DE GRUYTER
OPEN

DOI 10.2478/tmj-2018-0006

Some different type integral inequalities concerning twice differentiable generalized relative semi-$(r;m,h)$-preinvex mappings

Artion Kashuri and Rozana Liko

Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", Vlora, Albania
E-mail: artionkashuri@gmail.com, rozanaliko86@gmail.com

Abstract

In this article, we first present some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized relative semi-$(r;m,h)$-preinvex mappings. And then, a new identity concerning twice differentiable mappings defined on m-invex set is derived. By using the notion of generalized relative semi-$(r;m,h)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard, Ostrowski and Simpson type inequalities via fractional integrals are established. It is pointed out that some new special cases can be deduced from main results of the article.

2010 Mathematics Subject Classification. **26A33**. 26D07, 26D10, 26D15
Keywords. Hermite-Hadamard inequality, Ostrowski inequality, Simpson inequality, Hölder's inequality, power mean inequality, Minkowski inequality, fractional integrals, m-invex.

1 Introduction

The subsequent double inequality is known as Hermite-Hadamard inequality.

Theorem 1.1. Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex mapping on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the subsequent double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions (see [[2]-[25],[29],[30],[33],[36],[40],[41],[46],[47],[49],[50]]) and the references mentioned in these papers. Also the following result is known in the literature as the Ostrowski inequality [28], which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a,b]$.

Theorem 1.2. Let $f : I \longrightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a,b]$, then

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall x \in [a,b]. \tag{1.2}$$

The following inequality is well known in the literature as Simpson's inequality:

Tbilisi Mathematical Journal 11(1) (2018), pp. 79–97.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 10 September 2017.
Accepted for publication: 25 December 2017.

Unauthenticated
Download Date | 3/1/18 2:18 PM

Theorem 1.3. Let $f:[a,b]\longrightarrow\mathbb{R}$ be four time differentiable on the interval (a,b) and having the fourth derivative bounded on (a,b), that is $\|f^{(4)}\|_\infty=\sup_{x\in(a,b)}|f^{(4)}|<\infty$. Then, we have

$$\left|\int_a^b f(t)dt-\frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\leq\frac{1}{2880}\|f^{(4)}\|_\infty(b-a)^5. \tag{1.3}$$

Inequality (1.3) gives an error bound for the classical Simpson quadrature formula, which is one of the most used quadrature formulae in practical applications.

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Ostrowski type inequalities (see [[9],[10],[43]]). For other recent results concerning Simpson type inequalities (see [[27],[39]]).

Let us evoke some definitions as follows.

Definition 1.4. [48] A set $M_\varphi\subseteq\mathbb{R}^n$ is named as a relative convex (φ-convex) set, if and only if, there exists a function $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that,

$$t\varphi(x)+(1-t)\varphi(y)\in M_\varphi,\ \forall\ x,y\in\mathbb{R}^n:\varphi(x),\varphi(y)\in M_\varphi,t\in[0,1]. \tag{1.4}$$

Definition 1.5. [48] A function f is named as a relative convex (φ-convex) function on a relative convex (φ-convex) set M_φ, if and only if, there exists a function $\varphi:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that,

$$f(t\varphi(x)+(1-t)\varphi(y))\leq tf(\varphi(x))+(1-t)f(\varphi(y)), \tag{1.5}$$

$\forall\ x,y\in\mathbb{R}^n:\varphi(x),\varphi(y)\in M_\varphi,t\in[0,1]$.

Definition 1.6. [6] A non-negative function $f:I\subseteq\mathbb{R}\longrightarrow[0,+\infty)$ is said to be P-function, if

$$f(tx+(1-t)y)\leq f(x)+f(y),\quad\forall x,y\in I,\ t\in[0,1].$$

Definition 1.7. [1] A set $K\subseteq\mathbb{R}^n$ is said to be invex respecting the mapping $\eta:K\times K\longrightarrow\mathbb{R}^n$, if $x+t\eta(y,x)\in K$ for every $x,y\in K$ and $t\in[0,1]$.

Definition 1.8. [31] Let $h:[0,1]\longrightarrow\mathbb{R}$ be a non-negative function and $h\neq0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x+t\eta(y,x)\big)\leq h(1-t)f(x)+h(t)f(y) \tag{1.6}$$

for each $x,y\in K$ and $t\in[0,1]$ where $f(\cdot)>0$.

Clearly, when putting $h(t)=t$ in Definition 1.8, f becomes a preinvex function [38]. If the mapping $\eta(y,x)=y-x$ in Definition(1.8), then the non-negative function f reduces to h-convex mappings [45].

Definition 1.9. [44] Let $f:K\subseteq\mathbb{R}\longrightarrow\mathbb{R}$ be a non-negative function, a function $f:K\longrightarrow\mathbb{R}$ is said to be a tgs-convex function on K if the inequality

$$f\big((1-t)x+ty\big)\leq t(1-t)[f(x)+f(y)] \tag{1.7}$$

holds for all $x,y\in K$ and $t\in(0,1)$.

Unauthenticated
Download Date | 3/1/18 2:18 PM

Definition 1.10. [[4],[28]] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to MT-convex functions, if it is non-negative and $\forall\ x, y \in I$ and $t \in (0,1)$ satisfies the subsequent inequality:

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.8}$$

Definition 1.11. [34] A function: $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall x, y \in I$, and $t \in (0,1)$, among $m \in [0,1]$, satisfies the following inequality

$$f\big(tx + m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.9}$$

Definition 1.12. [37] Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$. A function $f : K \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0,1] \longrightarrow [0, +\infty)$, if

$$f\big(mx + t\eta(y,x,m)\big) \leq m h_1(t) f(x) + h_2(t) f(y) \tag{1.10}$$

is valid for all $x, y \in K$ and $t \in [0,1]$, together $m \in (0,1]$ is said to be generalized (m, h_1, h_2)-preinvex functions with respect to η.

