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Some sequence spaces of Invariant means and lacunary defined by a Musielak-Orlicz function over n-normed spaces

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Abstract

In the present paper we introduce some sequence spaces combining lacunary sequence, invariant means over n-normed spaces defined by Musielak-Orlicz function $\mathcal{M} = (M_k)$. We study some topological properties and also prove some inclusion results between these spaces.

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1 Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler [4] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [14]. Since then, many others have studied this concept and obtained various results, see Gunawan ([5],[6]) and Gunawan and Mashadi [7]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:ĉ

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$. For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

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defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
The notion of difference sequence spaces was introduced by Kızmaz [9], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et. and Çolak [2] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, v be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_v^m) = \{x = (x_k) \in w : (\Delta_v^m x_k) \in Z\},$$

where $\Delta_v^m x = (\Delta_v^m x_k) = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_v^m x_k = \sum_{s=0}^{m} (-1)^s \begin{pmatrix} m \\ s \end{pmatrix} x_{k+vs}.$$

Taking $v = 1$, we get the spaces which were introduced and studied by Et. and Çolak [2]. Taking $m = v = 1$, we get the spaces which were studied by Kızmaz [9].
Let σ be the mapping of the set of positive integers into itself. A continuous linear functional φ on l_∞, is said to be an invariant mean or σ-mean if and only if

1. $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k,
2. $\varphi(e) = 1$, where $e = (1, 1, 1, \cdots)$ and
3. $\varphi(x_{\sigma(k)}) = \varphi(x)$ for all $x \in l_\infty$.

If $x = (x_n)$, write $Tx = Tx_n = (x_{\sigma(n)})$. It can be shown in [31] that

$$V_\sigma = \Big\{x \in l_\infty : \lim_k t_{kn}(x) = l, \quad \text{uniformly in } n,\ l = \sigma - \lim x\Big\},$$

where

$$t_{kn}(x) = \frac{x_n + x_{\sigma^1 n} + ... + x_{\sigma^k n}}{k+1}.$$

In the case σ is the translation mapping $n \to n+1$, σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences see[10].
By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted

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by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence was defined by Freedman et al [3].
Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [33], Theorem 10.4.2, pp. 183). For more details about sequence spaces (see [1], [8], [12], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [26], [27], [28], [29], [30], [32]) and reference therein.
An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [11] used the idea of Orlicz function to define the sequence space,

$$l_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called as an Orlicz sequence space. Also l_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also, it was shown in [11] that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function (see [13],[25]). A sequence $\mathcal{N} = (N_k)$ is called a complementary function of a Musielak-Orlicz function $\mathcal{M}$

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, ...$$

For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_M(cx) < \infty, \quad \text{for some } \ c > 0\Big\},$$

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$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty, \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. By $S(n-X)$ we denote the space of all sequences defined over $(X, ||\cdot, \cdots, \cdot||)$. In this paper we define the following sequence spaces :

$w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0,$$

$$\text{uniformly in } n \text{ for some } \rho > 0\Big\},$$

$w_\sigma\Big[\mathcal{M}, u, p, ||., .||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0,$$

$$\text{uniformly in } n \text{ for some } l \text{ and } \rho > 0,\Big\}$$

and

$w_\sigma^\infty\Big[\mathcal{M}, u, p, ||., .||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \sup_{r,n} \frac{1}{h_r} \sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \infty,$$

$$\text{uniformly in } n \text{ for some } \rho > 0\Big\}.$$

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If we take $\mathcal{M}(x) = x$, we get the spaces
$w_\sigma^0\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} = 0,$$

$$\text{uniformly in } \ n \ \text{ for some } \ \rho > 0\Big\},$$

$w_\sigma\Big[u, p, ||., .||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} = 0,$$

$$\text{uniformly in } \ n \ \text{ for some } \ l \ \text{ and } \ \rho > 0\Big\}$$

