+\frac{1}{\alpha^2}h^{-1}(\nabla^{\varphi}_{*}\sigma,\sigma)^2\big]\sigma\Big)\Big]^V.
\end{aligned}
$$

∎

From Theorem 4.16 we obtain

Theorem 4.17. Let $\varphi:(M^m,g)\to(N^n,h)$ be a smooth map between Riemannian manifolds, $f:N\to]0,+\infty[$ be a strictly positive smooth function on N and (T^*N,h^f) the cotangent bundle of N equipped with the vertical rescaled Cheeger-Gromoll metric. If $\sigma:M\longrightarrow T^*N$ is a smooth map such that $\varphi=\pi_N\circ\sigma$, then σ is a harmonic if and only if the following conditions are verified

$$
\begin{aligned}
\tau(\varphi) = & \ trace_g\Big(\frac{1}{2\alpha}\big[h^{-1}(\nabla^{\varphi}_{*}\sigma,\nabla^{\varphi}_{*}\sigma)+h^{-1}(\nabla^{\varphi}_{*}\sigma,\sigma)^2\big]grad\, f \qquad (14)\\
&-\frac{f}{\alpha}R^N(\tilde{\sigma},\widetilde{\nabla^{\varphi}_{*}\sigma})d\varphi(*)\Big),\\
0 = & \ trace_g\Big((\nabla^{\varphi}_{*})^2\sigma+\big[\frac{1}{f}df(d\varphi(*))-\frac{2}{\alpha}h^{-1}(\nabla^{\varphi}_{*}\sigma,\sigma)\big]\nabla^{\varphi}_{*}\sigma \qquad (15)\\
&+\big[\frac{1+\alpha}{\alpha^2}h^{-1}(\nabla^{\varphi}_{*}\sigma,\nabla^{\varphi}_{*}\sigma)+\frac{1}{\alpha^2}h^{-1}(\nabla^{\varphi}_{*}\sigma,\sigma)^2\big]\sigma\Big).
\end{aligned}
$$

Corollary 4.18. Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds, $f : N \to]0, +\infty[$ be a strictly positive constant on N and (T^*N, h^f) the cotangent bundle of N equipped with the vertical rescaled Cheeger-Gromoll metric. If $\sigma : M \longrightarrow T^*N$ is a smooth map such that $\varphi = \pi_N \circ \sigma$, then σ is a harmonic if and only if the following conditions are verified

$$\tau(\varphi) = -trace_g\Big(\frac{f}{\alpha}R^N(\tilde{\sigma}, \widetilde{\nabla^{\varphi}_{*}\sigma})d\varphi(*)\Big), \quad (16)$$

$$\begin{aligned} 0 &= trace_g\Big((\nabla^{\varphi}_{*})^2\sigma - \frac{2}{\alpha}h^{-1}(\nabla^{\varphi}_{*}\sigma, \sigma)\nabla^{\varphi}_{*}\sigma \\ &\quad + \big[\frac{1+\alpha}{\alpha^2}h^{-1}(\nabla^{\varphi}_{*}\sigma, \nabla^{\varphi}_{*}\sigma) + \frac{1}{\alpha^2}h^{-1}(\nabla^{\varphi}_{*}\sigma, \sigma)^2\big]\sigma\Big). \end{aligned} \quad (17)$$

4.3 Harmonicity of the map $\phi : (TM, g^f) \longrightarrow N$

Lemma 4.19. Let (M^m, g) be a Riemannian manifold, $f : M \to]0, +\infty[$ be a strictly positive smooth function on M and (T^*M, g^f) its cotangent bundle equipped with the vertical rescaled Cheeger-Gromoll metric. The tension field of the canonical projection $\pi : (T^*M, g^f) \longrightarrow (M, g)$ is given by

$$\tau(\pi) = \frac{m}{2f}\,(grad\, f) \circ \pi.$$

Proof. Let $\xi = (x, p) \in T^*M$ and $\{E_i\}_{i=\overline{1,m}}$ be an orthonormal basis for the tangent space T_xM of M at x. Also, let $(\omega^i)_{i=\overline{1,m}}$ be a dual orthonormal basis for the cotangent spaces T^*_xM of M at x such that $\omega^1 = \frac{p}{\|p\|}$, then

$$\{E_i^H, \frac{1}{\sqrt{f}}(\omega^1)^V, \sqrt{\frac{\alpha}{f}}(\omega^j)^V\}_{i=\overline{1,m}, j=\overline{2,m}}$$

is an orthonormal basis for the cotangent space $T_\xi T^*M$ with respect to the vertical rescaled Cheeger-Gromoll metric.

