Undecidability of local structures of s-degrees and Q-degrees

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Abstract

We show that the first order theory of the Σ^0_2 s-degrees is undecidable. Via isomorphism of the s-degrees with the Q-degrees, this also shows that the first order theory of the Π^0_2 Q-degrees is undecidable. Together with a result of Nies, the proof of the undecidability of the Σ^0_2 s-degrees yields a new proof of the known fact (due to Downey, LaForte and Nies) that the first order theory of the c.e. Q-degrees is undecidable.

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1 Introduction

In [6], Cooper asks to characterize the degree of the first order theory of the Σ^0_2 s-degrees. We are not able to fully answer this question, but we are able to show that this theory is undecidable. Undecidability follows from the following two facts, which hold in the Σ^0_2 s-degrees: there is an independent antichain which is first order definable with three parameters (Theorem 2.2); and a suitable version of the Exact Degree Theorem of Nies (Theorem 2.1). In addition, Theorem 2.2 together with another suitable version of the Nies Exact Degree Theorem yields the undecidability of the Π^0_1 s-degrees. Via isomorphism of the s-degrees with the Q-degrees, this also gives undecidability of the structure of the Π^0_2 Q-degrees, and undecidability of the c.e. Q-degrees (a result of Downey, LaForte and Nies from [7]).

Positive reducibilities formalize models of relative computability which use only “positive ” oracle information. The most comprehensive positive reducibility is *enumeration reducibility*, denoted by $\leq_{\mathrm{e}}$. Intuitively a set A is enumeration reducible to a set B if there is some effective procedure

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for enumerating A given any enumeration of B. Following Friedberg and Rogers [9], this is made mathematically precise by defining $A \leq_{\mathrm{e}} B$ if there exists a c.e. set Φ such that

$$A = \{x : \text{there is a finite } F \text{ such that } \langle x, F\rangle \in \Phi \ \&\ F \subseteq B\}$$

(often denoted by $A = \Phi^B$) where finite sets are identified with their canonical indices. In this context a c.e. set Φ is also called an *enumeration operator*. According to this definition, a computation may enumerate a number x in A only upon retrieval of positive information about B, i.e., information of the form $F \subseteq B$, for some pair $\langle x, F\rangle \in \Phi$. Access to positive information about B is made possible via some enumeration of B.

1.1 s-reducibility

It is clear that given a set B, an enumeration operator Φ, and a given x, there is no bound to the number n of oracle questions which are needed to enumerate x in Φ^B, i.e. to the cardinality of a finite set F for which we need $F \subseteq B$, in order to have $x \in \Phi^B$. One can therefore introduce restricted versions of enumeration reducibility by requesting instead that there be such a bound. Although extreme, the case $n = 1$, in which for any given x we need at most *one* oracle question, is particularly interesting, and occurs often in practical applications of enumeration reducibility. This suggests the following definition:

Definition 1.1. An enumeration operator Φ is called an s-*operator* if for every $\langle x, F\rangle \in \Phi$, we have that F has at most one element.

It is straightforward to see that the s-operators (s stands for singleton) can be effectively listed, and give rise to a reducibility (called s-*reducibility*), denoted by $\leq_{\mathrm{s}}$. The corresponding degree structure, denoted by $\mathcal{D}_{\mathrm{s}}$, consists of the equivalence classes, called s-*degrees*, of the subsets of ω under the equivalence relation $\equiv_{\mathrm{s}}$ generated by $\leq_{\mathrm{s}}$. The s-degree of a set A will be denoted by $\deg_{\mathrm{s}}(A)$. The structure $\mathcal{D}_{\mathrm{s}}$ is an upper semilattice with least element $\mathbf{0}_{\mathrm{s}} = \deg_{\mathrm{s}}(\varnothing)$ consisting of the c.e. sets, and the operation of least upper bound is given by $\deg_{\mathrm{s}}(A) \cup \deg_{\mathrm{s}}(B) = \deg_{\mathrm{s}}(A \oplus B)$, where $\oplus$ denotes the usual disjoint union of sets. The reducibility $\leq_{\mathrm{s}}$ is properly contained in $\leq_{\mathrm{e}}$: As shown by Zacharov, [23], every nonzero e-degree contains at least two s-degrees. The reader is referred to the papers [5], [6], [19] for a survey of results on s-reducibility.

1.2 Q-reducibility

An apparently different but intimately related reducibility is Q-reducibility (due to Tennenbaum, as quoted by Rogers [20, p. 159]): A set A is *quasi-reducible* (Q-*reducible*) to a set B, $A \leq_{\mathrm{Q}} B$, if and only if there exists a total computable function f such that $A = \{x : W_{f(x)} \subseteq B\}$. (When dealing with $\leq_{\mathrm{Q}}$, the set ω should not be considered as lying in the universe of the reducibility, as $\omega <_{\mathrm{Q}} A$, for every set $A \neq \omega$.) In the usual way, the reducibility $\leq_{\mathrm{Q}}$ gives rise to a degree structure, denoted by $\mathcal{D}_{\mathrm{Q}}$; the elements of $\mathcal{D}_{\mathrm{Q}}$ are called Q-*degrees*; the Q-degree of a set $A \neq \omega$ will be denoted by $\deg_{\mathrm{Q}}(A)$. The structure $\mathcal{D}_{\mathrm{Q}}$ is an upper semilattice with least element $\mathbf{0}_{\mathrm{Q}} = \deg_{\mathrm{Q}}(\varnothing)$ consisting of the Π^0_1 sets, and the usual operation of least upper bound.

Several interesting applications of Q-reducibility to algebra are known. One can for instance quote Dobritsa's theorem (see [3]) stating that for every set X there is a word problem having the same Q-degree of X. Belegradek, [3], shows that a necessary condition for computably presented groups G and H to have G a subgroup of every algebraically closed group of which H is a subgroup, is that the word problem for G be Q-reducible to the word problem of H. It is worth noticing that this condition is also sufficient, [12], if $\leq_{\mathrm{Q}}$ is replaced by $\leq_{\mathrm{T}}$. But then since on c.e. sets Turing reducibility implies Q-reducibility, for computably presented groups with c.e. word problems the same condition is both necessary and sufficient. Q-reducibility has also been studied in connection with abstract complexity theoretic questions: Blum and Marques in [4] introduced the notions of subcreative and effectively speedable sets and they proved that a recursively enumerable set is subcreative if and only if it is effectively speedable. Gill and Morris in [10] gave a simple and interesting characterization of effectively speedable sets in terms of Q-complete sets. They proved that a c.e. set is effectively speedable if and only if it is Q-complete.