and
$w_\sigma^\infty\Big[u, p, ||., .||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \sup_{r,n} \frac{1}{h_r} \sum_{k\in I_r} u_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} < \infty,$$

$$\text{uniformly in } \ n \ \text{ for some } \ \rho > 0\Big\}.$$

If we take $p = (p_k) = 1$, we get the spaces
$w_\sigma^0\Big[\mathcal{M}, u, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0,$$

$$\text{uniformly in } \ n \ \text{ for some } \ \rho > 0\Big\},$$

$w_\sigma\Big[\mathcal{M}, u, ||., .||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0,$$

$$\text{uniformly in } \ n \ \text{ for some } \ l \ \text{ and } \ \rho > 0\Big\}$$

and
$w_\sigma^\infty\Big[\mathcal{M}, u, ||., .||\Big]_\theta(\Delta_v^m) =$

$$\Big\{x \in S(n-X) : \sup_{r,n} \frac{1}{h_r} \sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] < \infty,$$

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$$\text{uniformly in } n \text{ for some } \rho > 0\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main aim of the present paper is to study some topological properties and prove some inclusion relations between above defined sequence spaces.

2 Main results

Theorem 2.1 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. Then the classes of sequences*

$$w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m), \quad w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \quad \text{and} \quad w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$$

are linear spaces over the field of complex numbers $\mathbb{C}$.

Proof. The proof is obvious, so we omit it. Q.E.D.

Theorem 2.2 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. Then $w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$ is a topological linear space paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, n = 1, 2, \cdots\Big\},$$

where $H = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x) \geq 0$ for $x = (x_k) \in w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Conversely, suppose that $g(x) = 0$, then

$$\inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, n = 1, 2, \cdots\Big\} = 0.$$

This implies that for a given $\varepsilon > 0$, there exists some $\rho_\varepsilon (0 < \rho_\varepsilon < \varepsilon)$ such that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho_\varepsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1.$$

Thus

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\varepsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}$$

$$\leq \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho_\varepsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1,$$

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for each r and n. Suppose that $x_k \neq 0$ for each $k \in \mathbb{N}$. This implies that $t_{kn}(\Delta_v^m x_k) \neq 0$, for each $k, n \in \mathbb{N}$. Let $\varepsilon \to 0$, then $||\frac{t_{kn}(\Delta_v^m x_k)}{\varepsilon}, z_1, \cdots, z_{n-1}|| \to \infty$. It follows that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\varepsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \to \infty,$$

which is a contradiction. Therefore, $t_{kn}(\Delta_v^m x_k) = 0$ for each k and thus $\Delta_v^m x_k = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

and

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m y_k)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

for each r . Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have
$\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m(x_k+y_k))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}$

$$\begin{aligned}
&\leq \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k) + t_{kn}(\Delta_v^m y_k)}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\leq \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[\frac{\rho_1}{\rho_1+\rho_2}u_k M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big) \\
&+ \frac{\rho_2}{\rho_1+\rho_2}u_k M_k\Big(||\frac{t_{kn}(\Delta_v^m y_k)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m y_k)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\leq 1.
\end{aligned}$$

Since ρ's are non-negative, so we have
$g(x+y)$

$$\begin{aligned}
&= \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k) + t_{kn}(\Delta_v^m y_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\qquad\qquad \leq 1, \;\; r, n = 1, 2, \cdots\Big\} \\
&\leq \inf\Big\{\rho_1^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, n = 1, 2, \cdots\Big\} \\
&+ \inf\Big\{\rho_2^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m y_k)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, n = 1, 2, \cdots\Big\}.
\end{aligned}$$

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Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m \lambda x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, n = 1, 2, \cdots\Big\}.$$

Then

$$g(\lambda x_k) = \inf\Big\{(|\lambda|t)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r, n = 1, 2, \cdots\Big\},$$

where $t = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_r} \leq \max(1, |\lambda|^{\sup p_r})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{\sup p_r}) \inf\Big\{t^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}$$

$$\leq 1, r, n = 1, 2, \cdots\Big\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality.
This completes the proof of the theorem. Q.E.D.

