Approximation results on nonlinear operators by P_p−statistical convergence*

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Abstract

The first aim of this paper is to present P_p-statistical approximation theorem for max-product operators with the use of statistical convergence with respect to power series method which is incompatible with statistical convergence. The second aim is to obtain the P_p-statistical rate of this approximation. In the end of the paper, we construct special sequences of nonlinear operators which satisfy our results as an application.

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1 Introduction and preliminaries

In the development of approximation theory, the proof of Weierstrass's theorem which is dealing with the approximation of algebraic and trigonometric polynomials to a continuous function on a closed interval has played an important role [25]. Since the proof is long and complicated, many mathematicians have tried to present a simpler and more understandable proof. In this context a short proof has been given by using Bernstein polynomials which are well studied [15], [17], [19]. Instead of these polynomials, general operators have come to mind and then the approximation by positive linear operators has been studied. Notice that when working with these operators, we have a linear structure as an algebraic structure. Therefore the next question is: Do they have to be linear? This question has been answered by Bede and et. al. [2], [3], [4] with the use of various nonlinear approximation theorems. Of course, the limit used in the approximation theorems mentioned here is the classical limit of operators. What if the classical limit fails? There are many well known concepts of convergences which are effective to use when the classical limit fails. Therefore, Korovkin type approximation theory has been given via summability setting by statistical convergence, ideal convergence and summation process [6], [9], [11], [23]. Recently, some approximation operators that are not linear have also been studied by these various types of convergences [7], [12], [13], [24]. Besides this, approximation theory has also been used in feedforward neural networks (FFNs), ReLU networks [16], [20], [22]. For example in [20], a Kantorovich type variant of such operators has been introduced and has been used to present direct and inverse theorems of approximation in L^p with $1 \leq p < \infty$. Furthermore, deep learning which depends on structured deep neural networks has been successfully applied in different fields of science and technology.

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In the present paper, the first aim is to study approximation properties of max-product operators which are nonlinear by using statistical convergence with respect to power series method, shortly P_p-statistical convergence. We should point out that using P_p-statistical convergence in nonlinear approximation is the new idea of this paper. The second aim is to obtain the rate of this approximation. In the end of the paper, we construct special sequences of nonlinear operators which satisfy our results as an application. In [23], Ünver and Orhan have provided examples to show that statistical convergence and P_p-statistical convergence are incompatible.

Now we pause to collect basic definitions, notations and also earlier results which we need throughout the paper.

If

$$\delta(K) := \lim_{k\to\infty} \frac{1}{k+1} |\{n \le k : n \in K\}|$$

exists then it is said to be the density of the subset $K \subseteq \mathbb{N}_0$. Here the vertical bars denote the number of the elements of enclosed set and $\mathbb{N}_0$ is the set of all nonnegative integers. If for every $\varepsilon > 0$, $\delta(K_\varepsilon) = 0$ where $K_\varepsilon = \{n \in \mathbb{N}_0 : |x_n - l| \ge \varepsilon\}$ then $x = (x_n)$ is said to be statistically convergent to l [8], [10], [21]. Let (p_n) be nonnegative real sequence, $p_0 > 0$ and $p(t) := \sum_{n=0}^{\infty} p_n t^n$ has radius of convergence R with $0 < R \le \infty$. Then power series method is defined as follows :

Let also

$$C_p := \left\{ f : (-R, R) \to \mathbb{R} | \lim_{0<t\to R^-} \frac{1}{p(t)} f(t) \quad \text{exists} \right\}$$

and

$$C_{p_p} := \left\{ x = (x_n) | p_x(t) := \sum_{n=0}^{\infty} p_n t^n x_n \quad \text{has radius of convergence} \ge R \quad \text{and} \quad p_x \in C_p \right\}.$$

A power series method is defined by

$$P_p - \lim x = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{n=0}^{\infty} p_n t^n x_n$$

where the functional $P_p - \lim : C_{P_p} \to \mathbb{R}$ (shortly P_p) and then it is said to be that x is P_p-convergent [5], [14].

A power series method P_p is said to be regular if $P_p - \lim x = L$ provided that $\lim x = L$ [5]. The regularity of power series method is equivalent to

$$\lim_{t\to R^-} \frac{p_n t^n}{p(t)} = 0$$

holds for each $n \in \mathbb{N}_0$ [5].