We need the subsequent Riemann-Liouville fractional calculus background.

Definition 1.13. [29] Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+} f$ and $J^{\alpha}_{b-} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J^0_{a+} f(x) = J^0_{b-} f(x) = f(x)$.

Note that $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of Riemann-Liouville fractional integrals, many authors extended to research Riemann-Liouville fractional inequalities via different classes of convex mappings: for generalizations, variations and new inequalities for them, (see [[29]-[39]]).

Let us recall the Gauss-Jacobi type quadrature formula as follows.

$$\int_a^b (x-a)^p (b-x)^q f(x) dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R^{\star}_m |f|, \tag{1.11}$$

for certain $B_{m,k}, \gamma_k$ and rest $R^{\star}_m |f|$ (see [42]).

In [26], Liu obtained integral inequalities for P-function related to the left-hand side of (1.11), and in [35], Özdemir et al. also presented several integral inequalities concerning the left-hand side

Unauthenticated
Download Date | 3/1/18 2:18 PM

of (1.11) via some kinds of convexity.

Motivated by the above literatures, the main objective of this article is to establish integral inequalities using two lemmas as auxiliary results for the left side of Gauss-Jacobi type quadrature formula and some new estimates on Hermite-Hadamard, Ostrowski and Simpson type inequalities via fractional integrals associated with generalized relative semi-$(r;m,h)$-preinvex mappings. It is pointed out that some new special cases will be deduced from main results of the article.

2 Main results

The following definitions will be used in this section.

Definition 2.1. [7] A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

We next introduce generalized relative semi-$(r;m,h)$-preinvex mappings.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, $h : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. A mapping $f : K \longrightarrow (0,+\infty)$ is said to be generalized relative semi-$(r;m,h)$-preinvex, if

$$f\big(m\varphi(x) + t\eta(\varphi(y), \varphi(x), m)\big) \leq M_r(h(t); f(x), f(y), m) \tag{2.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$ and for some fixed $m \in (0,1]$, where

$$M_r(h(t); f(x), f(y), m) := \begin{cases} \Big[mh(1-t)f^r(x) + h(t)f^r(y)\Big]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \big[f(x)\big]^{mh(1-t)}\big[f(y)\big]^{h(t)}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(x)$ and $f(y)$.

Remark 2.4. In Definition 2.3, if we choose $m = r = 1$, these definition reduces to the definition considered by Noor in [32] and Preda et. al. in [8].

Remark 2.5. In Definition 2.3, if we choose $r = 1$ and $\varphi(x) = x$, then we get Definition 1.12.

Remark 2.6. For $r = 1$, let us discuss some special cases in Definition 2.3 as follows.

(I) If taking $h(t) = t$, then we get generalized relative semi-m-preinvex mappings.
(II) If taking $h(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Breckner-preinvex mappings.
(III) If taking $h(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex mappings.
(IV) If taking $h(t) = 1$, then we get generalized relative semi-(m,P)-preinvex mappings.
(V) If taking $h(t) = t(1-t)$, then we get generalized relative semi-(m,tgs)-preinvex mappings.
(VI) If taking $h(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex mappings.

Unauthenticated
Download Date | 3/1/18 2:18 PM

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

We claim the following integral identity.

Lemma 2.7. Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$, for $\eta(\varphi(b), \varphi(a), m) > 0$. Then for some fixed $m \in (0, 1]$ and any fixed $p, q > 0$, we have

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1 - t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + t\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p$$
$$\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - t\eta(\varphi(b), \varphi(a), m))^q$$
$$\times f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt$$
$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1 - t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.$$

This completes the proof of the lemma. Q.E.D.

With the help of Lemma 2.7, we have the following results.

Theorem 2.8. Let $k > 1$ and $0 < r \leq 1$. Suppose $h : [0, 1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, +\infty)$ is a continuous mapping on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$, for $\eta(\varphi(b), \varphi(a), m) > 0$. If $f^{\frac{k}{k-1}}$ is generalized relative semi-$(r; m, h)$-preinvex mappings on an open m-invex set K for some fixed $m \in (0, 1]$, then for any fixed $p, q > 0$, we have

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{1}{k}}(kp + 1, kq + 1)\left[mf^{\frac{rk}{k-1}}(a) + f^{\frac{rk}{k-1}}(b)\right]^{\frac{k-1}{rk}}$$
$$\times \left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{k-1}{k}}.$$

Unauthenticated
Download Date | 3/1/18 2:18 PM

Proof. Let $k > 1$ and $0 < r \leq 1$. Since $f^{\frac{k}{k-1}}$ is generalized relative semi-$(r; m, h)$-preinvex mappings on K, combining with Lemma 2.7, Hölder inequality, Minkowski inequality and properties of the modulus, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}}$$

$$\times \left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt\right]^{\frac{k-1}{k}}$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{1}{k}}(kp+1, kq+1)$$

$$\times \left[\int_0^1 \left[mh(1-t)f^{\frac{rk}{k-1}}(a) + h(t)f^{\frac{rk}{k-1}}(b)\right]^{\frac{1}{r}} dt\right]^{\frac{k-1}{k}}$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{1}{k}}(kp+1, kq+1)$$

$$\times \left\{ \left(\int_0^1 m^{\frac{1}{r}} h^{\frac{1}{r}}(1-t)f^{\frac{k}{k-1}}(a)dt\right)^r + \left(\int_0^1 h^{\frac{1}{r}}(t)f^{\frac{k}{k-1}}(b)dt\right)^r \right\}^{\frac{k-1}{rk}}$$

$$= \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{1}{k}}(kp+1, kq+1)\left[mf^{\frac{rk}{k-1}}(a) + f^{\frac{rk}{k-1}}(b)\right]^{\frac{k-1}{rk}}$$

$$\times \left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{k-1}{k}}.$$

So, the proof of this theorem is complete. Q.E.D.