Theorem 2.3 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. Then*

$$w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m).$$

Proof. The first inclusion is obvious. We will show that

$$w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m).$$

Let $x \in w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Then there exists some positive number ρ_1 such that

$$\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \to 0 \text{ as } r \to \infty \text{ uniformly in } n.$$

Define $\rho = 2\rho_1$. Since $\mathcal{M} = (M_k)$ is non-decreasing, convex and so by using inequality (1.1), we have

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$$\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

$$\leq \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$
$$+ \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{l}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$
$$\leq \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$
$$+ K\max\Big\{1, u_k\Big[M_k\Big(||\frac{l}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^H\Big\}.$$

Thus $x \in w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Q.E.D.

Theorem 2.4 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. If $\sup_k \Big[M_k(t)\Big]^{p_k} < \infty$ for all $t > 0$, then*

$$w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m).$$

Proof. Let $x \in w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. By using inequality (1.1), we have

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

$$\leq \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$
$$+ \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{l}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}.$$

Since $\sup_k \Big[M_k(t)\Big]^{p_k} < \infty$, we can take that $\sup_k \Big[M_k(t)\Big]^{p_k} = T$. Hence we get $x \in w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Q.E.D.

Theorem 2.5 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function which satisfies Δ_2-condition for all k, then*

$$w_\sigma\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m).$$

Proof. Let $x \in w_\sigma\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Then we have

$$\mathcal{T}_r = \frac{1}{h_r}\sum_{k\in I_r} u_k||t_{kn}(\Delta_v^m x_k - l), z_1, \cdots, z_{n-1}||^{p_k} \to \infty \text{ as } r \to \infty \text{ uniformly in } n, \text{ for some } l.$$

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Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \varepsilon$ for $0 \leq t \leq \delta$ for all k. So that
$\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$

$$\begin{aligned} &= \frac{1}{h_r}\sum_{k\in I_r, ||t_{kn}(x-l),z||\leq\delta}^{1} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\ &+ \frac{1}{h_r}\sum_{k\in I_r, ||t_{kn}(x-l),z||>\delta}^{2} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}. \end{aligned}$$

For the first summation in the right hand side of the above equation, we have $\sum^{1} \leq \varepsilon^H$ by using continuity of M_k for all k. For the second summation, we write

$$||t_{kn}(\Delta_v^m x_k - l), z_1, \cdots, z_{n-1}|| \leq 1 + ||\frac{t_{kn}(\Delta_v^m x_k - l)}{\delta}, z_1, \cdots, z_{n-1}||.$$

Since M_k is non-decreasing and convex for all k, it follows that
$u_k\Big[M_k(||t_{kn}(\Delta_v^m x_k - l), z_1, \cdots, z_{n-1}||)\Big]$

$$\begin{aligned} &< u_k\Big[M_k\Big(1 + ||\frac{t_{kn}(\Delta_v^m x_k - l)}{\delta}, z_1, \cdots, z_{n-1}||\Big)\Big] \\ &\leq \frac{1}{2}u_k\big(M_k(2)\big) + \frac{1}{2}u_k\Big[M_k\Big((2)||\frac{t_{kn}(\Delta_v^m x_k - l)}{\delta}, z_1, \cdots, z_{n-1}||\Big)\Big]. \end{aligned}$$

Since M_k satisfies Δ_2-condition for all k, we can write

$$\begin{aligned} u_k\Big[M_k(||t_{kn}(\Delta_v^m x_k - l), z_1, \cdots, z_{n-1}||) &\leq \frac{1}{2}L||\frac{t_{kn}(\Delta_v^m x_k - l)}{\delta}, z_1, \cdots, z_{n-1}||M_k(2) \\ &+ \frac{1}{2}L||\frac{t_{kn}(\Delta_v^m x_k - l)}{\delta}, z_1, \cdots, z_{n-1}||M_k(2) \\ &= L||\frac{t_{kn}(\Delta_v^m x_k - l)}{\delta}, z_1, \cdots, z_{n-1}||M_k(2). \end{aligned}$$

