Approximation properties of bivariate sampling Durrmeyer series in weighted spaces of functions

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Abstract

This paper deals with approximation properties of bivariate sampling Durrmeyer operators for functions belonging to weighted spaces of functions. After a short preliminaries and auxilary results we present well-definiteness of $(S_w^{\zeta,\varsigma})$. Main results of the paper includes pointwise and uniform convergence of the family of operators, rate of convergence via bivariate weighted modulus of continuity and quantitative Voronovskaja type theorem.

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1 Introduction

Whittaker-Kotel'nikov-Shannon (WKS) sampling theorem aims to reconstruct a function from its sample values and approximate version of WKS sampling theorem was one of the pioneering work done by P. L. Butzer and his school at RWTH Aachen, in the late 1970s. The generalized sampling series, in univariate case, defined in [20], is given by

$$\left(G_w^{\chi} f\right)(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\chi\left(wx-k\right), \quad x\in\mathbb{R}, w>0, \tag{1.1}$$

where $f:\mathbb{R}\to\mathbb{R}$ is any function for which the series is convergent for every $x\in\mathbb{R}$, and $\chi:\mathbb{R}\to\mathbb{R}$ (called the *kernel* of the operator) denotes a continuous, discrete approximate identity which satisfies suitable assumptions, and its generalized form can be found in [23].

The generalized sampling series has been widely studied since the 1980s ([21, 22, 35]) and also through the study of the series (1.1), various applications, particularly in signal theory, have been developed. For example, we quote here, applications to signal theory [35], to box splines [21], to image processing [15, 10]. Nevertheless, it should be noted that most real-world signals, including digital images, do not have a mathematical representation as continuous. Since generalized sampling operators are not bounded in $L^1(\mathbb{R})$ spaces, Kantorovich version of the generalized sampling operators was introduced in [13], by replacing the sample values in (1.1) with the mean values $\int_{k/w}^{(k+1)/w} f(u)\,du$ of the form:

$$\left(K_w^{\chi} f\right)(x) := \sum_{k\in\mathbb{Z}} \chi\left(wx-k\right)\left[w\int_{k/w}^{(k+1)/w} f(u)\,du\right], \ x\in\mathbb{R}, \tag{1.2}$$

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where f is a locally integrable function and χ is a kernel function, and its generalized form can be found in [26]. Kantorovich type sampling operators and their multivariate versions has an extensive working area such as linear prediction [3], image processing [12, 25, 27], inverse approximation [24, 28], non-linear approximation [42]. While the Kantorovich modification of the operators (1.1) presents an approximation method for functions belonging to L^1, the approximation method for functions belonging to L^p spaces is to construct Durrmeyer modification according to [32]. Durrmeyer modification of (1.1) was introduced in [14], using a general convolution integral instead of the integral means, by

$$\left(S_w^{\zeta,\zeta} f\right)(x) = \sum_{k=-\infty}^{+\infty} \zeta(wx-k)\, w \int_{\mathbb{R}} \zeta(wu-k) f(u)\, du, \quad x \in \mathbb{R}, \tag{1.3}$$

where ζ is a kernel function which satisfies suitable conditions and its generalized form can be found in [18]. The sampling Durrmeyer operators have been studied in several works, for some of them we refer the readers to [29, 30, 31, 16].

Among the all mentioned sampling type operators above, there is an important version of generalized sampling operators named exponential type sampling operators, firstly introduced by Bardaro et al. in [17] and they have been studied by many researchers, see [19, 40, 6, 7, 41].

In the context of studying generalized sampling series (1.1) in continuous function spaces, the focus is usually on the space $C^0(\mathbb{R})$, which consists of uniformly continuous and bounded functions on $\mathbb{R}$. However, in a recent paper [4], the authors investigated polynomial weighted spaces of continuous functions to enlarge the space of target functions and explore the approximation behaviors of generalized sampling series in a broader class of continuous functions, and for bivariate generalized sampling series in similar context see [8]. The similar approach was taken for generalized sampling Kantorovich series in [5], for generalized sampling Durrmeyer series in [9] and for generalized exponential sampling type series in [11].