We point out some special cases of Theorem 2.8.

Corollary 2.9. In Theorem 2.8 for $r = 1$ and $h(t) = t^s$ where $s \in [0, 1]$, we have the following inequality for generalized relative semi-(m, s)-Breckner-preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{1}{k}}(kp+1, kq+1)\left[\frac{mf^{\frac{k}{k-1}}(a) + f^{\frac{k}{k-1}}(b)}{s+1}\right]^{\frac{k-1}{k}}.$$

Unauthenticated
Download Date | 3/1/18 2:18 PM

Corollary 2.10. In Theorem 2.8 for $r = 1$ and $h(t) = t^{-s}$ where $s \in [0, 1)$, we get the following inequality for generalized relative semi-(m, s)-Godunova-Levin-Dragomir preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{1}{k}}(kp + 1, kq + 1)\left[\frac{mf^{\frac{k}{k-1}}(a) + f^{\frac{k}{k-1}}(b)}{1 - s}\right]^{\frac{k-1}{k}}.$$

Corollary 2.11. In Theorem 2.8 for $r = 1$ and $h(t) = t(1 - t)$, we obtain the following inequality for generalized relative semi-(m, tgs)-preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{1}{k}}(kp + 1, kq + 1)\left[\frac{mf^{\frac{k}{k-1}}(a) + f^{\frac{k}{k-1}}(b)}{6}\right]^{\frac{k-1}{k}}.$$

Corollary 2.12. In Theorem 2.8 for $r = 1$ and $h(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, we deduce the following inequality for generalized relative semi-m-MT-preinvex mappings:

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq \left(\frac{\pi}{4}\right)^{\frac{k-1}{k}} \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{1}{k}}(kp + 1, kq + 1)\left[mf^{\frac{k}{k-1}}(a) + f^{\frac{k}{k-1}}(b)\right]^{\frac{k-1}{k}}.$$

Theorem 2.13. Let $l \geq 1$ and $0 < r \leq 1$. Suppose $h : [0, 1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, +\infty)$ is a continuous mapping on K° with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$, for $\eta(\varphi(b), \varphi(a), m) > 0$. If f^l is generalized relative semi-$(r; m, h)$-preinvex mappings on an open m-invex set K for some fixed $m \in (0, 1]$, then for any fixed $p, q > 0$, we have

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{l-1}{l}}(p + 1, q + 1)$$

$$\times\left[mf^{rl}(a)I^r(h(t); r, p, q) + f^{rl}(b)I^r(h(t); r, q, p)\right]^{\frac{1}{rl}},$$

where

$$I(h(t); r, p, q) := \int_0^1 t^p(1 - t)^q h^{\frac{1}{r}}(1 - t)dt.$$

Unauthenticated
Download Date | 3/1/18 2:18 PM

Proof. Let $l \geq 1$ and $0 < r \leq 1$. Since f^l is generalized relative semi-$(r; m, h)$-preinvex mappings on K, combining with Lemma 2.7, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we get

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\
&= \eta(\varphi(b), \varphi(a), m)^{p+q+1} \int_0^1 \Big[t^p(1-t)^q\Big]^{\frac{l-1}{l}} \Big[t^p(1-t)^q\Big]^{\frac{1}{l}} \\
&\quad \times f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt \\
&\leq |\eta(\varphi(b), \varphi(a), m)|^{p+q+1} \left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}} \\
&\quad \times \left[\int_0^1 t^p(1-t)^q f^l(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt\right]^{\frac{1}{l}} \\
&\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{l-1}{l}}(p+1, q+1) \\
&\quad \times \left[\int_0^1 t^p(1-t)^q \Big[mh(1-t)f^{rl}(a) + h(t)f^{rl}(b)\Big]^{\frac{1}{r}} dt\right]^{\frac{1}{l}} \\
&\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{l-1}{l}}(p+1, q+1) \\
&\quad \times \left\{ \left(\int_0^1 m^{\frac{1}{r}} t^p(1-t)^q h^{\frac{1}{r}}(1-t) f^l(a)dt\right)^r + \left(\int_0^1 t^p(1-t)^q h^{\frac{1}{r}}(t) f^l(b)dt\right)^r \right\}^{\frac{1}{rl}} \\
&= \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{l-1}{l}}(p+1, q+1) \\
&\quad \times \Big[mf^{rl}(a)I^r(h(t); r, p, q) + f^{rl}(b)I^r(h(t); r, q, p)\Big]^{\frac{1}{rl}}.
\end{aligned}$$

So, the proof of this theorem is complete. Q.E.D.

Let us discuss some special cases of Theorem 2.13.

Corollary 2.14. In Theorem 2.13 for $r = 1$ and $h(t) = t^s$ with $s \in [0, 1]$, one can get the following inequality for generalized relative semi-(m, s)-Breckner-preinvex mappings:

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\
&\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{l-1}{l}}(p+1, q+1) \\
&\quad \times \Big[mf^l(a)\beta(p+1, q+s+1) + f^l(b)\beta(q+1, p+s+1)\Big]^{\frac{1}{l}}.
\end{aligned}$$

Unauthenticated
Download Date | 3/1/18 2:18 PM

Corollary 2.15. In Theorem 2.13 for $r = 1$ and $h(t) = t^{-s}$ with $s \in (0, 1]$, we deduce the following inequality for generalized relative semi-(m, s)-Godunova-Levin-Dragomir preinvex mappings:

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx\\ &\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{l-1}{l}}(p + 1, q + 1)\\ &\quad \times \left[mf^l(a)\beta(p + 1, q - s + 1) + f^l(b)\beta(q + 1, p - s + 1)\right]^{\frac{1}{l}}.\end{aligned}$$