So we write

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k - l)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq \varepsilon^H + [\max(1, LM_k(2))\delta]^H \mathcal{T}_r.$$

Letting $r \to \infty$, it follows that $x \in w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. This completes the proof. Q.E.D.

Theorem 2.6 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent :*
(i) $w_\sigma^\infty\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$,

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(ii) $w_\sigma^0\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^0\Big[\mathcal{M},u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m)$,

(iii) $\sup_r \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k(t)\Big]^{p_k} < \infty$ *for all* $t>0$.

Proof. (i) $\Longrightarrow$ (ii) We have only to show that $w_\sigma^0\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^\infty\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m)$. Let $x \in w_\sigma^0\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m)$. Then there exists $r \geq r_0$, for $\varepsilon>0$, such that

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)^{p_k} < \varepsilon.$$

Hence there exists $H>0$ such that

$$\sup_{r,n}\frac{1}{h_r}\sum_{k\in I_r} u_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)^{p_k} < H$$

for all n and r. So we get $x \in w_\sigma^\infty\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m)$.

(ii) $\Longrightarrow$ (iii) Suppose that (iii) does not hold. Then for some $t>0$

$$\sup_r \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k(t)\Big]^{p_k} = \infty$$

and therefore we can find a subinterval $I_{r(m)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(m)}}\sum_{k\in I_{r(m)}} u_k\Big[M_k(\frac{1}{m})\Big]^{p_k} > m, \;\; m=1,2,\cdots. \tag{2.1}$$

Let us define $x=(x_k)$ as follows, $x_k=\frac{1}{m}$ if $k\in I_{r(m)}$ and $x_k=0$ if $k\notin I_{r(m)}$. Then $x \in w_\sigma^0\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m)$ but by eqn.(2.1), $x \notin w_\sigma^0\Big[\mathcal{M},u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m)$, which contradicts (ii). Hence (iii) must hold.

(iii) $\Longrightarrow$ (i) Suppose (i) not holds, then for $x \in w_\sigma^\infty\Big[u,p,||\cdot,\cdots,\cdot||\Big]_\theta(\Delta_v^m)$, we have

$$\sup_{r,n}\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k} = \infty. \tag{2.2}$$

Let $t=||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1,\cdots,z_{n-1}||$ for each k and fixed n, so that eqn. (2.2) becomes

$$\sup_r \frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k(t)\Big]^{p_k} = \infty,$$

which contradicts (iii). Hence (i) must hold. Q.E.D.

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Theorem 2.7 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*
(i) $w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^0\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$,
(ii) $w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^\infty\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$,
(iii) $\inf\limits_r \sum\limits_{k\in I_r} u_k\Big[M_k(t)\Big]^{p_k} > 0$ *for all* $t > 0$.

Proof. (i) $\Longrightarrow$ (ii) : It is easy to prove.

(ii) $\Longrightarrow$ (iii) Suppose that (iii) does not hold. Then

$$\inf_r \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k(t)\Big]^{p_k} = 0 \quad \text{for some } \ t > 0,$$

and we can find a subinterval $I_{r(m)}$ of the set of interval I_r such that

$$\frac{1}{h_r} \sum_{k\in I_{r(m)}} u_k[M_k(m)]^{p_k} < \frac{1}{m}, \ \ m = 1, 2, \cdots . \tag{2.3}$$

Let us define $x_k = m$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Thus by eqn.(2.3), $x \in w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$ but $x \notin w_\sigma^\infty\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Longrightarrow$ (i) It is obvious. Q.E.D.