$$\begin{aligned} \tau(\pi)_\xi &= trace_{g^f}(\nabla d\pi)_\xi \\ \tau(\pi) &= \sum_{i=1}^{m}\Big\{\nabla^{\pi}_{E_i^H}d\pi(E_i^H) - d\pi(\nabla^f_{E_i^H}E_i^H)\Big\} \\ &\quad + \nabla^{\pi}_{(\frac{1}{\sqrt{f}}(\omega^1)^V)}d\pi(\frac{1}{\sqrt{f}}(\omega^1)^V) - d\pi(\nabla^f_{(\frac{1}{\sqrt{f}}(\omega^1)^V)}(\frac{1}{\sqrt{f}}(\omega^1)^V)) \\ &\quad + \sum_{j=2}^{m}\Big\{\nabla^{\pi}_{(\sqrt{\frac{\alpha}{f}}(\omega^j)^V)}d\pi(\sqrt{\frac{\alpha}{f}}(\omega^j)^V) - d\pi(\nabla^f_{(\sqrt{\frac{\alpha}{f}}(\omega^j)^V)}(\sqrt{\frac{\alpha}{f}}(\omega^j)^V))\Big\} \end{aligned}$$

as $d\pi((\omega^j)^V)=0$ and $d\pi(E_i^H)=E_i\circ\pi$ then

$$\begin{aligned}
\tau(\pi) &= \sum_{i=1}^{m}\Big\{(\nabla_{d\pi(E_i^H)}d\pi(E_i^H)) - d\pi(\nabla_{E_i}E_i)^H\Big\}\\
&\quad -\frac{1}{\sqrt{f}}d\pi\big[(\omega^1)^V(\frac{1}{\sqrt{f}})(\omega^1)^V+\frac{1}{\sqrt{f}}\nabla^f_{(\omega^1)^V}(\omega^1)^V\big]\\
&\quad -\sum_{j=2}^{m}\Big\{\sqrt{\frac{\alpha}{f}}d\pi\big[(\omega^j)^V(\sqrt{\frac{\alpha}{f}})(\omega^j)^V+\sqrt{\frac{\alpha}{f}}\nabla^f_{(\omega^j)^V}(\omega^j)^V\big]\Big\}\\
&= \sum_{i=1}^{m}\Big\{(\nabla_{E_i\circ\pi}E_i\circ\pi) - d\pi(\nabla_{E_i}E_i)^H\Big\}\\
&\quad -\frac{1}{f}d\pi(\nabla^f_{(\omega^1)^V}(\omega^1)^V)-\sum_{j=2}^{m}\Big\{\frac{\alpha}{f}d\pi(\nabla^f_{(\omega^j)^V}(\omega^j)^V)\Big\}\\
&= \sum_{i=1}^{m}\Big\{(\nabla_{E_i}E_i)\circ\pi - d\pi(\nabla_{E_i}E_i)^H\Big\}\\
&\quad +\frac{1}{2f}(grad\, f)\circ\pi+\frac{m-1}{2f}(grad\, f)\circ\pi\\
&= \frac{m}{2f}(\,grad\, f)\circ\pi.
\end{aligned}$$

∎

Theorem 4.20. Let $\varphi:(M^m,g)\to(N^n,h)$ be a smooth map between Riemannian manifolds, $f:M\to]0,+\infty[$ be a strictly positive smooth function on M and (T^*M,g^f) the cotangent bundle of M equipped with the vertical rescaled Cheeger-Gromoll metric. The tension field of the map

$$\begin{aligned}
\phi:(T^*M,g^f) &\longrightarrow (N,h)\\
(x,p) &\longmapsto \varphi(x)
\end{aligned}$$

is given by

$$\tau(\phi)=\big[\tau(\varphi)+\frac{m}{2f}d\varphi(grad\, f)\big]\circ\pi.$$

Proof. Let $\xi=(x,p)\in T^*M$ and $\{E_i\}_{i=\overline{1,m}}$ be an orthonormal basis for the tangent space T_xM of M at x. Also, let $(\omega^i)_{i=\overline{1,m}}$ be a dual orthonormal basis for the cotangent spaces T^*_xM of M at x such that $\omega^1=\frac{p}{\|p\|}$, then