In this paper, we study bivariate generalized sampling Durrmeyer series defined by

$$\left(S_w^{\zeta,\zeta} f\right)(x_1,x_2) := \sum_{(k_1,k_2)\in\mathbb{Z}^2} \zeta\left(w(x_1,x_2)-(k_1,k_2)\right) w^2 \int_{\mathbb{R}^2} \zeta\left(w(u_1,u_2)-(k_1,k_2)\right) f(u_1,u_2)\, du_1 u_2 \tag{1.4}$$

for functions that belong to the polynomial weighted space of bivariate continuous functions, suitable kernel functions ζ and $w > 0$. We state some notations and auxilary results in Section 2. Main results begin with the well definiteness of $\left(S_w^{\zeta,\zeta}\right)$ which is given in Section 3. We continue with pointwise and uniform convergence in Section 4 and present rate of convergence of $\left(S_w^{\zeta,\zeta}\right)$ in Section 5. At the end we illustrate a quantitative Voronovskaja type theorem and corresponding to this theorem a qualitative form of Voronovskaja type theorem.

2 Preliminaries

Let us denote by $\mathbb{N}^2, \mathbb{N}_0^2$ and $\mathbb{Z}^2$ the sets of vectors $\mathrm{k} = (k_1,k_2)$ positive integers, non-negative integers and integers, respectively. We set $|\mathrm{k}| := k_1 + k_2$. Furthermore, by $\mathbb{R}^2$ we will denote the 2−dimensional Euclidean space consisting of all vectors $(x_1,x_2) \in \mathbb{R}^2$.

Let $\underline{\mathrm{x}} = (x_1,x_2)\,, \underline{\mathrm{y}} = (y_1,y_2)$. We say that $\underline{\mathrm{x}} > \underline{\mathrm{y}}$ if and only if $x_i > y_i$ for $i = 1,2$ and we will denote by $\underline{0} := (0,0)$ and by $\mathbb{R}_+^2$ the space of all vectors $\underline{\mathrm{x}} > \underline{0}$. Given $\underline{\mathrm{x}}, \underline{\mathrm{y}} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ the usual operations are given by

$$\underline{x} + \underline{y} := (x_1 + y_1, x_2 + y_2),$$

and

$$\lambda \underline{x} := (\lambda x_1, \lambda x_2).$$

We will put by $\langle \underline{x} \rangle = x_1 x_2$ and we write $\underline{k}! = k_1! k_2!$. Moreover, the product and division of two vectors of $\mathbb{R}^2$ are given by

$$\underline{x}\underline{y} := (x_1 y_1, x_2 y_2),$$

and

$$\frac{\underline{x}}{\underline{y}} := \left(\frac{x_1}{y_1}, \frac{x_2}{y_2} \right), \quad (y_i \neq 0 \text{ for all } i = 1, 2).$$

The norm of a vector $\underline{x} := (x_1, x_2) \in \mathbb{R}^2$ is given by $\|\underline{x}\| = \|(x_1, x_2)\| := \sqrt{x_1^2 + x_2^2}$, and the Euclidean distance is $d(\underline{x}, \underline{y}) := \|x - y\|$ for $\underline{x}, \underline{y} \in \mathbb{R}^2$.

A function $\widetilde{\rho}$ is called a weight function if it is a positive continuous function on the whole $\mathbb{R}^2$. In this paper, we consider the weight function

$$\widetilde{\rho}(x, y) := \frac{1}{1 + x^2 + y^2}, \qquad x, y \in \mathbb{R}.$$

We denote the space of functions whose product with the weight function $\widetilde{\rho}$ on $\mathbb{R}^2$ is bounded by $B_{\widetilde{\rho}}(\mathbb{R}^2)$, that is,

$$B_{\widetilde{\rho}}(\mathbb{R}^2) = \left\{ f : \mathbb{R}^2 \to \mathbb{R} : \sup_{x, y \in \mathbb{R}} \widetilde{\rho}(x, y) |f(x, y)| \in \mathbb{R} \right\}.$$

We denote the space of continuous function on the whole $\mathbb{R}^2$ by $C^0(\mathbb{R}^2)$. We can also consider the following natural subspaces of $B_{\widetilde{\rho}}(\mathbb{R}^2)$:

$$\begin{aligned}
C_{\widetilde{\rho}}(\mathbb{R}^2) &:= C^0(\mathbb{R}^2) \cap B_{\widetilde{\rho}}(\mathbb{R}^2), \\
C^*_{\widetilde{\rho}}(\mathbb{R}^2) &:= \left\{ f \in C_{\widetilde{\rho}}(\mathbb{R}^2) : \exists \lim_{\|(x,y)\| \to \pm\infty} \widetilde{\rho}(x, y) f(x, y) \in \mathbb{R} \right\}, \\
U_{\widetilde{\rho}}(\mathbb{R}^2) &:= \left\{ f \in C_{\widetilde{w}}(\mathbb{R}^2) : \widetilde{\rho} f \text{ is uniformly continuous} \right\}.
\end{aligned}$$

The linear space of functions $B_{\widetilde{\rho}}(\mathbb{R}^2)$, and its above subspaces are normed spaces with the norm

$$\|f\|_{\widetilde{\rho}} := \sup_{x, y \in \mathbb{R}} \widetilde{\rho}(x, y) |f(x, y)|$$

see [1, 2, 33, 34, 36].