There is an extensive bibliography on Q-reducibility: For c.e. Q-degrees see for instance [8], [7] and [18]; Arslanov and Omanadze in [2] study the Q-degrees of n-c.e. sets.

1.3 s-reducibility or Q-reducibility?

The following is a useful result due to Gill and Morris [10] relating s-reducibility to Q-reducibility, where given a set $X \subseteq \omega$, $\overline{X}$ denotes its complement.

Lemma 1.2 (Isomorphism Lemma)**.** For any sets A and $B \neq \omega$, $A \leq_{\mathrm{Q}} B$ if and only if $\overline{A} \leq_{\mathrm{s}} \overline{B}$.

Proof. Suppose A and B are given, with $B \neq \omega$. If $A \leq_{\mathrm{Q}} B$ via a computable function f, then define

$$\Gamma = \{\langle x, \{y\}\rangle : y \in W_{f(x)}\}.$$

Then Γ is an s-operator and $\overline{A} = \Gamma^{\overline{B}}$.

On the other hand, suppose that $\overline{A} \leq_{\mathrm{s}} \overline{B}$ via the s-operator Γ. Let $b \notin B$, and let

$$W_{f(x)} = \begin{cases} \{y : \langle x, \{y\}\rangle \in \Gamma\} & \text{if } \langle x, \varnothing\rangle \notin \Gamma, \\ \{y : \langle x, \{y\}\rangle \in \Gamma\} \cup \{b\} & \text{otherwise.} \end{cases}$$

Then $A \leq_{\mathrm{Q}} B$ via the computable function f. Q.E.D.

Notwithstanding this isomorphism, s-reducibility and Q-reducibility have lived so far quite independent lives. Most of the papers on s-reducibility do not mention Q-reducibility, and viceversa most of the papers on Q-reducibility do not mention s-reducibility. An additional bit of confusion comes perhaps from an early unusual variety of approaches to s-reducibility: Friedberg and Rogers originally defined $A \leq_{\mathrm{s}} B$ if $A = \{x : W_{f(x)} \cap B \neq \varnothing\}$ for some computable function f; $\leq_{\mathrm{s}}$ appears as $\leq_{\mathrm{se}}$ in [13]; the branch finite version of $\leq_{\mathrm{s}}$ (i.e., the reducibility given by s-operators Φ in which for every x there are only finitely many axioms $\langle x, F\rangle \in \Phi$) appears as $\leq_{\mathrm{Q}}$ in [15]. Our formalization of s-reducibility, and the notion of s-operator, derive from [10].

Following [19], one can define a jump operation on the s-degrees, for which the jump of the least element $\mathbf{0}_{\mathrm{s}}$ is given by the s-degree $\mathbf{0}'_{\mathrm{s}} = \deg_{\mathrm{s}}(\overline{K})$, where K denotes the halting set. In the following, we denote $\mathcal{L}_{\mathrm{s}} = \mathcal{D}_{\mathrm{s}}(\leq_{\mathrm{s}} \mathbf{0}'_{\mathrm{s}})$. The structure $\mathcal{L}_{\mathrm{s}}$ is called the *local structure* of the s-degrees, studied for instance in [19] and [22]. It is straightforward to show that for every set A, $A \leq_{\mathrm{s}} \overline{K}$ if and only if $A \in \Sigma^0_2$. Thus the elements of $\mathcal{L}_{\mathrm{s}}$ are exactly the Σ^0_2 s-degrees, and consist only of Σ^0_2 sets.

Via the isomorphism of Lemma 1.2, this gives also a jump operation on the Q-degrees, so that for the first jump $\mathbf{0}'_{\mathrm{Q}}$ we have $\mathbf{0}'_{\mathrm{Q}} = \deg_{\mathrm{Q}}(K)$, and the *local structure* of the Q degrees consists exactly of the Π^0_2 Q-degrees.

Our notations and terminology for computability theory are standard, and can be found in [16], [17], [20], and [21].

2 The theorems

The following result shows how to "code" any Σ^0_4 set in an independent family of s-degrees below $\mathbf{0}'_{\mathrm{s}} = \deg_{\mathrm{s}}(\overline{K})$. Recall that in an upper semilattice $\langle U, \leq, \vee\rangle$, a countable $A \subseteq U$ is called *independent* if for every $a \in A$ and any finite $F \subseteq A$, we have

$$a \leq \bigvee F \Rightarrow a \in F.$$

Theorem 2.1 (Exact Degree Theorem for the Σ^0_2 s-degrees)**.** Suppose that $\{A_i\}_{i\in\omega}$ is a uniformly Σ^0_2 sequence of sets such that the family $\{\deg_{\mathrm{s}}(A_i)\}_{i\in\omega}$

is independent. Then, for each Σ^0_4 set S, there exists a Σ^0_2 set B such that

$$i \in S \Leftrightarrow A_i \leq_{\mathrm{s}} B.$$

Moreover, the result holds uniformly: a Σ^0_2 index for B can be uniformly found starting from any Σ^0_4 index of S.

Proof. An examination of the proof by Nies in [14] of the Exact Degree Theorem for Σ^0_2 e-degrees shows that he actually proved that given S one can uniformly find B such that

$$\begin{aligned} i \in S &\quad\Rightarrow\quad A_i \leq_{\mathrm{m}} B \\ i \notin S &\quad\Rightarrow\quad A_i \nleq_{\mathrm{e}} B. \end{aligned}$$

Thus is it straightforward to adapt the proof to the Σ^0_2 s-degrees. Q.E.D.

The Exact Degree Theorem turns out to be quite useful. If we could show that there exists a uniformly Σ^0_2 sequence of sets whose s-degrees form an independent family $\{\mathbf{a}_i\}_{i\in\omega}$ of Σ^0_2 s-degrees, which is first order definable with parameters (in the language of partial orders), then this would yield that the first order theory of the Σ^0_2 s-degrees is undecidable. This is done in the following manner. Assume that $\alpha(v, \overline{p})$ is a first order relation with parameters $\overline{p}$ that defines the elements of an independent family. By Theorem 2.1, every Σ^0_4 set S can be uniformly associated with an s-degree $\mathbf{b}$ such that

$$S = S_{\mathbf{b}} = \{i : \mathbf{a}_i \leq_{\mathrm{s}} \mathbf{b}\}.$$

Thus

$$S_{\mathbf{b}} \subseteq S_{\mathbf{c}} \Leftrightarrow \mathcal{L}_{\mathrm{s}} \models \forall \mathbf{a}\, [(\alpha(\mathbf{a}, \overline{\mathbf{p}})\ \&\ \mathbf{a} \leq \mathbf{b}) \rightarrow \mathbf{a} \leq \mathbf{c}].$$

Hence the first order theory of the poset $\langle \{A : A \text{ is } \Sigma^0_4\}, \subseteq \rangle$ (which is known to be hereditarily undecidable, see [11]) is elementarily definable with parameters in the Σ^0_2 s-degrees, giving undecidability of the local structure $\mathcal{L}_{\mathrm{s}}$, as stated in Corollary 2.4.