Corollary 2.16. In Theorem 2.13 for $r = 1$ and $h(t) = t(1 - t)$, one can obtain the following inequality for generalized relative semi-(m, tgs)-preinvex mappings:

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx\\ &\leq \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{l-1}{l}}(p + 1, q + 1)\beta^{\frac{1}{l}}(p + 2, q + 2)\left[mf^l(a) + f^l(b)\right]^{\frac{1}{l}}.\end{aligned}$$

Corollary 2.17. In Theorem 2.13 for $r = 1$ and $h(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, we derive the following inequality for generalized relative semi-m-MT-preinvex mappings:

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx\\ &\leq \left(\frac{1}{2}\right)^{\frac{1}{l}} \eta^{p+q+1}(\varphi(b), \varphi(a), m)\beta^{\frac{l-1}{l}}(p + 1, q + 1)\\ &\quad \times \left[mf^l(a)\beta\left(p + \frac{1}{2}, q + \frac{3}{2}\right) + f^l(b)\beta\left(q + \frac{1}{2}, p + \frac{3}{2}\right)\right]^{\frac{1}{l}}.\end{aligned}$$

For establishing our second main results regarding generalizations of Hermite-Hadamard, Ostrowski and Simpson type inequalities associated with generalized relative semi-$(r; m, h)$-preinvexity via fractional integrals, we need the following new crucial lemma.

Lemma 2.18. Let $\varphi : I \longrightarrow K$ be a continuous function. Suppose $K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$ and let $\eta(\varphi(b), \varphi(a), m) > 0$. Assume that $f : K \longrightarrow \mathbb{R}$ be a twice differentiable mapping on K° and $f'' \in L_1(K)$. Then for any $\lambda \in [0, 1]$ and $\alpha > 0$, the following identity holds:

$$\begin{aligned}&I_{f,\eta,\varphi}(x; \lambda, \alpha, m, a, b)\\ &= \frac{\eta^{\alpha+2}(\varphi(x), \varphi(a), m)}{\eta(\varphi(b), \varphi(a), m)} \int_0^1 t(\lambda - t^\alpha) f''(m\varphi(a) + t\eta(\varphi(x), \varphi(a), m))dt\\ &+ \frac{\eta^{\alpha+2}(\varphi(x), \varphi(b), m)}{\eta(\varphi(b), \varphi(a), m)} \int_0^1 t(\lambda - t^\alpha) f''(m\varphi(b) + t\eta(\varphi(x), \varphi(b), m))dt\end{aligned}$$

Unauthenticated
Download Date | 3/1/18 2:18 PM

$$= \frac{\lambda-1}{\eta(\varphi(b),\varphi(a),m)}\Big\{\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))$$
$$+\eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))\Big\}$$
$$+\frac{1+\alpha-\lambda}{\eta(\varphi(b),\varphi(a),m)}\Big\{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))$$
$$+\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))\Big\}$$
$$+\frac{\lambda}{\eta(\varphi(b),\varphi(a),m)}\Big\{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a))+\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b))\Big\}$$
$$-\frac{\Gamma(\alpha+2)}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\Big[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))^-}f(m\varphi(a))+J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))^-}f(m\varphi(b))\Big]. \tag{2.2}$$

Proof. A simple proof of the equality (2.2) can be done by performing two integration by parts in the integrals above and changing the variable. The details are left to the interested reader. This completes the proof of the lemma. Q.E.D.

Using Lemma 2.18, we now state the following theorems for the corresponding version for power of second derivative.

Theorem 2.19. Suppose $h:[0,1]\longrightarrow[0,+\infty)$ and $\varphi: I\longrightarrow K$ are continuous functions. Let $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$ and let $\eta(\varphi(b),\varphi(a),m)>0$. Assume that $f: K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow(0,+\infty)$ be a twice differentiable mapping on K° (the interior of K). If $0<r\leq 1$ and $(f''(x))^q$ is generalized relative semi-$(r;m,h)$-preinvex mappings on K, $q>1$, $p^{-1}+q^{-1}=1$, then for any $\lambda\in[0,1]$ and $\alpha>0$, the following inequality for fractional integrals holds:

$$|I_{f,\eta,\varphi}(x;\lambda,\alpha,m,a,b)|\leq\frac{B^{\frac{1}{p}}(\alpha,\lambda,p)}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{1}{q}}$$
$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m(f''(a))^{rq}+(f''(x))^{rq}\Big]^{\frac{1}{rq}} \tag{2.3}$$
$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m(f''(b))^{rq}+(f''(x))^{rq}\Big]^{\frac{1}{rq}}\Bigg\},$$

where

$$B(\alpha,\lambda,p):=\int_0^1|t(\lambda-t^{\alpha})|^p dt. \tag{2.4}$$

Proof. Suppose that $q>1$ and $0<r\leq 1$. Using relation (2.2), generalized relative semi-$(r;m,h)$-preinvexity of $(f''(x))^q$, Hölder inequality, Minkowski inequality and properties of the modulus, we have

Unauthenticated
Download Date | 3/1/18 2:18 PM

$$
\begin{aligned}
&|I_{f,\eta,\varphi}(x;\lambda,\alpha,m,a,b)| \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |t(\lambda-t^\alpha)||f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |t(\lambda-t^\alpha)||f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 |t(\lambda-t^\alpha)|^p dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 (f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m)))^q\,dt\right)^{\frac{1}{q}} \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 |t(\lambda-t^\alpha)|^p dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 (f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m)))^q\,dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}B^{\frac{1}{p}}(\alpha,\lambda,p) \\
&\times\left(\int_0^1 \left[mh(1-t)(f''(a))^{rq}+h(t)(f''(x))^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}} \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}B^{\frac{1}{p}}(\alpha,\lambda,p) \\
&\times\left(\int_0^1 \left[mh(1-t)(f''(b))^{rq}+h(t)(f''(x))^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}B^{\frac{1}{p}}(\alpha,\lambda,p) \\
&\times\left\{\left(\int_0^1 m^{\frac{1}{r}}h^{\frac{1}{r}}(1-t)(f''(a))^q dt\right)^r+\left(\int_0^1 h^{\frac{1}{r}}(t)(f''(x))^q dt\right)^r\right\}^{\frac{1}{rq}} \\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}B^{\frac{1}{p}}(\alpha,\lambda,p) \\
&\times\left\{\left(\int_0^1 m^{\frac{1}{r}}h^{\frac{1}{r}}(1-t)(f''(b))^q dt\right)^r+\left(\int_0^1 h^{\frac{1}{r}}(t)(f''(x))^q dt\right)^r\right\}^{\frac{1}{rq}}
\end{aligned}
$$