By combining these concepts, Ünver and Orhan [23] have recently introduced P_p-statistical convergence and have presented a Korovkin type theorem for a sequence of positive linear operators defined on $C[0, 1]$, the space of all continuous functions on $[0, 1]$.

Now we are ready to recall the statistical convergence with respect to power series method. Let P_p be regular and $K \subseteq \mathbb{N}_0$. If

$$\delta_{P_p}(K) := \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{n\in K} p_n t^n$$

exists then it is said to be P_p-density of K. One can immediately observe that $0 \leq \delta_{P_p}(K) \leq 1$ whenever it exists [23].
Let $x = (x_n)$ be a real sequence and let P_p be regular. If for every $\varepsilon > 0$

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{n\in K_\varepsilon} p_n t^n = 0$$

that is, $\delta_{P_p}(K_\varepsilon) = 0$ for every $\varepsilon > 0$, then x is said to be P_p-statistically convergent to l and we denote it by $st_{P_p} - \lim x = l$ [23].

Let (X, d) be a compact metric space and let P_p be regular. The space of all nonnegative continuous functions on X is denoted by $C(X, [0, \infty))$. Now consider the following operators of max-product type:

$$T_n(f; x) = \bigvee_{k=0}^{n} K_n(x, x_k) \cdot f(x_k),\ x \in X \ and\ f \in C(X, [0, \infty)) \tag{1.1}$$

where $x_k \in X, k = 0, 1, ..., n$, are the knots and $K_n(x, x_k)$ are nonnegative continuous functions on X having relatively simple expression such that, for any $x \in X$,

$$\delta_{P_p}\left(\left\{n : \bigvee_{k=0}^{n} K_n(x, x_k) = 1\right\}\right) = 1 \tag{1.2}$$

holds. Also, through the paper, we define $T_n(f; x) = f(x)$ for $n = 0$. This definition makes the operator well-defined for the knots $x_k = \frac{k}{n}$. Notice that these operators are nonlinear. In fact, T_n is pseudo-linear, i.e.,

$$T_n(a \cdot f_1 \bigvee b \cdot f_2; x) = a \cdot T_n(f_1; x) \bigvee b \cdot T_n(f_2; x)$$

holds for every $f_1, f_2 \in C(X, [0, \infty))$ and for any nonnegative numbers a, b.
It is useful to recall the following lemma.

Lemma 1.1. [3]

$$\left|\bigvee_{k=0}^{n} a_k - \bigvee_{k=0}^{n} b_k\right| \leq \bigvee_{k=0}^{n} |a_k - b_k|$$

holds for any $a_k,\ b_k \in [0, \infty),\ k = 0, 1, 2, ..., n$.

2 Main results

This section is devoted to presentation of our main results dealing with the approximation properties of max-product operators and the corresponding P_p-statistical rate of approximation.

Theorem 2.1. Let (X, d) be a compact metric space and let P_p be regular. If

$$st_{P_p} - \lim\left\{\bigvee\{|T_n(\varphi_x; x)| : x \in X\}\right\} = 0 \ with\ \varphi_x(t) = d^2(t, x), \tag{2.1}$$

holds for the operators T_n given by (1.1), (1.2) then, we have, for all $f \in C(X, [0, \infty))$

$$st_{P_p} - \lim\left\{\bigvee\{|T_n(f; x) - f(x)| : x \in X\}\right\} = 0.$$

Proof. Let $x \in X$ and $f \in C(X,[0,\infty))$ be fixed. Since f is continuous and X is compact, we have that f is uniform continuous on X. Therefore, we can write that for a given $\varepsilon > 0$ $|f(t)-f(x)| < \varepsilon$ holds for every $x, t \in X$ for a number $\delta > 0$ such that $d(x,t) < \delta$. Therefore,

$$|f(t)-f(x)| \leq \varepsilon + \frac{2H}{\delta^2}\varphi_x(t) \tag{2.2}$$

holds for all $t \in X$, where $H := \bigvee\{|f(t)| : t \in X\}$. Now write

$$E := \left\{ n \in \mathbb{N}_0 : \bigvee_{k=0}^{n} K_n(x,x_k) = 1 \right\}. \tag{2.3}$$