Next theorem shows the existence of an independent set of Σ^0_2 s-degrees which is definable with parameters.

Theorem 2.2. There is an independent set of Σ^0_2 s-degrees that is first order definable with parameters. More specifically, there exist 2-c.e. s-degrees $\{\mathbf{g_i}\}_{i\in\omega}$, $\mathbf{g}$, $\mathbf{a}$, $\mathbf{b}$, such that the $\mathbf{g_i}$'s form an independent set and are the minimal solutions of the inequalities

$$\mathbf{x} \leq_{\mathrm{s}} \mathbf{g}\ \&\ \mathbf{a} \leq_{\mathrm{s}} \mathbf{x} \cup \mathbf{b},$$

i.e., for every i, $\mathbf{g_i} \leq_{\mathrm{s}} \mathbf{g}$, $\mathbf{a} \leq_{\mathrm{s}} \mathbf{g}_i \cup \mathbf{b}$, and for every $\mathbf{x}$,

$$\mathbf{x} \leq_{\mathrm{s}} \mathbf{g}\ \&\ \mathbf{a} \leq_{\mathrm{s}} \mathbf{x} \cup \mathbf{b} \Rightarrow (\exists i)[\mathbf{g}_i \leq_{\mathrm{s}} \mathbf{x}].$$

Before proving this theorem, we state a few corollaries. We first recall the following theorem.

Theorem 2.3 (Ambos-Spies, Nies & Shore; [1]). Let $\mathfrak{P} = \langle P, \leq, \vee, 0\rangle$ be an upper semilattice such that, for some $n \geq 1$, the partial order of Σ^0_n sets under inclusion is first order definable with parameters in $\mathfrak{P}$. Then the first order theory of $\mathfrak{P}$ is undecidable.

It now follows that

Corollary 2.4. The first order theory of the Σ^0_2 s-degrees is undecidable.

Proof. This is clear from Theorem 2.3 and the discussion at the beginning of this section. Q.E.D.

Next, recall the following lemma:

Lemma 2.5 (Omanadze & Sorbi; [19]). For every 2-c.e. set C there exists a Π^0_1 set D such that $C \equiv_{\mathrm{s}} D$.

Proof. See [19]. Q.E.D.

As a consequence we have

Theorem 2.6. There is an independent set of Π^0_1 s-degrees that is first order definable with parameters. More specifically, there exist Π^0_1 s-degrees $\{\mathbf{g_i}\}_{i\in\omega}$, $\mathbf{g}$, $\mathbf{a}$, $\mathbf{b}$, such that the $\mathbf{g_i}$'s form an independent set and are the minimal solutions of the inequalities

$$\mathbf{x} \leq_{\mathrm{s}} \mathbf{g} \;\&\; \mathbf{a} \leq_{\mathrm{s}} \mathbf{x} \cup \mathbf{b}$$

i.e., for every i, $\mathbf{g_i} \leq_{\mathrm{s}} \mathbf{g}$, $\mathbf{a} \leq_{\mathrm{s}} \mathbf{g}_i \cup \mathbf{b}$, and for every Π^0_1 degree $\mathbf{x}$,

$$\mathbf{x} \leq_{\mathrm{s}} \mathbf{g} \;\&\; \mathbf{a} \leq_{\mathrm{s}} \mathbf{x} \cup \mathbf{b} \Rightarrow (\exists i)[\mathbf{g}_i \leq_{\mathrm{s}} \mathbf{x}].$$

Proof. By Theorem 2.2, and Lemma 2.5. Q.E.D.

Lastly, we recall the following version of the Exact Degree Theorem.

Theorem 2.7 (Nies; [14]). Let $\{A_i\}_{i\in\omega}$ be a uniformly Π^0_1 sequence of sets such that their s-degrees form an independent family. Then for every Σ^0_4 set S, there uniformly exists a Π^0_1 set C such that, for every i,

$$i \in S \Leftrightarrow A_i \leq_{\mathrm{s}} C.$$

Proof. Nies proves the Exact Degree Theorem for c.e. Q-degrees: Namely, he shows that starting from any c.e. set G such that the Q-degrees of the columns $\{G^{[i]}\}_{i\in\omega}$ of G form an independent family in the Q-degrees, then for every Σ^0_4 set S, one can uniformly find a c.e. set C such that, for every i,

$$i \in S \Leftrightarrow G^{[i]} \leq_{\mathrm{Q}} C.$$

Then the result translates to an Exact Degree Theorem for Π^0_1 s-degrees by the isomorphism between c.e. Q-degrees and Π^0_1 s-degrees established by Lemma 1.2. Q.E.D.

This gives us the last two corollaries.

Corollary 2.8. The Π^0_1 s-degrees are undecidable.

Proof. By Theorem 2.6, and Theorem 2.7. Q.E.D.

Hence, we get as a corollary a different proof of a result due to Downey, LaForte and Nies, [7]:

Corollary 2.9 (Downey, Laforte & Nies; [7])**.** The first order of the c.e. Q-degrees is undecidable.

Proof. Again, by the isomorphism between c.e. Q-degrees and Π^0_1 s-degrees, established by Lemma 1.2. Q.E.D.

3 A first order definable independent antichain

In this section, we prove Theorem 2.2 which gives us a first order definable independent set of Σ^0_2 s-degrees.

We aim at constructing 2-c.e. sets A, B, and G_i, with $i \in \omega$, such that the following requirements are satisfied, where $G = \bigoplus_i G_i$. We will guarantee that $G_i \subseteq \omega^{[i]}$, so that in fact $G = \bigoplus_{i\in\omega} G_i$ can be taken to be $\bigcup_{i\in\omega} G_i$.