Unauthenticated
Download Date | 3/1/18 2:18 PM

$$= \frac{B^{\frac{1}{p}}(\alpha,\lambda,p)}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{1}{q}}$$
$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\left[m(f''(a))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right.$$
$$\left.+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\left[m(f''(b))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right\}.$$

So, the proof of this theorem is complete. Q.E.D.

Let us discuss some special cases of Theorem 2.19.

Corollary 2.20. In Theorem 2.19, if we choose $x=\frac{a+b}{2}$, $m=1$ and $\eta(\varphi(y),\varphi(x),1)=\varphi(y)-\varphi(x)$, we get the following generalized Simpson type inequality for fractional integrals:

$$\left|I_{f,\varphi}\left(\frac{a+b}{2};\lambda,\alpha,1,a,b\right)\right| \le \frac{B^{\frac{1}{p}}(\alpha,\lambda,p)}{(\varphi(b)-\varphi(a))}\left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{1}{q}}$$
$$\times\left\{\left(\varphi\left(\frac{a+b}{2}\right)-\varphi(a)\right)^{\alpha+2}\left[(f''(a))^{rq}+\left(f''\left(\frac{a+b}{2}\right)\right)^{rq}\right]^{\frac{1}{rq}}\right.$$
$$\left.+\left(\varphi(b)-\varphi\left(\frac{a+b}{2}\right)\right)^{\alpha+2}\left[(f''(b))^{rq}+\left(f''\left(\frac{a+b}{2}\right)\right)^{rq}\right]^{\frac{1}{rq}}\right\}. \tag{2.5}$$

Corollary 2.21. In Theorem 2.19, if we choose $\lambda=m=1$ and $\eta(\varphi(y),\varphi(x),1)=\varphi(y)-\varphi(x)$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:

$$\left|I_{f,\varphi}(x;1,\alpha,1,a,b)\right| \le \left(\frac{\alpha}{2(\alpha+2)}\right)^{\frac{1}{p}}\left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{1}{q}}\frac{1}{(\varphi(b)-\varphi(a))}$$
$$\times\left\{(\varphi(x)-\varphi(a))^{\alpha+2}\left[(f''(a))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right.$$
$$\left.+(\varphi(b)-\varphi(x))^{\alpha+2}\left[(f''(b))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right\}. \tag{2.6}$$

Corollary 2.22. In Theorem 2.19, if we choose $\lambda=0$, $m=1$ and $\eta(\varphi(y),\varphi(x),1)=\varphi(y)-\varphi(x)$, we get the following generalized Ostrowski type inequality for fractional integrals:

$$\left|I_{f,\varphi}(x;0,\alpha,1,a,b)\right| \le \frac{1}{\left[p(\alpha+1)+1\right]^{\frac{1}{p}}}\left(\int_0^1 h^{\frac{1}{r}}(t)dt\right)^{\frac{1}{q}}\frac{1}{(\varphi(b)-\varphi(a))}$$
$$\times\left\{(\varphi(x)-\varphi(a))^{\alpha+2}\left[(f''(a))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\right.$$

Unauthenticated
Download Date | 3/1/18 2:18 PM

$$+(\varphi(b)-\varphi(x))^{\alpha+2}\left[(f''(b))^{rq}+(f''(x))^{rq}\right]^{\frac{1}{rq}}\Bigg\}. \tag{2.7}$$

Theorem 2.23. Suppose $h:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous functions. Let $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$ and let $\eta(\varphi(b),\varphi(a),m)>0$. Assume that $f:K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow(0,+\infty)$ be a twice differentiable mapping on K°. If $0<r\leq 1$ and $(f''(x))^q$ is generalized relative semi-$(r;m,h)$-preinvex mappings on K, $q\geq 1$, then for any $\lambda\in[0,1]$ and $\alpha>0$, the following inequality for fractional integrals holds:

$$\begin{aligned}|I_{f,\eta,\varphi}(x;\lambda,\alpha,m,a,b)|&\leq\frac{E^{1-\frac{1}{q}}}{\eta(\varphi(b),\varphi(a),m)}\\&\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m(f''(a))^{rq}(F_1+F_2)^r+(f''(x))^{rq}(G_1+G_2)^r\Big]^{\frac{1}{rq}}\\&+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m(f''(b))^{rq}(F_1+F_2)^r+(f''(x))^{rq}(G_1+G_2)^r\Big]^{\frac{1}{rq}}\Bigg\},\end{aligned} \tag{2.8}$$

where

$$E=E(\alpha,\lambda):=\frac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\frac{\lambda}{2};$$

$$F_1=F_1(h(t);\alpha,\lambda,r):=\int_0^{\lambda^{\frac{1}{\alpha}}}t(\lambda-t^{\alpha})h^{\frac{1}{r}}(1-t)dt;$$

$$F_2=F_2(h(t);\alpha,\lambda,r):=\int_{\lambda^{\frac{1}{\alpha}}}^1t(t^{\alpha}-\lambda)h^{\frac{1}{r}}(1-t)dt;$$

$$G_1=G_1(h(t);\alpha,\lambda,r):=\int_0^{\lambda^{\frac{1}{\alpha}}}t(\lambda-t^{\alpha})h^{\frac{1}{r}}(t)dt;$$

$$G_2=G_2(h(t);\alpha,\lambda,r):=\int_{\lambda^{\frac{1}{\alpha}}}^1t(t^{\alpha}-\lambda)h^{\frac{1}{r}}(t)dt.$$