Then by (1.2), we may write that $\delta_{P_p}(E) = 1$ and $\delta_{P_p}(\mathbb{N}_0 \setminus E) = 0$.
Hence by (1.2), (2.2) and Lemma 1, one can write that

$$\begin{aligned}
|T_n(f;x) - f(x)| &= \left| \bigvee_{k=0}^{n} K_n(x,x_k)\cdot f(x_k) - \bigvee_{k=0}^{n} K_n(x,x_k)\cdot f(x) \right| \\
&\leq \bigvee_{k=0}^{n} K_n(x,x_k)\cdot |f(x_k)-f(x)| \\
&\leq \bigvee_{k=0}^{n} K_n(x,x_k)\cdot \left(\varepsilon + \frac{2H}{\delta^2}\varphi_x(x_k)\right) \\
&\leq \varepsilon + \frac{2H}{\delta^2}\bigvee_{k=0}^{n} K_n(x,x_k)\cdot \varphi_x(x_k) \\
&= \varepsilon + \frac{2H}{\delta^2} T_n(\varphi_x;x)
\end{aligned}$$

holds for all $n \in E$.

If we take maximum overall $x \in X$, then

$$\bigvee \{|T_n(f;x)-f(x)| : x \in X\} \leq \varepsilon + \frac{2H}{\delta^2}\bigvee\{|T_n(\varphi_x;x)| : x \in X\} \tag{2.4}$$

holds for all $n \in E$ by the above inequality. For a given $\varepsilon^{'} > 0$, we can pick $\varepsilon > 0$ such that $\varepsilon < \varepsilon^{'}$, and then define

$$\begin{aligned}
G &:= \left\{ n \in \mathbb{N}_0 : \left(\bigvee\{|T_n(f;x)-f(x)| : x \in X\}\right) \geq \varepsilon^{'} \right\}, \\
G' &:= \left\{ n \in \mathbb{N}_0 : \left(\bigvee\{|T_n(\varphi_x;x)| : x \in X\}\right) \geq \frac{(\varepsilon^{'}-\varepsilon)\delta^2}{2H} \right\}.
\end{aligned}$$

By (2.4), we have that

$$G \cap E \subset G' \cap E.$$

Therefore

$$\frac{1}{p(t)}\sum_{n\in G\cap E} p_n t^n \leq \frac{1}{p(t)}\sum_{n\in G'\cap E} p_n t^n \leq \frac{1}{p(t)}\sum_{n\in G'} p_n t^n$$

holds and by taking limit in both sides, we obtain

$$\lim_{t\to R^-}\frac{1}{p(t)}\sum_{n\in G\cap E}p_nt^n=0$$

which means $\delta_{P_p}(G\cap E)=0$.

This implies that

$$\frac{1}{p(t)}\sum_{n\in G}p_nt^n=\frac{1}{p(t)}\sum_{n\in G\cap E}p_nt^n+\frac{1}{p(t)}\sum_{n\in G\cap(\mathbb{N}_0\setminus E)}p_nt^n\le\frac{1}{p(t)}\sum_{n\in G\cap E}p_nt^n+\frac{1}{p(t)}\sum_{n\in(\mathbb{N}_0\setminus E)}p_nt^n,$$

and again by taking limit in both sides, since $\delta_{P_p}(\mathbb{N}_0\setminus E)=0$ we have

$$\delta_{P_p}(G)=0.$$

This implies

$$st_{P_p}-\lim\left\{\bigvee\{|T_n(f;x)-f(x)|:x\in X\}\right\}=0$$

and completes the proof. Q.E.D.

In order to study the rate of approximation, one of the main tools is modulus of continuity and it is given by

$$\omega(f,\delta)=\bigvee\{|f(x)-f(t)|:d(x,t)\le\delta,\ x,t\in X\}$$

where δ is a positive constant and $f\in C(X,[0,\infty)$. It is important to mention from [18] that if (X,d) is compact convex linear metric space then for any $\gamma>0$

$$\omega(f,\gamma\delta)\le(1+\gamma)\omega(f,\delta).$$

The rates of convergence in A-statistical sense have also been defined in [6], [9]. Under the light of these studies, P_p- statistical rate has been defined in [1].

Definition 2.2. Let (a_n) be a positive non-increasing sequence of real numbers and let P_p be regular. If for every $\varepsilon>0$

$$\lim_{0<t\to R^-}\left[\frac{1}{p\,(t)}\sum_{n:|x_n-l|\ge\varepsilon a_n}p_nt^n\right]=0$$

holds then $x=(x_n)$ is said to be P_p-statistically convergent to the number l with rate $o\,(a_n)$ and we denote it by $x_n-l=st_{P_p}-o\,(a_n)$, as $n\to\infty$.