In Section 5 we aim to study rate of convergence of the operators $S_w^{\zeta,\zeta}$. To follow this aim we mention the weighted modulus of continuity for bivariate functions. It was defined in [38] for $f \in C^*_{\widetilde{w}}(\mathbb{R}^2)$ by

$$\Omega(f; \delta_1, \delta_2) = \sup_{\substack{|u| < \delta_1, |v| < \delta_2 \\ (x,y) \in \mathbb{R}^2}} \frac{|f(x + u, y + v) - f(x, y)|}{(1 + u^2 + v^2)(1 + x^2 + y^2)}. \tag{2.1}$$

The weighted modulus of continuity has following properties (as in one dimensional case):

$$\Omega\left(f;\delta_1,\delta_2\right)\rightarrow 0 \text{ for } \delta_1\rightarrow 0,\delta_2\rightarrow 0 \tag{2.2}$$

and for $\lambda_1>0,\lambda_2>0$,

$$\Omega\left(f;\lambda_1\delta_1,\lambda_2\delta_2\right)\leq 4\left(1+\lambda_1\right)\left(1+\lambda_2\right)\left(1+\delta_1^2\right)\left(1+\delta_2^2\right)\Omega\left(f,\delta_1,\delta_2\right). \tag{2.3}$$

To get more information about bivariate weighted modulus of continuity, we direct readers to see [37, 38]. Here, we present an auxiliary result that will be utilized in the following section.

Remark 2.1 ([8])**.** In the inequality (2.3) if we replace $\lambda_1=\frac{|x_2-x_1|}{\delta_1},\lambda_2=\frac{|y_2-y_1|}{\delta_2},(x_1,y_1)\in\mathbb{R}^2,(x_2,y_2)\in\mathbb{R}^2,\delta_1,\delta_2>0$ and consider the definition of the weighted modulus of continuity, the inequality

$$\begin{aligned}&\left|f\left(x_2,y_2\right)-f\left(x_1,y_1\right)\right|\\ \leq&16\left(1+\delta_1^2\right)^2\left(1+\delta_2^2\right)^2\left(1+x_1^2+y_1^2\right)\Omega\left(f;\delta_1,\delta_2\right)\left[1+\frac{\left|y_2-y_1\right|^3}{\delta_2^3}+\frac{\left|x_2-x_1\right|^3}{\delta_1^3}+\frac{\left|y_2-y_1\right|^3}{\delta_2^3}\frac{\left|x_2-x_1\right|^3}{\delta_1^3}\right].\end{aligned}$$

holds. Finally we obtain

$$\begin{aligned}&\left|f\left(x_2,y_2\right)-f\left(x_1,y_1\right)\right|\\ \leq&256\left(1+x_1^2+y_1^2\right)\Omega\left(f;\delta_1,\delta_2\right)\left[1+\frac{\left|y_2-y_1\right|^3}{\delta_2^3}+\frac{\left|x_2-x_1\right|^3}{\delta_1^3}+\frac{\left|y_2-y_1\right|^3}{\delta_2^3}\frac{\left|x_2-x_1\right|^3}{\delta_1^3}\right]\end{aligned} \tag{2.4}$$

with the choice of $\delta_1\leq 1$ and $\delta_2\leq 1$.