The requirements. The construction aims at satisfying the following requirements:

$$\begin{aligned}
D_i &\quad : \quad (\exists \Delta_i)[A = \Delta_i^{G_i \oplus B}] \\
I_{i,\Phi} &\quad : \quad G_i \neq \Phi^{G_{\neq i}} \\
M_{\Phi,\Psi} &\quad : \quad A = \Phi^{\Psi^G \oplus B} \Rightarrow (\exists i)(\exists \Gamma_i)[G_i = \Gamma_i^{\Psi^G}]
\end{aligned}$$

where Φ, Ψ are given s-operators, and Δ_i, Γ_i are s-operators built by us, and $G_{\neq i} = \bigoplus_{j\neq i} G_j$. Then it is easy to show that the s-degrees $\mathbf{g} = \deg_{\mathrm{s}}(G)$, $\mathbf{g_i} = \deg_{\mathrm{s}}(G_i)$, $\mathbf{a} = \deg_{\mathrm{s}}(A)$, and $\mathbf{b} = \deg_{\mathrm{s}}(B)$ satisfy the claim.

Informal description of the strategies. Before giving the formal construction we give some intuition underlying the strategies used to meet the requirements.

The strategy for requirement D_i. The strategy here consists in contributing to the definition of a correct s-operator Δ_i such that $A = \Delta_i^{G_i \oplus B}$. Imagine we have placed this strategy on a tree of strategies: Let us call this strategy α. (For the sake of definiteness, we employ here terminology and notions concerning trees, which will be fully introduced later.) Then strategy α defines suitable axioms of the form $\langle x, \varnothing \oplus \varnothing \rangle \in \Delta_i$[1] for all those numbers x that higher priority strategies (i.e., strategies $\beta < \alpha$) want to restrain in A. When α acts, it initializes all strategies $\beta >_{\mathrm{L}} \alpha$, thus assuming that any witness x used by any such β before its initialization will maintain its A-membership state forever (i.e., $x \in A$ if and only if currently $x \in A$), and for any such witness $x \in A$, α defines the axiom $\langle x, \varnothing \oplus \varnothing \rangle \in \Delta_i$. Finally α lets strategies $\beta \supset \alpha$ maintain a correct definition of Δ_i with respect to the elements that these β's are using: More specifically, such a β may define an axiom of the form $\langle x, \{g\} \oplus \varnothing \rangle \in \Delta_i$ defining $g \in G_i$, and then later possibly an axiom of the form $\langle x, \varnothing \oplus \{b\} \rangle \in \Delta_i$ defining $b \in B$. If later β wants to extract x from A, then β also needs to extract g from G_i and b from B. If β is initialized before ever extracting x, then α correctly assumes that $x \in A$, and adds an axiom $\langle x, \varnothing \oplus \varnothing \rangle \in \Delta_i$.

The strategy for requirement $I_{i,\Phi}$. This is a more or less obvious version for s-reducibility of the classical Friedberg-Muchnick strategy:

1. Appoint a new witness $g \in G_i$;
2. Await $g \in \Phi^{G_{\neq i}}$. If and when this happens, through say an axiom $\langle g, F \rangle \in \Phi$, with $F \subseteq G_{\neq i}$, then extract g from G_i, and restrain $F \subseteq G_{\neq i}$: This is possible since $g \notin F$.

The strategy for requirement $M_{\Phi,\Psi}$. Suppose that Φ and Ψ are given s-operators. At first we try to diagonalize, and to define A, G, and B in such a way as to have, for some x, $A(x) \neq \Phi^{\Psi^G \oplus B}(x)$. So the first attempt consists in trying to execute the following actions:

1. Appoint a witness x, and temporarily let $x \in A$;
2. await $x \in \Phi^{\Psi^G \oplus B}$; (this will be referred to as outcome w;)
3. extract x from A, and restrain $x \in \Phi^{\Psi^G \oplus B}$. (This will be referred to as outcome d.)

Unfortunately, it might not be possible to proceed with item (3) of the previous naive strategy. Indeed, the following could happen, as a consequence of the interaction of our strategy with D-strategies having higher priority: Axioms of the form $\langle x, \{y\} \oplus \varnothing \rangle \in \Phi$, and $\langle y, \{g\} \rangle \in \Psi$ might appear, with

[1] Note that $\varnothing \oplus \varnothing$ is just a more informative way of denoting the empty set $\varnothing$!

$g \in G$, which makes $x \in \Phi^{\Psi^G \oplus B}$, but on the other hand there is a higher priority strategy D_i, with an axiom $\langle x, \{g\} \oplus \varnothing \rangle \in \Delta_i$ already defined, thus with $g \in G_i$, so it is not possible to restrain $g \in G$ (which would make $x \in \Delta_i^{G_i \oplus B}$), and extract x from A, without injuring D_i. So, unless later axioms of a different form appear for y in Ψ (for instance: $\langle y, \varnothing \rangle \in \Psi$; or $\langle y, \{g'\} \rangle \in \Psi$, with $g' \neq g$ such that g' can be restrained without preventing D_i from extracting x; or later axioms of the form $\langle x, \varnothing \oplus \varnothing \rangle \in \Phi$, or $\langle x, \varnothing \oplus \{b\} \rangle \in \Phi$ with $\langle x, \varnothing \oplus \{b\} \rangle \notin \Delta_i$), we do the following: We define an s-operator Γ_i, by enumerating the axiom $\langle g, \{y\} \rangle \in \Gamma_i$; and extract from G all those numbers $\hat{g}$ such that there are axioms $\langle x, \{\hat{g}\} \oplus \varnothing \rangle \in \Delta_j$ for all strategies D_j, with $j \neq i$, of higher priority than $M_{\Phi,\Psi}$. If a new axiom $\langle y, \{g'\} \rangle \in \Psi$ (with $g' \in G$) as before appears, then g' is different from g and the $\hat{g}$'s, and we are free to diagonalize as explained above, by restraining $g' \in G$, extracting x from A, and mantaining all Δ_j's correct. On the other hand, if no new such axiom appears then we have $g \in G_i$ if and only if $y \in \Psi^G$, and thus $g \in G_i$ if and only if $g \in \Gamma_i^{\Psi^G}$. The idea is then to "pass on" g (through a sort of stream of elements) to lower priority strategies for their own use. Whatever they do with g, they can not destroy correctness of Γ_i at g. Unfortunately, if no further action is taken, this would make $x \notin \Delta_j^{G_j \oplus B}$ for $j \neq i$, even if $x \in A$. To set $A(x) = \Delta_j^{G_j \oplus B}(x)$ for all relevant j, we select a new element b, define $b \in B$, and enumerate the axiom $\langle x, \varnothing \oplus \{b\} \rangle \in \Delta_j$. Of course if later we are able to diagonalize by extracting x from A, then we must extract b from B, together with g, in order to preserve $A(x) = \Delta_j^{G_j \oplus B}(x)$.