Theorem 2.8 *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. Then* $w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^0\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$ *if and only if*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k(t)\Big]^{p_k} = \infty. \tag{2.4}$$

Proof. Let $w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^0\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Suppose that eqn. (2.4) does not hold. Therefore there is a subinterval $I_{r(m)}$ of the set of interval I_r and a number $t_0 > 0$, where $t_0 = ||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots z_{n-1}||$ for all k and n, such that

$$\frac{1}{h_{r(m)}} \sum_{k\in I_{r(m)}} u_k[M_k(t_0)]^{p_k} \le M < \infty, \ \ m = 1, 2, \cdots . \tag{2.5}$$

Let us define $x_k = t_0$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Then, by eqn. (2.5), $x \in w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. But $x \notin w_\sigma^0\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Hence eqn. (2.5) must hold.

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Conversely, suppose that eqn. (2.5) hold and that $x \in w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Then for each r and n

$$\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq M < \infty. \tag{2.6}$$

Now suppose that $x \notin w_\sigma^0\Big[u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Then for some number $\varepsilon > 0$ and for a subinterval I_{ri} of the set of interval I_r, there is k_0 such that $||t_{kn}(\Delta_v^m x_k), z_1, \cdots, z_{n-1}||^{p_k} > \varepsilon$ for $k \geq k_0$. From the properties of sequence of Orlicz functions, we obtain

$$u_k\Big[M_k\Big(\frac{\varepsilon}{\rho}\Big)\Big]^{p_k} \leq u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

which contradicts eqn.(2.5), by using eqn. (2.6). This completes the proof. Q.E.D.

Theorem 2.9 *Let $m \geq 1$ be a fixed integer. Then*
(i) $w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^{m-1}) \subset w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$;
(ii) $w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^{m-1}) \subset w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$;
(iii) $w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^{m-1}) \subset w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$.

Proof. The proof of the inclusions follows from the following inequality

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} u_k\Big[M_k(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||)\Big]^{p_k} \\ \leq \quad \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k||\frac{t_{kn}(\Delta_v^{m-1} x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big]^{p_k} \\ + \quad \frac{K}{h_r}\sum_{k\in I_r} u_k\Big[M_k||\frac{t_{kn}(\Delta_v^{m-1} x_{k+1})}{\rho}, z_1, \cdots, z_{n-1}||\Big]^{p_k}.\end{aligned}$$

Q.E.D.

Theorem 2.10 *Let $\mathcal{M} = (M_k)$ and $\mathcal{M}' = (M_k')$ are Musielak-Orlicz functions. Then*
(i) $w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \cap w_\sigma^\infty\Big[\mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$
$\subset w_\sigma^\infty\Big[\mathcal{M} + \mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$;
(ii) $w_\sigma\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \cap w_\sigma\Big[\mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$
$\subset w_\sigma\Big[\mathcal{M} + \mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$;
(iii) $w_\sigma^0\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \cap w_\sigma^0\Big[\mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$
$\subset w_\sigma^0\Big[\mathcal{M} + \mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$.

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Proof. Let $x \in w_\sigma^\infty\Big[\mathcal{M}, u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \cap w_\sigma^\infty\Big[\mathcal{M}', u, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Then

$$\sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \infty \text{ uniformly in } n$$

and

$$\sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[M_k'\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \infty \text{ uniformly in } n.$$

Thus by using inequality (1.1) we have

$$\begin{aligned} u_k\Big[(M_k + M_k')\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} &\leq K\Big[u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big] \\ &+ K\Big[u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big] \end{aligned}$$

$$\begin{aligned} \Longrightarrow \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[(M_k + M_k')\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\ \leq \frac{K}{h_r} \sum_{k \in I_r} u_k \Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\ + \frac{K}{h_r} \sum_{k \in I_r} u_k \Big[M_k'\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \text{ uniformly in } n. \end{aligned}$$

This completes the proof. Similarly, we can prove (ii) and (iii). Q.E.D.