Let ζ be a function belonging to $L^1\left(\mathbb{R}^2\right)$, such that ζ is bounded in a neighborhood of the origin, and satisfies

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2}\zeta\left(\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)=1, \qquad \text{for every } \left(u_1,u_2\right)\in\mathbb{R}^2 \tag{2.5}$$

and

$$\int_{\mathbb{R}^2}\zeta\left(u_1,u_2\right)du_1du_2=1. \tag{2.6}$$

We recall that for any $j\geq 0$, the discrete and continuous absolute moments of order j are defined by

$$M_j\left(\zeta\right):=\sup_{(u_1,u_2)\in\mathbb{R}^2}\sum_{(k_1,k_2)\in\mathbb{Z}^2}\zeta\left(\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\left\|\left(u_1,u_2\right)-\left(k_1,k_2\right)\right\|^j$$

and

$$\widetilde{M}_j\left(\zeta\right):=\int_{\mathbb{R}^2}\left|\zeta\left(u_1,u_2\right)\right|\left\|\left(u_1,u_2\right)\right\|^j du_1du_2,$$

respectively. Throughout out the paper, ζ will be called kernel if it satisfies the conditions (2.5), (2.6) such that there exist $\alpha,\beta>0$ with $M_j\left(\zeta\right)<+\infty$ and $\widetilde{M}_j\left(\zeta\right)<+\infty$.

Lemma 2.2 ([26])**.** Let ζ be a kernel with some $\beta > 0$ and continuous on $\mathbb{R}^2$.

1. For every $\eta > 0$ there holds:

$$\lim_{w\to\infty} \sum_{\|(k_1,k_2)-w(x_1,x_2)\|>w\eta} |\zeta(w(x_1,x_2)-(k_1,k_2))| = 0,$$

uniformly with respect to $(x_1, x_2) \in \mathbb{R}^2$.

2. For every $\xi > 0$ and $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$\int_{\|(u_1,u_2)\|>C} w^2 |\zeta(w(u_1,u_2)-(k_1,k_2))| \, du_1 du_2 < \varepsilon,$$

for sufficiently large $w > 0$ and (k_1, k_2) such that $\|(k_1, k_2)\| \leq \xi w$.

3 Well definiteness of the operators $S_w^{\zeta,\zeta}$

Proposition 3.1. Let ζ be kernel with $\alpha = \beta = 2$. Moreover, we denote by $p(x_1, x_2) := \frac{1}{\widetilde{\rho}(x_1,x_2)} = 1 + x_1^2 + x_2^2$, $(x_1, x_2) \in \mathbb{R}^2$. Then

$$\left|\left(S_w^{\zeta,\zeta} p\right)(x_1,x_2)\right| \leq M_0(\zeta)\left(\widetilde{M}_0(\zeta) + \frac{2}{w^2}\widetilde{M}_2(\zeta)\right) + 4\widetilde{M}_0(\zeta)\left(\frac{1}{w^2}M_2(\zeta) + M_0(\zeta)\|(x_1,x_2)\|^2\right)$$

holds.

Proof. Using definition of the operators $S_w^{\zeta,\zeta}$ we have

$$\begin{aligned}
&\left|\left(S_w^{\zeta,\zeta} p\right)(x_1,x_2)\right| \\
\leq &\sum_{(k_1,k_2)\in\mathbb{Z}^2} |\zeta(w(x_1,x_2)-(k_1,k_2))|\, w^2 \int_{\mathbb{R}^2} |\zeta(w(u_1,u_2)-(k_1,k_2))| \left(1+u_1^2+u_2^2\right) du_1 du_2 \\
:= &I_1 + I_2.
\end{aligned}$$

It is easy to see that $I_1 \leq M_0(\zeta)\widetilde{M}_0(\zeta)$. Let us estimate I_2. By simple calculation we get:

$$\begin{aligned}
I_2 &= \sum_{(k_1,k_2)\in\mathbb{Z}^2} |\zeta(w(x_1,x_2)-(k_1,k_2))|\, w^2 \int_{\mathbb{R}^2} |\zeta(w(u_1,u_2)-(k_1,k_2))| \, \|(u_1,u_2)\|^2 du_1 du_2 \\
&= \sum_{(k_1,k_2)\in\mathbb{Z}^2} |\zeta(w(x_1,x_2)-(k_1,k_2))| \\
&\times \int_{\mathbb{R}^2} |\zeta(w(u_1,u_2)-(k_1,k_2))| \, \|w(u_1,u_2)-(k_1,k_2)+(k_1,k_2)\|^2 du_1 du_2 \\
&\leq \sum_{(k_1,k_2)\in\mathbb{Z}^2} |\zeta(w(x_1,x_2)-(k_1,k_2))| \\
&\times 2\int_{\mathbb{R}^2} |\zeta(w(u_1,u_2)-(k_1,k_2))| \left[\|w(u_1,u_2)-(k_1,k_2)\|^2 + \|(k_1,k_2)\|^2\right] du_1 du_2 \\
&\leq \frac{2}{w^2} M_0(\zeta)\widetilde{M}_2(\zeta)
\end{aligned}$$