Having lost x as a diagonalization witness, we then appoint a new witness x' in a new attempt at diagonalization as before. Proceeding as outlined above, if all our attempts at diagonalization fail, then since there are only finitely many strategies D_j having higher priority than $M_{\Phi,\Psi}$, the conclusion must be that there is a least i such that we define infinitely many axioms of the form $\langle g, \{y\} \rangle \in \Gamma_i$, and the elements of the infinite set (stream) of these g's can be used as witnesses by lower priority strategies. There are of course other stratagems that one has to employ here. In particular, Γ_i is a priori correct only on the g's that are in the stream. We must make sure that Γ_i is correct also on numbers which are not in the stream. This is no problem as regards numbers used as witnesses by higher priority strategies. On the other hand when we define Γ_i we initialize all strategies of lower priority that may use numbers not in the stream, and our strategy assumes that these numbers will maintain their G_i-membership, thus defining an axiom $\langle g', \varnothing \rangle \in \Gamma_i$ for those relevant g''s that are currently in G_i.

The tree of strategies. We work with a tree of strategies

$$T \subseteq (\omega \cup \{d, w\})^{<\omega}$$

where the set of *outcomes* (i.e., the elements of $\omega \cup \{w, d\}$) are ordered as follows:

$$d < 0 < 1 < \cdots < w.$$

We use the standard notations and terminology on strings. In particular, given strings $\alpha, \beta \in T$: $|\alpha|$ denotes the length of α; $\alpha \subseteq \beta$ means that α is an initial segment of β; $\alpha <_{\mathrm{L}} \beta$ means that there is a string γ such that $\gamma \subset \alpha, \beta$ and $\alpha(|\gamma|) < \beta(|\gamma|)$; $\alpha \leq \beta$ means $\alpha \subseteq \beta$ or $\alpha <_{\mathrm{L}} \beta$; we say that α has higher priority than β if $\alpha < \beta$; the empty string is denoted by λ.

We will refer to some computable *requirement assignment* R of requirements to the elements of T (i.e., finite strings of outcomes; the strings in T are also called *nodes*, or *strategies*), i.e., a function R mapping nodes to requirements (we will denote by R_α the requirement assigned to node α), in such a way that along any infinite path of T, R is in fact a bijection. We say that a strategy α is a *D-strategy* (*I-strategy*, or *M-strategy*, respectively) if $R_\alpha = D_i$ for some i ($R_\alpha = I_{i,\Phi}$ for some i and s-operator Φ, or $R_\alpha = M_{\Phi,\Psi}$, for some pair of s-operators Φ, Ψ, respectively). We assume that if $\beta \subset \beta'$ are D-strategies and $D_\beta = D_i$, $D_{\beta'} = D_{i'}$ then $i < i'$. If β is a strategy such that, e.g., $R_\beta = D_i$, then we also write G_β for G_i; and similar other abuses of notations will be allowed, hopefully without affecting clearness and readability of the proof.

During the construction, we define approximations to the sets G_i, A, B. We also define several additional parameters, including witnesses, auxiliary sets, and s-operators. In particular:

- for every D-node α we define an s-operator Δ_α; for every M-node α and any i, an s-operator $\Gamma_{\alpha,i}$;
- for every I-node we define a witness g_α; for every M-node α we define witnesses $x_\alpha(0), x_\alpha(1), \ldots$, and parameters $b_\alpha(0), b_\alpha(1), \ldots$;
- for every α we define a set (called *stream*) S_α, which is given, stage by stage, by specifying its elements.

At stage s of the construction, in addition to the approximations $G_{i,s}$, A_{s}, B_{s} to the sets G_i, A, B, respectively, we define approximations to the above mentioned parameters, thus defining $\Delta_{\alpha,s}$, $\Gamma_{\alpha,i,s}$, $g_{\alpha,s}$, $x_{\alpha,s}(t)$, $b_{\alpha,s}(t)$, $S_{\alpha,s}$.
The desired sets G_i, A, B, and G will eventually be defined by

$$G_i = \{y : (\exists t)(\forall s \geq t)[y \in G_{i,s}]\},$$
$$A = \{y : (\exists t)(\forall s \geq t)[y \in A_s]\},$$
$$B = \{y : (\exists t)(\forall s \geq t)[y \in B_s]\},$$

and, as already remarked, $G = \bigcup_{i \in \omega} G_i$.

Definition 3.1. When we *initialize* a strategy α at stage s, we discard the current version of the relative parameters, i.e., we set $\Delta_{\alpha,s} = \Gamma_{\alpha,i,s} = S_{\alpha,s} = \varnothing$, $g_{\alpha,s} =\uparrow$ (undefined), and $x_{\alpha,s}(t) = b_{\alpha,s}(t) =\uparrow$ for any t. (Hence when we initialize α we discard the current values of the parameters, waiting to define new values if needed later. Notice that upon discarding the value of a parameter, the construction will not change its current membership state, thus for instance $x_{\alpha,s}(t)$ will stay forever in A if currently in A, or it will be forever $x_{\alpha,s}(t) \notin A$ if currently not in A, etc.)

In order to define $S_{\alpha,s}$, we will in fact define $S^{[j]}_{\alpha,s}$, i.e., $S_{\alpha,s} \cap \omega^{[j]}$, for every j: The idea underlining the set $S^{[j]}_{\alpha}$ is that the only elements that strategies $\beta \supseteq \alpha$ may use in order to define G_j are taken from $S^{[j]}_{\alpha}$.

Definition 3.2. During the construction we say that at a stage $s+1$ a number y is *new* for strategy α if either

1. y needs to be chosen to be enumerated into A or B, and y is bigger than any number that has been used so far by any strategy; or

2. y needs to be chosen to be enumerated into G_i, for some i, and $y \in S^{[i]}_{\alpha,s+1} - S^{[i]}_{\alpha,s}$.

In the construction below, any parameter retains the same value as at the previous stage unless otherwise specified. Moreover, for any α if o is the current outcome of α at stage s, any new element entering $S^{[j]}_{\alpha,s}$ will also be enumerated into $S^{[j]}_{\alpha^\frown\langle o\rangle,s}$ unless otherwise specified.