Proof. Suppose that $q\geq 1$ and $0<r\leq 1$. Using relation (2.2), generalized relative semi-$(r;m,h)$-preinvexity of $(f''(x))^q$, the well-known power mean inequality, Minkowski inequality and properties of the modulus, we have

$$\begin{aligned}&|I_{f,\eta,\varphi}(x;\lambda,\alpha,m,a,b)|\\&\leq\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1|t(\lambda-t^{\alpha})||f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt\\&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\int_0^1|t(\lambda-t^{\alpha})||f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt\end{aligned}$$

Unauthenticated
Download Date | 3/1/18 2:18 PM

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 |t(\lambda-t^\alpha)|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 |t(\lambda-t^\alpha)|\left(f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))\right)^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 |t(\lambda-t^\alpha)|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 |t(\lambda-t^\alpha)|\left(f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))\right)^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}E^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 |t(\lambda-t^\alpha)|\left[mh(1-t)(f''(a))^{rq}+h(t)(f''(x))^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}E^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 |t(\lambda-t^\alpha)|\left[mh(1-t)(f''(b))^{rq}+h(t)(f''(x))^{rq}\right]^{\frac{1}{r}}dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}E^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}h^{\frac{1}{r}}(1-t)|t(\lambda-t^\alpha)|(f''(a))^q dt\right)^r+\left(\int_0^1 h^{\frac{1}{r}}(t)|t(\lambda-t^\alpha)|(f''(x))^q dt\right)^r\right\}^{\frac{1}{rq}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}E^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1 m^{\frac{1}{r}}h^{\frac{1}{r}}(1-t)|t(\lambda-t^\alpha)|(f''(b))^q dt\right)^r+\left(\int_0^1 h^{\frac{1}{r}}(t)|t(\lambda-t^\alpha)|(f''(x))^q dt\right)^r\right\}^{\frac{1}{rq}}$$

$$=\frac{E^{1-\frac{1}{q}}}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\left[m(f''(a))^{rq}\left(F_1+F_2\right)^r+(f''(x))^{rq}\left(G_1+G_2\right)^r\right]^{\frac{1}{rq}}\right.$$

$$\left.+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\left[m(f''(b))^{rq}\left(F_1+F_2\right)^r+(f''(x))^{rq}\left(G_1+G_2\right)^r\right]^{\frac{1}{rq}}\right\}.$$

So, the proof of this theorem is complete. Q.E.D.

Unauthenticated
Download Date | 3/1/18 2:18 PM

Let us discuss some special cases of Theorem 2.23.

Corollary 2.24. In Theorem 2.23, if we choose $x = \dfrac{a+b}{2}$, $m = 1$ and $\eta(\varphi(y), \varphi(x), 1) = \varphi(y) - \varphi(x)$, we get the following generalized Simpson type inequality for fractional integrals:

$$\left| I_{f,\varphi}\left(\frac{a+b}{2}; \lambda, \alpha, 1, a, b\right)\right| \leq \frac{E^{1-\frac{1}{q}}}{(\varphi(b)-\varphi(a))}$$

$$\times\Bigg\{\left(\varphi\left(\frac{a+b}{2}\right)-\varphi(a)\right)^{\alpha+2}\left[(f''(a))^{rq}(F_1+F_2)^r+\left(f''\left(\frac{a+b}{2}\right)\right)^{rq}(G_1+G_2)^r\right]^{\frac{1}{rq}}$$

$$+\left(\varphi(b)-\varphi\left(\frac{a+b}{2}\right)\right)^{\alpha+2}\left[(f''(b))^{rq}(F_1+F_2)^r+\left(f''\left(\frac{a+b}{2}\right)\right)^{rq}(G_1+G_2)^r\right]^{\frac{1}{rq}}\Bigg\}. \tag{2.9}$$

Corollary 2.25. In Theorem 2.23, if we choose $\lambda = m = 1$ and $\eta(\varphi(y), \varphi(x), 1) = \varphi(y) - \varphi(x)$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:

$$\left|I_{f,\varphi}(x; 1, \alpha, 1, a, b)\right| \leq \left(\frac{\alpha}{2(\alpha+2)}\right)^{1-\frac{1}{q}} \frac{1}{(\varphi(b)-\varphi(a))}$$

$$\times\Bigg\{(\varphi(x)-\varphi(a))^{\alpha+2}\left[(f''(a))^{rq}F^r+(f''(x))^{rq}G^r\right]^{\frac{1}{rq}}$$

$$+(\varphi(b)-\varphi(x))^{\alpha+2}\left[(f''(b))^{rq}F^r+(f''(x))^{rq}G^r\right]^{\frac{1}{rq}}\Bigg\}, \tag{2.10}$$

where

$$F = F(h(t); \alpha, r) := \int_0^1 t(1-t^\alpha)h^{\frac{1}{r}}(1-t)dt;$$

$$G = G(h(t); \alpha, r) := \int_0^1 t(1-t^\alpha)h^{\frac{1}{r}}(t)dt.$$

Corollary 2.26. In Theorem 2.23, if we choose $\lambda = 0$, $m = 1$ and $\eta(\varphi(y), \varphi(x), 1) = \varphi(y) - \varphi(x)$, we get the following generalized Ostrowski type inequality for fractional integrals:

$$\left|I_{f,\varphi}(x; 0, \alpha, 1, a, b)\right| \leq \frac{1}{(\alpha+2)^{1-\frac{1}{q}}} \frac{1}{(\varphi(b)-\varphi(a))}$$

$$\times\Bigg\{(\varphi(x)-\varphi(a))^{\alpha+2}\left[(f''(a))^{rq}\overline{F}^r+(f''(x))^{rq}\overline{G}^r\right]^{\frac{1}{rq}}$$

$$+(\varphi(b)-\varphi(x))^{\alpha+2}\left[(f''(b))^{rq}\overline{F}^r+(f''(x))^{rq}\overline{G}^r\right]^{\frac{1}{rq}}\Bigg\}, \tag{2.11}$$