$$+\frac{2}{w^2}\widetilde{M}_0(\zeta)\sum_{(k_1,k_2)\in\mathbb{Z}^2}|\zeta(w(x_1,x_2)-(k_1,k_2))|\,\|(k_1,k_2)-w(x_1,x_2)+w(x_1,x_2)\|^2$$
$$\leq\frac{2}{w^2}M_0(\zeta)\widetilde{M}_2(\zeta)$$
$$+\frac{4}{w^2}\widetilde{M}_0(\zeta)\sum_{(k_1,k_2)\in\mathbb{Z}^2}|\zeta(w(x_1,x_2)-(k_1,k_2))|\left[\|(k_1,k_2)-w(x_1,x_2)\|^2+\|w(x_1,x_2)\|^2\right]$$
$$\leq\frac{2}{w^2}M_0(\zeta)\widetilde{M}_2(\zeta)+\frac{4}{w^2}\widetilde{M}_0(\zeta)\left(M_2(\zeta)+w^2M_0(\zeta)\|(x_1,x_2)\|^2\right).$$

At the end, by combining estimates I_1 and I_2 we have

$$\left|\left(S_w^{\zeta,\zeta}p\right)(x_1,x_2)\right|\leq M_0(\zeta)\left(\widetilde{M}_0(\zeta)+\frac{2}{w^2}\widetilde{M}_2(\zeta)\right)+4\widetilde{M}_0(\zeta)\left(\frac{1}{w^2}M_2(\zeta)+M_0(\zeta)\|(x_1,x_2)\|^2\right)$$

which is desired. Q.E.D.

Theorem 3.2. Let ζ be kernel with $\alpha=\beta=2$. Then the inequality

$$\left\|S_w^{\zeta,\zeta}\right\|_{B_{\widetilde{\rho}}(\mathbb{R}^2)\to B_{\widetilde{\rho}}(\mathbb{R}^2)}\leq M_0(\zeta)\left(\widetilde{M}_0(\zeta)+\frac{2}{w^2}\widetilde{M}_2(\zeta)\right)+4\widetilde{M}_0(\zeta)\left(\frac{1}{w^2}M_2(\zeta)+M_0(\zeta)\right)$$

holds. In particular, $S_w^{\zeta,\zeta}$ is a linear operator from $B_{\widetilde{\rho}}\left(\mathbb{R}^2\right)$ to $B_{\widetilde{\rho}}\left(\mathbb{R}^2\right)$ for any fixed $w>0$.

Proof. By using the definition of $\left(S_w^{\zeta,\zeta}f\right)$ and Proposition 3.1 we have

$$\left|\left(S_w^{\zeta,\zeta}f\right)(x_1,x_2)\right|$$
$$\leq\sum_{(k_1,k_2)\in\mathbb{Z}^2}|\zeta(w(x_1,x_2)-(k_1,k_2))|\,w^2\int_{\mathbb{R}^2}|\zeta(w(u_1,u_2)-(k_1,k_2))|\,|f(u_1,u_2)|\,du_1du_2$$
$$=\sum_{(k_1,k_2)\in\mathbb{Z}^2}|\zeta(w(x_1,x_2)-(k_1,k_2))|\,w^2\int_{\mathbb{R}^2}|\zeta(w(u_1,u_2)-(k_1,k_2))|\,\widetilde{\rho}(x_1,x_2)\,|f(u_1,u_2)|\,p(x_1,x_2)\,du_1du_2$$
$$\leq\|f\|_{\widetilde{\rho}}\left[M_0(\zeta)\left(\widetilde{M}_0(\zeta)+\frac{2}{w^2}\widetilde{M}_2(\zeta)\right)+4\widetilde{M}_0(\zeta)\left(\frac{1}{w^2}M_2(\zeta)+M_0(\zeta)\|(x_1,x_2)\|^2\right)\right].$$

If we multiply both sides with $\widetilde{\rho}(x_1,x_2)$, we have

$$\widetilde{\rho}(x_1,x_2)\left|\left(S_w^{\zeta,\zeta}f\right)(x_1,x_2)\right|\leq\|f\|_{\widetilde{\rho}}\left[M_0(\zeta)\left(\widetilde{M}_0(\zeta)+\frac{2}{w^2}\widetilde{M}_2(\zeta)\right)+4\widetilde{M}_0(\zeta)\left(\frac{1}{w^2}M_2(\zeta)+M_0(\zeta)\right)\right].$$