The construction. By stages: We define at stage s a string δ_s, which is the current approximation to what will be called the *true path.*

Step 0. Let $\delta_0 = \lambda$. Initialize all strategies.

Step $s+1$. For the sake of simplicity we will often write p (where p is a parameter) instead of p_{s}, or p_{s+1}, to denote the most recent value of p that has been defined, or is being defined, during the construction. We will also often omit the strategy to which the parameter refers, thus writing p for p_α, when the strategy is clearly understood from the context. Similarly, we omit specifying s when writing $x \in A$, meaning $x \in A_s$, etc.

Suppose we have already defined $\alpha = \delta_{s+1} \upharpoonright n$, and $S^{[j]}_{\alpha}$, for every j, having defined $S_{\lambda,s+1} = \{\langle x,y\rangle : y \leq s, x \in \omega\}$.

We act on α according to the requirement R_α.

$R_\alpha = D_i$. For any $x \in A$ enumerated into A by any $\beta \not\supseteq \alpha$, or $x \in A$ and x enumerated by $\beta \supset \alpha$ which has been later initialized, add the axiom $\langle x, \varnothing \oplus \varnothing\rangle \in \Delta_\alpha$. Let $\alpha^\frown\langle 0\rangle$ be eligible to act next.

$R_\alpha = I_{i,\Phi}$. We distinguish the following cases:

1. There is no appointed witness: Appoint a new witness g, i.e., let $g_\alpha = g$, define $g \in G_i$, and end the stage.

2. $g = g_\alpha$ is defined and $g \in G_i - \Phi^{G_{\neq i}}$: Let $\alpha^\frown\langle w\rangle$ be eligible to act next.

3. $g \in \Phi^{G_{\neq i}}$: Define $g \notin G_i$. If this is the first time we have taken this case since the last initialization of α, end the stage. (This has the effect of restraining $g \in \Phi^{G_{\neq i}}$ if α is never again initialized.) Otherwise, let $\alpha^\frown\langle d\rangle$ be eligible to act next.

$R_\alpha = M_{\Phi,\Psi}$. We first give the following definition which allows us to identify an x for which we can force $x \in \Phi^{\Psi^G \oplus B} - A$, and still make $A(x) = \Delta_\beta^{G_\beta \oplus B}(x)$ for each D-strategy $\beta \subset \alpha$.

Definition 3.3. We say that a number x is *eligible to act at* α if one of the following holds:

1. There is an axiom $\langle x, \varnothing \oplus \varnothing\rangle \in \Phi$.

2. There is an axiom $\langle x, \varnothing \oplus \{b\}\rangle \in \Phi$ such that $b \in B$, $\langle x, \varnothing \oplus \{b\}\rangle \notin \Delta_\beta$, for any $\beta \subset \alpha$.

3. There is an axiom $\langle x, \{y\} \oplus \varnothing\rangle \in \Phi$ and an axiom $\langle y, \varnothing\rangle \in \Psi$.

4. There is an axiom $\langle x, \{y\} \oplus \varnothing\rangle \in \Phi$ and an axiom $\langle y, \{g\}\rangle \in \Psi$ with $g \in G$, such that there is no D-node $\beta \subset \alpha$, with the axiom $\langle x, \{g\} \oplus \varnothing\rangle \in \Delta_\beta$.

We now proceed with the strategy. Suppose that since last initialization of α we have already defined $x_\alpha(t)$ and $b_\alpha(t)$, with $t < n$. We distinguish the following cases:

1. If some $x = x_\alpha(t)$ is eligible to act, do the following *actions*: Extract x from A, i.e., define $x \notin A$. Correct Δ_β for $\beta \subset \alpha$: If $\langle x, \{g\} \oplus \varnothing\rangle \in \Delta_\beta$ then extract g from G_β, and if $\langle x, \varnothing \oplus \{b\}\rangle \in \Delta_\beta$ then extract b from B. If this is the first time we have taken this case since α's last initialization, end the current stage. Otherwise, let $\alpha^\frown\langle d\rangle$ be eligible to act next.

2. $n > 0$ and $x_\alpha(n-1) \in A - \Phi^{\Psi^G \oplus B}$: let $\alpha^\frown\langle w\rangle$ be eligible to act next.

3. $n > 0$ and $x_\alpha(n-1) \in \Phi^{\Psi^G \oplus B} \cap A$. Denote again for simplicity $x = x_\alpha(n-1)$. We further distinguish two cases:

(a) $b_\alpha(n-1)$ is undefined. Notice that there are only axioms of the form $\langle x, \{y\} \oplus \varnothing \rangle \in \Phi$, such that for all axioms $\langle y, \{g\} \rangle \in \Psi$ there are a D-node $\beta \subset \alpha$, with $g \in G_\beta$, and an axiom $\langle x, \{g\} \oplus \varnothing \rangle \in \Delta_\beta$, so that we can not restrain $g \in G_\beta$, and extract x from A without making it impossible to achieve $A(x) = \Delta_\beta^{G_\beta \oplus B}$. Pick the least such β, and suppose that $R_\beta = D_i$: For all D-strategies $\beta' \neq \beta$ such that $\beta' \subset \alpha$, define $g' \notin G_{\beta',s+1}$, where g' is such that there is an axiom $\langle x, \{g'\} \oplus \varnothing \rangle \in \Delta_{\beta'}$. Pick a new $b = b_\alpha(n-1)$, define $b \in B$, and add the axiom $\langle x, \varnothing \oplus \{b\} \rangle \in \Delta_{\beta'}$, for any D-node $\beta' \subset \alpha$.

Add the axiom $\langle g, \{y\} \rangle \in \Gamma_{\alpha,i}$. For each g' such that $g' \in G_i$ and g' has been enumerated by some strategy $\beta \not\supseteq \alpha^\frown\langle i \rangle$, define the axiom $\langle g', \varnothing \rangle \in \Gamma_{\alpha,i}$. Define $S^{[i]}_{\alpha^\frown\langle i\rangle,s+1} = S^{[i]}_{\alpha^\frown\langle i\rangle,s} \cup \{g\}$, and let $\alpha^\frown\langle i \rangle$ be eligible to act next.