Unauthenticated
Download Date | 3/1/18 2:18 PM

where

$$\overline{F} = \overline{F}(h(t);\alpha,r) := \int_0^1 t^{\alpha+1} h^{\frac{1}{r}}(1-t)dt;$$

$$\overline{G} = \overline{G}(h(t);\alpha,r) := \int_0^1 t^{\alpha+1} h^{\frac{1}{r}}(t)dt.$$

Remark 2.27. In Theorems 2.19 and 2.23, if taking $r=1$ and $h(t)=t^s, h(t)=t^{-s}, h(t)=t(1-t)$ or $h(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then one can get some special fractional integral inequalities for generalized relative semi-(m,s)-Breckner-preinvex mappings, generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex mappings, generalized relative semi-(m,tgs)-preinvex mappings and generalized relative semi-m-MT-preinvex mappings, respectively. Also, using Theorems 2.19 and 2.23 some interesting applications to special means can be deduced. The details are left to the interested reader.

3 Conclusions

In this article, we first presented some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized relative semi-$(r;m,h)$-preinvex mappings. And then, a new identity concerning twice differentiable mappings defined on m-invex set is derived. By using the notion of generalized relative semi-$(r;m,h)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard, Ostrowski and Simpson type inequalities via fractional integrals are established. It is pointed out that some new special cases are deduced from main results of the article. Motivated by this new interesting class of generalized relative semi-$(r;m,h)$-preinvex mappings we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard, Ostrowski and Simpson type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k-fractional integrals, local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p,q)-calculus, time scale calculus and conformable fractional integrals.

References

[1] T. Antczak, Mean value in invexity analysis, *Nonlinear Anal.*, **60**, (2005), 1473-1484.

[2] F. Chen, A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals, *Ital. J. Pure Appl. Math.*, **33**, (2014), 299-306.

[3] F. X. Chen and S. H. Wu, Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions, *J. Nonlinear Sci. Appl.*, **9**(2), (2016), 705-716.

[4] Y. M. Chu, M. A. Khan, T. U. Khan and T. Ali, Generalizations of Hermite-Hadamard type inequalities for MT-convex functions, *J. Nonlinear Sci. Appl.*, **9**(5), (2016), 4305-4316.

[5] Y. M. Chu, G. D. Wang and X. H. Zhang, Schur convexity and Hadamard's inequality, *Math. Inequal. Appl.*, **13**(4), (2010), 725-731.

Unauthenticated
Download Date | 3/1/18 2:18 PM

[6] S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.*, **21**, (1995), 335-341.

[7] T. S. Du, J. G. Liao and Y. J. Li, Properties and integral inequalities of Hadamard-Simpson type for the generalized (s,m)-preinvex functions, *J. Nonlinear Sci. Appl.*, **9**, (2016), 3112-3126.

[8] C. Fulga and V. Preda, Nonlinear programming with φ-preinvex and local φ-preinvex functions, *Eur. J. Oper. Res.*, **192**, (2009), 737-743.

[9] A. Kashuri and R. Liko, Ostrowski type fractional integral inequalities for generalized (s,m,φ)-preinvex functions, *Aust. J. Math. Anal. Appl.*, **13**(1), (2016), Article 16, 1-11.

[10] A. Kashuri and R. Liko, Ostrowski type fractional integral inequalities for generalized (g,s,m,φ)-preinvex functions, *Extr. Math.*, **32**(1), (2017), 105-123.

[11] A. Kashuri and R. Liko, Generalizations of Hermite-Hadamard and Ostrowski type inequalities for MT_m-preinvex functions, *Proyecciones*, **36**(1), (2017), 45-80.

[12] A. Kashuri and R. Liko, Hermite-Hadamard type inequalities for MT_m-preinvex functions, *Fasc. Math.*, **58**(58), (2017), 77-96.

[13] A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for generalized $(r;s,m,\varphi)$-preinvex functions, *Eur. J. Pure Appl. Math.*, **10**(3), (2017), 495-505.

[14] A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for twice differentiable generalized (s,m,φ)-preinvex functions, *Konuralp J. Math.*, **5**(2), (2017), 228-238.

[15] A. Kashuri and R. Liko, Some new Ostrowski type fractional integral inequalities for generalized $(r;g,s,m,\varphi)$-preinvex functions via Caputo k-fractional derivatives, *Int. J. Nonlinear Anal. Appl.*, **8**(2), (2017), 109-124.

[16] A. Kashuri and R. Liko, Hermite-Hadamard type inequalities for generalized (s,m,φ)-preinvex functions via k-fractional integrals, *Tbil. Math. J.*, **10**(4), (2017), 73-82.

[17] A. Kashuri and R. Liko, Ostrowski type fractional integral operators for generalized $(r;s,m,\varphi)$-preinvex functions, *Appl. Appl. Math.*, **12**(2), (2017), 1017-1035.

[18] A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for $MT_{(m,\varphi)}$-preinvex functions, *Stud. Univ. Babeş-Bolyai, Math.*, **62**(4), (2017), 439-450.

[19] A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for twice differentiable generalized beta-preinvex functions, *J. Fract. Calc. Appl.*, **9**(1), (2018), 241-252.