By assumption, since ζ is a kernel with $\alpha=\beta=2$, we conclude $\left\|S_w^{\zeta,\zeta}\right\|_{\widetilde{\rho}}<+\infty$, that is $S_w^{\zeta,\zeta}f\in B_{\widetilde{\rho}}\left(\mathbb{R}^2\right)$. Now taking supremum over $(x_1,x_2)\in\mathbb{R}^2$ and the supremum with respect to $f\in B_{\widetilde{\rho}}\left(\mathbb{R}^2\right)$ with $\|f\|\leq 1$ it turns out that

$$\left\|S_w^{\zeta,\zeta}\right\|_{B_{\widetilde{\rho}}(\mathbb{R}^2)\to B_{\widetilde{\rho}}(\mathbb{R}^2)}\leq M_0(\zeta)\left(\widetilde{M}_0(\zeta)+\frac{2}{w^2}\widetilde{M}_2(\zeta)\right)+4\widetilde{M}_0(\zeta)\left(\frac{1}{w^2}M_2(\zeta)+M_0(\zeta)\right)$$

which completes the proof. Q.E.D.

4 Convergence of the family of operators $S_w^{\zeta,\zeta}$

In this section, we present pointwise and uniform convergence of the sampling Durrmeyer operators in weighted spaces of bivariate functions.

Theorem 4.1. Let $f \in C_{\widetilde{\rho}}\left(\mathbb{R}^2\right)$ be fixed and ζ kernel with $\alpha = \beta = 2$. Then

$$\lim_{w \to +\infty} \left(S_w^{\zeta,\zeta} f\right)(x_1, x_2) = f(x_1, x_2) \tag{4.1}$$

holds for $(x_1,x_2) \in \mathbb{R}^2$. Moreover if $f \in U_{\widetilde{\rho}}\left(\mathbb{R}^2\right)$, then

$$\lim_{w \to +\infty} \left\| S_w^{\zeta,\zeta} f - f \right\|_{\widetilde{\rho}} = 0. \tag{4.2}$$

Proof. For all $(x_1, x_2), (u_1, u_2) \in \mathbb{R}^2$, the inequality

$$\begin{aligned} |f(u_1, u_2) - f(x_1, x_2)| \leq \widetilde{\rho}(u_1, u_2) |f(u_1, u_2)| \left| \frac{1}{\widetilde{\rho}(u_1, u_2)} - \frac{1}{\widetilde{\rho}(x_1, x_2)} \right| \\ + \frac{1}{\widetilde{\rho}(x_1, x_2)} \left| \widetilde{\rho}(u_1, u_2) f(u_1, u_2) - \widetilde{\rho}(x_1, x_2) f(x_1, x_2) \right| \end{aligned}$$

holds. By using the above inequality, linearity of $\left(S_w^{\zeta,\zeta} f\right)$ and definition of kernel we have

$$\begin{aligned} &\left| \left(S_w^{\zeta,\zeta} f\right)(x_1, x_2) - f(x_1, x_2) \right| \\ &\leq \sum_{(k_1,k_2) \in \mathbb{Z}^2} |\zeta(w(x_1, x_2) - (k_1,k_2))| \, w^2 \int_{\mathbb{R}^2} |\zeta(w(u_1, u_2) - (k_1, k_2))| \, |f(u_1, u_2) - f(x_1, x_2)| \, du_1 du_2 \\ &\leq \sum_{(k_1,k_2) \in \mathbb{Z}^2} |\zeta(w(x_1, x_2) - (k_1,k_2))| \, w^2 \\ &\times \int_{\mathbb{R}^2} |\zeta(w(u_1, u_2) - (k_1, k_2))| \, \widetilde{\rho}(u_1, u_2) |f(u_1, u_2)| \left| \frac{1}{\widetilde{\rho}(u_1, u_2)} - \frac{1}{\widetilde{\rho}(x_1, x_2)} \right| du_1 du_2 \\ &+ \sum_{(k_1,k_2) \in \mathbb{Z}^2} |\zeta(w(x_1, x_2) - (k_1,k_2))| \, w^2 \\ &\times \int_{\mathbb{R}^2} |\zeta(w(u_1, u_2) - (k_1, k_2))| \frac{1}{\widetilde{\rho}(x_1, x_2)} \left| \widetilde{\rho}(u_1, u_2) f(u_1, u_2) - \widetilde{\rho}(x_1, x_2) f(x_1, x_2) \right| du_1 du_2 \\ &= \sum_{(k_1,k_2) \in \mathbb{Z}^2} |\zeta(w(x_1, x_2) - (k_1,k_2))| \, w^2 \\ &\times \int_{\mathbb{R}^2} |\zeta(w(u_1, u_2) - (k_1, k_2))| \, \widetilde{\rho}(u_1, u_2) |f(u_1, u_2)| \left| u_1^2 + u_2^2 - x_1^2 - x_2^2 \right| du_1 du_2 \\ &+ \sum_{(k_1,k_2) \in \mathbb{Z}^2} |\zeta(w(x_1, x_2) - (k_1,k_2))| \, w^2 \\ &\times \int_{\mathbb{R}^2} |\zeta(w(u_1, u_2) - (k_1, k_2))| \frac{1}{\widetilde{\rho}(x_1, x_2)} \left| \widetilde{\rho}(u_1, u_2) f(u_1, u_2) - \widetilde{\rho}(x_1, x_2) f(x_1, x_2) \right| du_1 du_2 \\ &= I_1 + I_2. \end{aligned}$$