$$\{E_i^H,\frac{1}{\sqrt{f}}(\omega^1)^V,\sqrt{\frac{\alpha}{f}}(\omega^j)^V\}_{i=\overline{1,m},j=\overline{2,m}}$$

is an orthonormal basis for the cotangent space $T_\xi T^*M$ with respect to the vertical rescaled Cheeger-Gromoll metric.
As the ϕ is defined by

$$\begin{aligned}
\phi:(T^*M,g^f) &\xrightarrow{\ \pi\ } (M,g)\xrightarrow{\ \varphi\ }(N,h)\\
(x,p) &\longmapsto x\longmapsto\varphi(x)
\end{aligned}$$

i.e., $\phi = \varphi \circ \pi$, we have

$$\begin{aligned}\tau(\phi)_\xi &= \tau(\varphi\circ\pi)_\xi \\ &= d\varphi(\tau(\pi))_\xi + trace_{g^f}\nabla d\varphi(d\pi, d\pi)_\xi\end{aligned}$$

$$\begin{aligned}trace_{g^f}\nabla d\varphi(d\pi,d\pi) &= \sum_{i=1}^{m}\Big\{\nabla^{\varphi}_{d\pi(E_i^H)}d\varphi(d\pi(E_i^H)) - d\varphi(\nabla_{d\pi(E_i^H)}d\pi(E_i^H))\Big\} \\ &\quad +\nabla^{\varphi}_{d\pi(\frac{1}{\sqrt{f}}(\omega^1)^V)}d\varphi(d\pi(\frac{1}{\sqrt{f}}(\omega^1)^V)) \\ &\quad -d\varphi(\nabla_{d\pi(\frac{1}{\sqrt{f}}(\omega^1)^V)}d\pi(\frac{1}{\sqrt{f}}(\omega^1)^V)) \\ &\quad +\sum_{j=2}^{m}\Big\{\nabla^{\varphi}_{d\pi(\sqrt{\frac{\alpha}{f}}(\omega^j)^V)}d\varphi(d\pi(\sqrt{\frac{\alpha}{f}}(\omega^j)^V)) \\ &\quad -d\varphi(\nabla_{d\pi(\sqrt{\frac{\alpha}{f}}(\omega^j)^V)}d\pi(\sqrt{\frac{\alpha}{f}}(\omega^j)^V))\Big\} \\ &= \sum_{i=1}^{m}\Big\{(\nabla^{\varphi}_{E_i\circ\pi}d\varphi(E_i\circ\pi)) - d\varphi(\nabla_{E_i\circ\pi}E_i\circ\pi)\Big\} \\ &= \sum_{i=1}^{m}\Big\{\nabla^{\varphi}_{E_i\circ\pi}d\varphi(E_i)\circ\pi - d\varphi((\nabla_{E_i}E_i)\circ\pi)\Big\} \\ &= \sum_{i=1}^{m}\Big\{\nabla^{\varphi}_{E_i}d\varphi(E_i) - d\varphi(\nabla_{E_i}E_i)\Big\}\circ\pi \\ &= \tau(\varphi)\circ\pi.\end{aligned}$$

Using Lemma 4.19, we obtain

$$\tau(\phi) = \big[\tau(\varphi) + \frac{m}{2f}d\varphi(grad\, f)\big]\circ\pi.$$

∎

Remark 4.21.

1. If f is constant then ϕ is harmonic if and only if φ is harmonic.

2. If φ is a harmonic map, then the following properties are equivalent
$$\begin{cases} i) & gradf \in \ker d\varphi \\ ii) & \phi\ is\ \ harmonic.\end{cases}$$

3. If φ is a harmonic Riemannian submersion, then the following properties are equivalent
$$\begin{cases} i) & grad f \text{ is tangent to the fibres of } \varphi \\ ii) & \phi \text{ is harmonic.} \end{cases}$$

Theorem 4.22. Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds, $f : M \to]0, +\infty[$ be a strictly positive smooth function on M and (T^*M, g^f) the cotangent bundle of M equipped with the vertical rescaled Cheeger-Gromoll metric. The map

$$\begin{array}{rcl} \phi : (T^*M, g^f) & \longrightarrow & (N, h) \\ (x, p) & \longmapsto & \varphi(x) \end{array}$$

is harmonic if and only if
$$\tau(\varphi) = -\frac{m}{2f} d\varphi(grad\, f).$$

Acknowledgement. We would like to thank the anonymous referee for his/her valuable and insightful comments, which have improved our manuscript substantially.

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