[20] A. Kashuri, R. Liko and T. S. Du, Ostrowski type fractional integral operators for generalized beta (r,g)-preinvex functions, *Khayyam J. Math.*, **4**(1), (2018), 39-58.

[21] H. Kavurmaci, M. Avci and M. E. Özdemir, New inequalities of Hermite-Hadamard type for convex functions with applications, *arXiv:1006.1593v1 [math. CA]*, (2010), 1-10.

Unauthenticated
Download Date | 3/1/18 2:18 PM

[22] H. Kavurmaci, M. Avci and M. E. Özdemir, New inequalities of Hermite-Hadamard type for convex functions with applications, *J. Inequal. Appl.*, **86**, (2011), 1–11.

[23] M. Adil Khan, Y. Khurshid and T. Ali, Hermite-Hadamard inequality for fractional integrals via η-convex functions, *Acta Math. Univ. Comenianae*, **79**(1), (2017), 153-164.

[24] M. Adil Khan, Y. Khurshid, T. Ali and N. Rehman, Inequalities for three times differentiable functions, *J. Math., Punjab Univ.*, **48**(2), (2016), 35-48.

[25] M. Adil Khan, Y. M. Chu, A. Kashuri, R. Liko and G. Ali, New Hermite-Hadamard inequalities for conformable fractional integrals, *J. Funct. Spaces*, (In press).

[26] W. Liu, New integral inequalities involving beta function via P-convexity, *Miskolc Math. Notes*, **15**(2), (2014), 585-591.

[27] W. J. Liu, Some Simpson type inequalities for h-convex and (α, m)-convex functions, *J. Comput. Anal. Appl.*, **16**(5), (2014), 1005-1012.

[28] W. Liu, W. Wen and J. Park, Ostrowski type fractional integral inequalities for MT-convex functions, *Miskolc Math. Notes*, **16**(1), (2015), 249-256.

[29] W. Liu, W. Wen and J. Park, Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals, *J. Nonlinear Sci. Appl.*, **9**, (2016), 766-777.

[30] C. Luo, T. S. Du, M. A. Khan, A. Kashuri and Y. Shen, Some k-fractional integrals inequalities through generalized φ-m-MT-preinvexity, *J. Comput. Anal. Appl.*, **240**, (2019), (Accepted paper).

[31] M. Matłoka, Inequalities for h-preinvex functions, *Appl. Math. Comput.*, **234**, (2014), 52-57.

[32] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, *J. Math. Anal. Approx. Theory*, **2**, (2007), 126-131.

[33] M. A. Noor, K. I. Noor, M. U. Awan and S. Khan, Hermite-Hadamard type inequalities for differentiable h_φ-preinvex functions, *Arab. J. Math.*, **4**, (2015), 63-76.

[34] O. Omotoyinbo and A. Mogbodemu, Some new Hermite-Hadamard integral inequalities for convex functions, *Int. J. Sci. Innovation Tech.*, **1**(1), (2014), 1-12.

[35] M. E. Özdemir, E. Set and M. Alomari, Integral inequalities via several kinds of convexity, *Creat. Math. Inform.*, **20**(1), (2011), 62-73.

[36] B. G. Pachpatte, On some inequalities for convex functions, *RGMIA Res. Rep. Coll.*, **6**, (2003).

[37] C. Peng, C. Zhou and T. S. Du, Riemann-Liouville fractional Simpson's inequalities through generalized (m, h_1, h_2)-preinvexity, *Ital. J. Pure Appl. Math.*, **38**, (2017), 345-367.

[38] R. Pini, Invexity and generalized convexity, *Optimization*, **22**, (1991), 513-525.

[39] F. Qi and B. Y. Xi, Some integral inequalities of Simpson type for $GA-\varepsilon$-convex functions, *Georgian Math. J.*, **20**(5), (2013), 775-788.

Unauthenticated
Download Date | 3/1/18 2:18 PM

[40] E. Set, M. A. Noor, M. U. Awan and A. Gözpinar, Generalized Hermite-Hadamard type inequalities involving fractional integral operators, *J. Inequal. Appl.*, **169**, (2017), 1-10.

[41] H. N. Shi, Two Schur-convex functions related to Hadamard-type integral inequalities, *Publ. Math. Debrecen*, **78**(2), (2011), 393-403.

[42] D. D. Stancu, G. Coman and P. Blaga, Analiză numerică şi teoria aproximării, *Cluj-Napoca: Presa Universitară Clujeană.*, **2**, (2002).

[43] M. Tunç, Ostrowski type inequalities for functions whose derivatives are MT-convex, *J. Comput. Anal. Appl.*, **17**(4), (2014), 691-696.

[44] M. Tunç, E. Göv and Ü. Şanal, On *tgs*-convex function and their inequalities, *Facta Univ. Ser. Math. Inform.*, **30**(5), (2015), 679-691.

[45] S. Varošanec, On h-convexity, *J. Math. Anal. Appl.*, **326**(1), (2007), 303-311.

[46] Y. Wang, M. M. Zheng and F. Qi, Integral inequalities of Hermite-Hadamard type for functions whose derivatives are α-preinvex, *J. Inequal. Appl.*, **97**, (2014), 1-10.

[47] J. Wang, C. Zhu and Y. Zhou, New generalized Hermite-Hadamard type inequalities and applications to special means, *J. Inequal. Appl.*, **325**, (2013), 1-15.

[48] E. A. Youness, E-convex sets, E-convex functions, and E-convex programming, *J. Optim. Theory Appl.*, **102**, (1999), 439-450.

[49] Y. Zhang and J. Wang, On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals, *J. Inequal. Appl.*, **220**, (2013), 1-27.

[50] X. M. Zhang, Y. M. Chu and X. H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, *J. Inequal. Appl.*, (2010), Article ID 507560, 11 pages.

Unauthenticated
Download Date | 3/1/18 2:18 PM