Firstly, we consider I_1. Similar calculation given in [9], we have

$$\begin{aligned} &\left|u_1^2+u_2^2-x_1^2-x_2^2\right| \\ \leq &\frac{1}{w^2}\left[\left|wu_1-k_1\right|^2+\left|wu_1-k_1\right|\left|k_1+wx_1\right|+\left|k_1-wx_1\right|\left|wu_1-k_1\right|\right. \\ &\left.+\left|k_1-wx_1\right|\left|k_1+wx_1\right|\right] \\ +&\frac{1}{w^2}\left[\left|wu_2-k_2\right|^2+\left|wu_2-k_2\right|\left|k_2+wx_2\right|+\left|k_2-wx_2\right|\left|wu_2-k_2\right|\right. \\ &\left.+\left|k_2-wx_2\right|\left|k_2+wx_2\right|\right]. \end{aligned} \tag{4.3}$$

Using (4.3) we obtain

$$\begin{aligned} I_1 \leq \|f\|_{\widetilde{\rho}} &\left[\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right|\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|\left(\left|wu_1-k_1\right|^2+\left|wu_2-k_2\right|^2\right)du_1du_2\right. \\ &+\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right|\left|k_1+wx_1\right|\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|\left|wu_1-k_1\right|du_1du_2 \\ &+\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right|\left|k_1-wx_1\right|\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|\left|wu_1-k_1\right|du_1du_2 \\ &+\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right|\left|k_1-wx_1\right|\left|k_1+wx_1\right|\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|du_1du_2 \\ &+\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right|\left|k_2+wx_2\right|\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|\left|wu_2-k_2\right|du_1du_2 \\ &+\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right|\left|k_2-wx_2\right|\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|\left|wu_2-k_2\right|du_1du_2 \\ &\left.+\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right|\left|k_2-wx_2\right|\left|k_2+wx_2\right|\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|du_1du_2\right] \\ \leq &\frac{\|f\|_{\widetilde{\rho}}}{w^2}\left[M_0\left(\zeta\right)\widetilde{M}_2\left(\zeta\right)+4M_1\left(\zeta\right)\widetilde{M}_1\left(\zeta\right)+2wM_0\left(\zeta\right)\widetilde{M}_1\left(\zeta\right)\left[\left|x_1\right|+\left|x_2\right|\right]\right. \\ &\left.+2M_2\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)+2wM_1\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)\left[\left|x_1\right|+\left|x_2\right|\right]\right]. \end{aligned}$$

Now, we estimate I_2 by rewriting it into three parts:

$$\begin{aligned} I_2 = \frac{1}{\widetilde{\rho}\left(x_1,x_2\right)} &\left[\sum_{\|w(x_1,x_2)-(k_1,k_2)\|\leq\frac{w\delta}{2}}\left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right|w^2\right. \\ \times &\int_{\|w(u_1,u_2)-(k_1,k_2)\|\leq\frac{w\delta}{2}}\left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|\left|\widetilde{\rho}\left(u_1,u_2\right)f\left(u_1,u_2\right)-\widetilde{\rho}\left(x_1,x_2\right)f\left(x_1,x_2\right)\right|du_1du_2 \end{aligned}$$

$$+\sum_{\|w(x_1,x_2)-(k_1,k_2)\|\le\frac{w\delta}{2}}\left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right|w^2$$