Theorem 2.11 *Let $0 < p_k \leq q_k$ for each k and $\left(\frac{q_k}{p_k}\right)$ be bounded. Then*
(i) $w_\sigma^\infty\Big[\mathcal{M}, q, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^\infty\Big[\mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$;
(ii) $w_\sigma\Big[\mathcal{M}, q, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma\Big[\mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$;
(iii) $w_\sigma^0\Big[\mathcal{M}, q, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m) \subset w_\sigma^0\Big[\mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$.

Proof. (i) Let $x \in w_\sigma^\infty\Big[\mathcal{M}, q, ||\cdot, \cdots, \cdot||\Big]_\theta(\Delta_v^m)$. Then

$$\sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} u_k \Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{q_k} < \infty \text{ uniformly in } n.$$

Write $\mu_{k,n} = u_k\Big[M_k\Big(||\frac{t_{kn}(\Delta_v^m x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{q_k}$ and $\lambda_k = \frac{p_k}{q_k}$. Since $p_k \leq q_k$ therefore $0 < \lambda < \lambda_k \leq 1$. Define $y_{k,n} = \mu_{k,n}$, $y_{k,n} = 0$ if $\mu_{k,n} \geq 1$ and $z_{k,n} = \mu_{k,n}$, $z_{k,n} = 0$ if $\mu_{k,n} \geq 1$. So $\mu_{k,n} = y_{k,n} + z_{k,n}$ and $\mu_{k,n}^{\lambda_k} = y_{k,n}^{\lambda_k} + z_{k,n}^{\lambda_k}$. Now it follows that $y_{k,n}^{\lambda_k} \leq y_{k,n} \leq z_{k,n}$ and $z_{k,n}^{\lambda_k} \leq z_{k,n}^{\lambda}$. Therefore

$$\frac{1}{h_r} \sum_{k \in I_r} \mu_{k,n}^{\lambda_k} = \frac{1}{h_r} \sum_{k \in I_r} \left(y_{k,n}^{\lambda_k} + z_{k,n}^{\lambda_k}\right) \leq \frac{1}{h_r} \sum_{k \in I_r} y_{k,n} + \frac{1}{h_r} \sum_{k \in I_r} z_{k,n}^{\lambda}.$$

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Since $\lambda < 1$ so that $\frac{1}{\lambda} > 1$, for each n and by using Holder's inequality, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} z_{k,n}^{\lambda} &= \sum_{k\in I_r}\Big(\frac{1}{h_r}z_{k,n}\Big)^{\lambda}\Big(\frac{1}{h_r}\Big)^{1-\lambda}\\ &\leq \Big(\sum_{k\in I_r}\Big[\big(\frac{1}{h_r}z_{k,n}\big)^{\lambda}\Big]^{\frac{1}{\lambda}}\Big)^{\lambda}\Big(\sum_{k\in I_r}\Big[\big(\frac{1}{h_r}\big)^{1-\lambda}\Big]^{\frac{1}{(1-\lambda)}}\Big)^{1-\lambda}\\ &= \big(\frac{1}{h_r}\sum_{k\in I_r} z_{k,n}\big)^{\lambda}.\end{aligned}$$

Thus, we have

$$\frac{1}{h_r}\sum_{k\in I_r}\mu_{k,n}^{\lambda_k} \leq \frac{1}{h_r}\sum_{k\in I_r}\mu_{k,n} + \big[\frac{1}{h_r}\sum_{k\in I_r} z_{k,n}\big]^{\lambda}.$$

Hence $x \in w_{\sigma}^{\infty}\Big[\mathcal{M}, p, ||\cdot, \cdots, \cdot||\Big]_{\theta}(\Delta_v^m)$. This completes the proof of (i). Similarly, we can prove (ii) and (iii). Q.E.D.

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