Here it is noteworthy to mention that the terms of the sequence (x_n) are controlling the rate. In order to continue, let us recall the following.

Lemma 2.3. [7] For every $a_k,b_k\ge0$ $k=0,1,...,n$, we have

$$\bigvee_{k=0}^{n}a_kb_k\le\left(\bigvee_{k=0}^{n}a_k^2\right)^{1/2}\left(\bigvee_{k=0}^{n}b_k^2\right)^{1/2}.$$

Now we can present the following theorem on the P_p- statistical rate of the approximation.

Theorem 2.4. Let (X, d) be a compact convex linear metric space and let P_p be regular. If the operators T_n given by (1.1) and (1.2) satisfy that

$$\omega(f, \delta_n) = st_{P_p} - o(a_n) \ as\ n \to \infty\ for\ f \in C(X, [0, \infty)), \tag{2.5}$$

then,

$$\bigvee\{|T_n(f; x) - f(x)| : x \in X\} = st_{P_p} - o(b_n) \ as\ n \to \infty \tag{2.6}$$

holds for any sequence (b_n) of positive increasing real numbers such that $b_n \geq a_n$ for all $n \in \mathbb{N}_0$ where (a_n) is a sequence of positive non-increasing real numbers. Here (δ_n) is defined by

$$\delta_n := \sqrt{\bigvee\{T_n(\varphi_x; x) : x \in X\}} \ with\ \varphi_x(t) = d^2(t, x). \tag{2.7}$$

Proof. Let $x \in X$ and $f \in C(X, [0, \infty))$ be fixed. Consider the set E as in (2.3), then one can write that

$$\begin{aligned}
|T_n(f; x) - f(x)| &\leq \bigvee_{k=0}^{n} K_n(x, x_k) \cdot |f(x_k) - f(x)| \\
&\leq \bigvee_{k=0}^{n} K_n(x, x_k) \cdot \omega(f, d(x_k, x)) \\
&\leq \omega(f, \delta) \bigvee_{k=0}^{n} K_n(x, x_k) \left(1 + \frac{d(x_k, x)}{\delta}\right) \\
&\leq \omega(f, \delta) \left\{1 + \frac{1}{\delta} \bigvee_{k=0}^{n} K_n(x, x_k) \cdot d(x_k, x)\right\} \\
&= \omega(f, \delta) \left\{1 + \frac{1}{\delta} \bigvee_{k=0}^{n} \left[K_n^{1/2}(x, x_k)\right] \cdot \left[K_n^{1/2}(x, x_k) d(x_k, x)\right]\right\}
\end{aligned}$$

holds for every $n \in E$ and for any $\delta > 0$. Now, by Lemma 2, we have that

$$|T_n(f; x) - f(x)| \leq \omega(f, \delta) \left\{1 + \frac{1}{\delta} \sqrt{T_n(d^2(\cdot, x); x)}\right\}$$

holds for every $n \in E$ and for any $\delta > 0$. Therefore,

$$\bigvee \{|T_n(f; x) - f(x)| : x \in X\} \leq \omega(f, \delta) \left\{1 + \frac{\delta_n}{\delta}\right\} \tag{2.8}$$

holds for the same n and δ. Now picking $\delta := \delta_n$ by (2.6), it follows from (2.8) that

$$\bigvee \{|T_n(f; x) - f(x)| : x \in X\} \leq 2\omega(f, \delta_n). \tag{2.9}$$

For any ε, define

$$F := \left\{ n \in \mathbb{N}_0 : \bigvee \{|T_n(f;x) - f(x)| : x \in X\} \geq \varepsilon b_n \right\},$$
$$F' := \left\{ n \in \mathbb{N}_0 : \omega(f, \delta_n) \geq \frac{\varepsilon a_n}{2} \right\}.$$

Then, by (2.9), we get

$$F \cap E \subseteq F' \cap E. \tag{2.10}$$

Therefore, we immediately obtain that

$$\frac{1}{p(t)} \sum_{n \in F \cap E} p_n t^n \leq \frac{1}{p(t)} \sum_{n \in F' \cap E} p_n t^n \leq \frac{1}{p(t)} \sum_{n \in F'} p_n t^n$$

which gives

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{n \in F \cap E} p_n t^n = 0. \tag{2.11}$$

It implies that

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{n \in F} p_n t^n = 0.$$

Hence

$$\bigvee \{|T_n(f;x) - f(x)| : x \in X\} = st_{P_p} - o(b_n) \ as \ n \to \infty$$

holds and the proof is completed. Q.E.D.