(b) $b = b_\alpha(n-1)$ is defined, or $n = 0$: Choose a new $x_\alpha(n)$; define $x_\alpha(n) \in A$; for every D-strategy $\beta \subset \alpha$, with say $D_\beta = D_j$, appoint a new number $g_j \in S^{[j]}_\alpha$, define $g_j \in G_j$, add the axiom $\langle x, \{g_j\} \oplus \varnothing \rangle \in \Delta_\beta$, and end the stage.

4 Verification of the construction

Lemma 4.1. The sets G_i, A and B are 2-c.e.

Proof. A careful look at the construction shows that if X is any of the sets G_i, A, B, then for every z, at stage 0 we have $X_0(z) = 0$.

Consider first the case $X = A$. An element x can enter A only if enumerated by some M-strategy α, i.e., $x = x_\alpha(t)$, for some t. But then it can only be extracted just through Case 1 of the same strategy α. After this x is never again enumerated into A.

A similar argument applies to B. An element b can enter B if enumerated by some M-strategy α, and can be extracted again only by α upon giving outcome d.

Finally assume that $X = G_i$. An element g can be enumerated in G_i a first time by an M-strategy α, in correspondence to some witness x, i.e., α enumerates x into A, defines the axiom $\langle x, \{g\} \oplus \varnothing \rangle \in \Delta_\alpha$ and defines $g \in G_i$. Then it can only be extracted by the same strategy α, when moving to outcome d or to outcome j with $j \neq i$; or it can be extracted by some strategy $\gamma \supseteq \alpha^\frown\langle i \rangle$, after g has been put in the set $S_{\alpha^\frown\langle i\rangle}$. After being extracted g is not used anymore. Q.E.D.

Lemma 4.2. For every n the following hold: $\alpha_n = \liminf_s \delta_s \upharpoonright n$ exists; α_n is eventually never initialized; after the last initialization of α_n there are infinitely many α_n-true stages s (i.e., stages at which $\alpha_n \subseteq \delta_s$) and at each such stage $S^{[j]}_{\alpha_n,s}$ contains a new element for every j; witnesses g_{α_n}, and $x_{\alpha_n}(t)$, $b_{\alpha_n}(t)$ reach a limit.

Proof. The proof is by induction on n. For $n = 0$ the claim is obvious. Suppose now that $\alpha_n = \liminf_{\mathrm{s}} \delta_{\mathrm{s}} \upharpoonright n$ exists, and the inductive claim is true of n. For simplicity, let $\alpha = \alpha_n$. Let t be a stage such that at no $s \geq t$ do we act on any $\beta <_{\mathrm{L}} \alpha$. Notice that the inductive assumption on $S^{[j]}_{\alpha}$ allows us to conclude that if α needs to appoint some new element $g \in S^{[j]}_{\alpha}$ in order to define the set G_j, then it is allowed to do so.

We now distinguish three cases according to whether α is a D-strategy, or an I-strategy, or an M-strategy, respectively.

$R_\alpha = D_i$. We first notice that when α acts, we give outcome 0, and we never end the stage after acting. So

$$\alpha_{n+1} = \liminf_{\mathrm{s}} \delta_{\mathrm{s}} \upharpoonright n+1 = \alpha^\frown\langle 0\rangle.$$

On the other hand the inductive claim on $S^{[j]}_{\alpha_{n+1}}$ clearly carries through.

$R_\alpha = I_{i,\Phi}$. Using the inductive assumption on $S^{[i]}_{\alpha}$, at some stage $s \geq t$ we appoint a final witness g_α, after which we end the current stage at most twice: once in Case 1 and once in Case 3. Again the inductive claim on $S^{[j]}_{\alpha_{n+1}}$ trivially carries through.

$R_\alpha = M_{\Phi,\Psi}$. Since α is never initialized after stage t, whenever we appoint a number $x_\alpha(t)$ at some stage $s \geq t$, this will never change again. The same conclusion holds for $b_\alpha(t)$. Clearly there exists a greatest $m \in \omega \cup \{\omega\}$ such that for every $t < m$, $x_\alpha(t)$ is eventually appointed. If $m \in \omega$ then we eventually have outcome w or d, which are both finitary, and the inductive claim on $S^{[j]}_{\alpha_{n+1}}$ carries through.

Thus assume that $m = \omega$ and i is the least such that $\alpha^\frown\langle i\rangle$ is visited infinitely often. Notice that whenever we visit $\alpha^\frown\langle i\rangle$ we add a new element g to the set $S^{[i]}_{\alpha^\frown\langle i\rangle}$. On the other hand the inductive claim on $S^{[j]}_{\alpha_{n+1}}$ clearly carries through. Q.E.D.

Let $f = \bigcup_n \alpha_n$ be the *true path*, defined by

$$\alpha_n = \liminf_{\mathrm{s}} \delta_{\mathrm{s}} \upharpoonright n.$$

Lemma 4.3. For every n, R_{α_n} is satisfied.

Proof. Let $\alpha = \alpha_n$, for some n, be given, and suppose by the previous lemma that t_n is the last stage at which α is initialized.

$R_\alpha = D_i$: Let x be given. In order to check that $A(x) = \Delta^{G_i \oplus B}_{\alpha}(x)$, we need only check this for those numbers x such that there are β and t

with $x = x_\beta(t)$. Only strategy β is responsible for keeping x in or out of A. Without loss of generality, we may assume that $t \geq t_n$.

Case 1. $\beta \not\supseteq \alpha$. At the first α-stage $s > t$ if $x \in A$ then we enumerate the axiom $\langle x, \varnothing \oplus \varnothing \rangle \in \Delta_\alpha$, which makes $x \in \Delta_\alpha^{G_i \oplus B}$. Otherwise, if $x \notin A$ then at no α-stage s after last initialization of α do we have $x \in A_{\mathrm{s}}$, hence we do not define any Δ_α-axiom for x.

Case 2. $\beta \supseteq \alpha$. If β appoints x and β is initialized before ever extracting x, then $x \in A$, but on the other hand at the first α-stage after initialization of β, we enumerate the axiom $\langle x, \varnothing \oplus \varnothing \rangle \in \Delta_\alpha$, which makes $x \in \Delta_\alpha^{G_i \oplus B}$. Otherwise, at stage t, when β appoints x, β enumerates also an axiom $\langle x, \{g\} \oplus \varnothing \rangle\rangle \in \Delta_\alpha$, letting $g \in G_i$, which makes $x \in \Delta_\alpha^{G_i \oplus B}$ as long as β takes outcome w, waiting for $x \in \Phi_\beta^{\Psi_\beta^G \oplus B}$. Then either β jumps immediately from outcome w to outcome d, extracts x from A and g from G_i, which makes $A(x) = \Delta_\alpha^{G_i \oplus B}(x)$; or β takes some outcome $j \in \omega$, keeps $x \in A$, adds an axiom $\langle x, \varnothing \oplus \{b\} \rangle \in \Delta_\alpha$ letting $b \in B$, which keeps $x \in \Delta_\alpha^{G_i \oplus B}$ even if some lower priority strategy extracts g from G_i; until β takes, if this is ever the case, outcome d and thus extracts x from A, g from G_i, and b from B, making $x \notin \Delta_\alpha^{G_i \oplus B}$. In all cases $A(x) = \Delta_\alpha^{G_i \oplus B}(x)$.