3 Application

This section is devoted to the construction of special sequences of nonlinear operators which satisfy our results.

Example 3.1. Define the sequences (p_n) and (u_n) as follows:

$$p_n = \begin{cases} 1 & , \quad n = 2k \\ 0 & , \quad n = 2k+1 \end{cases}, \qquad u_n = \begin{cases} 0 & , \quad n = 2k \\ 1 & , \quad n = 2k+1 \end{cases}.$$

One can obtain that the method P_p is regular and also observe that

$$K_\varepsilon = \{n \in \mathbb{N}_0 : |u_n - 0| \geq \varepsilon\} \subseteq \{n = 2k+1 : k \in \mathbb{N}_0\}$$

holds for every $\varepsilon > 0$. Then we have

$$\delta_{P_p}(K_\varepsilon) = \lim_{0 < t \to R^-} \frac{1}{p(t)} \sum_{n \in K_\varepsilon} p_n t^n = 0$$

i.e., (u_n) is P_p-statistically convergent to 0. Now, let (X, d) be compact metric space and consider the Shepard-type max-product operators (see [4]) as follows:

$$S_n^{\lambda}(f;x)=\bigvee_{k=0}^{n}\left(\frac{\frac{1}{d^{\lambda}(x,x_k)}}{\bigvee_{j=0}^{n}\frac{1}{d^{\lambda}(x,x_j)}}\right)\cdot f(x_k)=\frac{\bigvee_{k=0}^{n}\frac{f(x_k)}{d^{\lambda}(x,x_k)}}{\bigvee_{j=0}^{n}\frac{1}{d^{\lambda}(x,x_j)}} \tag{3.1}$$

where $x\in X,\lambda,n\in\mathbb{N}$ and $f\in C(X,[0,\infty))$.
It is known that for all $f\in C(X,[0,\infty))$, the sequence $\{S_n^{\lambda}(f)\}$ converges uniformly to f on X.

Let $(u_n),(p_n)$ given above and define

$$T_n(f;x)=(1+u_n)S_n^{\lambda}(f;x),\ x\in X\ and\ f\in C(X,[0,\infty)).$$

Notice that T_n satisfies the conditions of our theorem. Therefore,

$$st_{P_p}-\lim\left\{\bigvee\{|T_n(f,x)-f(x)|:x\in X\}\right\}=0$$

holds for all $f\in C(X,[0,\infty))$.
However, notice that it is not possible to approximate f by using $T_n(f)$ since the sequence (u_n) is not convergent in the ordinary sense. Furthermore it is still possible to approximate f by using $T_n(f)$ via P_p-statistical convergence since (u_n) is P_p-statistically convergent to 0.

Example 3.2. Construct the sequence of operators T_n by

$$T_n(f;x)=u_nS_n^{\lambda}(f;x)$$

where (S_n^{λ}) is Shepard-type max-product operators and

$$u_n=\begin{cases}1 & , & n=2k\\ 0 & , & n=2k+1\end{cases},\qquad p_n=\begin{cases}1 & , & n=2k\\ 0 & , & n=2k+1\end{cases}.$$

One can obtain that the method P_p is regular. Also for every $\varepsilon>0$ we have $K_{\varepsilon}=\{n\in\mathbb{N}_0:|u_n-1|\geq\varepsilon\}\subseteq\{n=2k+1:k\in\mathbb{N}_0\}$. Then, we obtain that

$$\delta_{P_p}(K_{\varepsilon})=\lim_{0<t\to R^-}\frac{1}{p(t)}\sum_{n\in K_{\varepsilon}}p_nt^n=0.$$

Again notice that it is not possible to approximate f by using $T_n(f)$ since the sequence (u_n) is not convergent in the ordinary sense. Furthermore it is still possible to approximate f by using $T_n(f)$ via P_p-statistical convergence since (u_n) is P_p-statistically convergent to 1.

As a conclusion, using P_p-statistical convergence gives us the oppurtunity to approximate a function in the case that the classical limit fails. In these recent studies [16], [20], [22], especially [20], while approximating by $S_{n,\sigma}(f,x)$, we guess that using different types of convergences will be benefical as in our paper. Therefore, it will be interesting to apply this idea in recent literature.

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