$R_\alpha = I_{i,\Phi}$. Let t be a stage after which α does not change g_α anymore. By Lemma 4.2 such a stage exists. If at no future α-stage do we have $g_\alpha \in \Phi^{G_{\neq i}}$ then $\alpha_{n+1} = \alpha^\frown\langle w \rangle$ and the requirement is satisfied. Otherwise at some future α-stage we have that $g_\alpha \in \Phi^{G_{\neq i}}$. As explained in the construction, at the first such stage, we restrain $g_\alpha \in \Phi^{G_{\neq i}}$, and we extract g_α from G_i, thus letting $g_\alpha \in \Phi^{G_{\neq i}} - G_i$.

$R_\alpha = M_{\Phi,\Psi}$. If $\alpha_{n+1} = \alpha^\frown\langle w \rangle$ then there exists n such that $x = x_\alpha(n)$ is defined, no $x_\alpha(m)$ is ever defined for $m > n$, and $x \in A - \Phi^{\Psi^G \oplus B}$. If $\alpha_{n+1} = \alpha^\frown\langle d \rangle$ then there is some $x = x_\alpha(t)$ (among finitely many witnesses $x_\alpha(0), \ldots, x_\alpha(n)$ which β has defined after last initialization) such that $x \in \Phi^{\Psi^G \oplus B} - A$.

It remains to consider the case when $\alpha_{n+1} = \alpha^\frown\langle i \rangle$ for some $i \in \omega$. We claim in this case that $G_i = \Gamma_{\alpha,i}^{\Psi^G}$, where $\Gamma_{\alpha,i}$ is the s-operator, as enumerated by α after the last initialization of α.

If g is eventually used by a strategy $\beta \leq \alpha$, then either $g \notin G_i$, and in this case there is no axiom $\langle g, F \rangle \in \Gamma_{\alpha,i}$, or $g \in G_i$, in which case by construction we add an axiom $\langle g, \varnothing \rangle \in \Gamma_{\alpha,i}$.

Next, for every g which is ever used by any strategy $\beta >_{\mathrm{L}} \alpha^\frown\langle i \rangle$, we have (at the moment when we discard g by initialization) either $g \notin G_i$, in which case we have $G_i(g) = \Gamma_{\alpha,i}^{\Psi^G}(g)$ since we never define any axiom in $\Gamma_{\alpha,i}$ for these g's, or we have $g \in G_i$, in which case we add an axiom $\langle g, \varnothing \rangle \in \Gamma_{\alpha,i}$.

So we need only show that for every g such that g is enumerated into $S^{[i]}_{\alpha^\frown\langle i\rangle}$ at some $\alpha^\frown\langle i\rangle$-stage,

$$g \in G_i \Leftrightarrow g \in \Gamma_i^{\Psi^G}.$$

The reason we have enumerated g into $S^{[i]}_{\alpha^\frown\langle i\rangle}$ at some stage $t' \geq t_n$ is that we have found an axiom $\langle y, \{g\}\rangle \in \Psi$, with $g \in G_i$, in correspondence with some witness x, for which there is an axiom $\langle x, \{g\} \oplus \varnothing\rangle \in \Delta_\beta$ (where $\beta \subset \alpha$ is such that $R_\beta = D_i$). Moreover there is no other axiom $\langle y, \{g'\}\rangle \in \Psi$ with $g' \in G$. Indeed, such an axiom can not appear after t' since in this case we would be able to diagonalize and give outcome d. If it is present at stage t', then since we give outcome i at t' there must be an axiom $\langle x, \{g'\} \oplus \varnothing\rangle \in \Delta_j$ with $j > i$ (here $D_j = D_{\beta'}$, for some β' such that $\beta \subset \beta' \subset \alpha$), but in this case we extract g' from G by construction.

We are therefore able to conclude

$$g \in G_i \Leftrightarrow y \in \Psi^G \Leftrightarrow g \in \Gamma_i^{\Psi^G}$$

as desired. Q.E.D.

Lemma 4.4. Let $\mathbf{g}_i = \deg_s(G_i)$. The set $\{\mathbf{g}_i : i \in \omega\}$ is first order definable with parameters $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Proof. Let $\alpha(x)$ be the following formula with parameters g, a, b in the language of partial orders (where $\vee$ and $<$ are obvious abbreviations):

$$\alpha(x): \quad x \leq g \;\&\; a \leq x \vee b$$

and let

$$\varphi(x): \quad \alpha(x) \;\&\; \neg(\exists w < x)\alpha(w).$$

Now, if the parameters g, a, b are interpreted with the s-degrees $\mathbf{g} = \deg_s(G)$, $\mathbf{a} = \deg_s(A)$, $\mathbf{b} = \deg_s(B)$, respectively, then in the structure $\mathcal{L}_s$ of the Σ^0_2 s-degrees, we have that

$$\mathcal{L}_s \models \varphi(\mathbf{x}, \mathbf{g}, \mathbf{a}, \mathbf{b}) \Leftrightarrow \mathbf{x} \in \{\mathbf{g}_i : i \in \omega\}.$$

Indeed, the formula is certainly satisfied when $\mathbf{x} = \mathbf{g}_i$, any i, since each $\mathbf{g}_i$ is incomparable with all the others. On the other hand, if $\mathbf{x} \leq_s \mathbf{g}$ and $\mathbf{a} \leq_s \mathbf{x} \vee \mathbf{b}$ and there is no $\mathbf{y} <_s \mathbf{x}$ such that $\mathbf{y} \leq_s \mathbf{g}$ and $\mathbf{a} \leq_s \mathbf{y} \vee \mathbf{b}$, then since $\mathbf{g}_i \leq_s \mathbf{x}$ for some i, we have that $\mathbf{g}_i = \mathbf{x}$.

Q.E.D.

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