$$\times\int_{\|w(u_1,u_2)-(k_1,k_2)\|>\frac{w\delta}{2}}\left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|\left|\widetilde{\rho}\left(u_1,u_2\right)f\left(u_1,u_2\right)-\widetilde{\rho}\left(x_1,x_2\right)f\left(x_1,x_2\right)\right|du_1du_2$$

$$+\sum_{\|w(x_1,x_2)-(k_1,k_2)\|>\frac{w\delta}{2}}\left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right|w^2$$

$$\left.\times\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|\left|\widetilde{\rho}\left(u_1,u_2\right)f\left(u_1,u_2\right)-\widetilde{\rho}\left(x_1,x_2\right)f\left(x_1,x_2\right)\right|du_1du_2\right]$$

$$=\frac{1}{\widetilde{\rho}\left(x_1,x_2\right)}\left[I_{2,1}+I_{2,2}+I_{2,3}\right].$$

For $(u_1,u_2)\in\mathbb{R}^2$ with the property $\left|w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right|\le\frac{w\delta}{2}$ if we also have $\left|w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right|\le\frac{w\delta}{2}$ then we have

$$\left|(u_1,u_2)-(x_1,x_2)\right|\le\left|(u_1,u_2)-\left(\frac{k_1}{w},\frac{k_2}{w}\right)\right|+\left|\left(\frac{k_1}{w},\frac{k_2}{w}\right)-(x_1,x_2)\right|.$$

Since $\widetilde{\rho}f\in C_{\widetilde{\rho}}\left(\mathbb{R}^2\right)$, we have $I_{2,1}\le\varepsilon M_0\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)$. For $I_{2,2}$, taking supremum for $(u_1,u_2)\in\mathbb{R}^2$, we have

$$I_{2,2}\le 2\left\|f\right\|_{\widetilde{\rho}}\sum_{\|w(x_1,x_2)-(k_1,k_2)\|\le\frac{w\delta}{2}}\left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right|w^2$$
$$\times\int_{\|w(u_1,u_2)-(k_1,k_2)\|>\frac{w\delta}{2}}\left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|du_1du_2$$

and in view of Lemma 2.2

$$w^2\int_{\|w(u_1,u_2)-(k_1,k_2)\|>\frac{w\delta}{2}}\left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|du_1du_2$$
$$=\int_{\|(y_1,y_2)\|>\frac{w\delta}{2}}\left|\zeta\left(\left(y_1,y_2\right)\right)\right|dy_1dy_2\to 0\text{ as }w\to+\infty$$

for sufficiently large w, so, we have

$$I_{2,2}\le 2\left\|f\right\|_{\widetilde{\rho}}M_0\left(\zeta\right)\varepsilon.$$

Finally, by Lemma 2.2, since for every $\eta>0$, we have

$$\lim_{w\to+\infty}\sum_{|w(x_1,x_2)-(k_1,k_2)|>\eta}\left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right|=0$$

uniformly with respect to $(x_1, x_2) \in \mathbb{R}^2$ we get

$$I_{2,3} \leq 2 \|f\|_{\widetilde{\rho}} \widetilde{M}_0(\zeta) \varepsilon$$

for sufficiently large w. Hence, combining all obtained estimates we have

$$\begin{aligned}\left|\left(S_w^{\zeta,\zeta} f\right)(x_1, x_2) - f(x_1, x_2)\right| \leq \frac{\|f\|_{\widetilde{\rho}}}{w^2} \Big[& M_0(\zeta) \widetilde{M}_2(\zeta) + 4M_1(\zeta) \widetilde{M}_1(\zeta) + 2wM_0(\zeta) \widetilde{M}_1(\zeta) [|x_1| + |x_2|] \\ & + 2M_2(\zeta) \widetilde{M}_0(\zeta) + 2wM_1(\zeta) \widetilde{M}_0(\zeta) [|x_1| + |x_2|] \Big] \\ & + \frac{\varepsilon}{\widetilde{\rho}(x_1, x_2)} \left[M_0(\zeta) \widetilde{M}_0(\zeta) + 2 \|f\|_{\widetilde{\rho}} \left(M_0(\zeta) + \widetilde{M}_0(\zeta) \right) \right].\end{aligned} \tag{4.4}$$

By taking limit as $w \to +\infty$ we get (4.1). Let us consider $f \in U_{\widetilde{w}}\left(\mathbb{R}^2\right)$. If we multiply both sides of (4.4) with $\widetilde{\rho}(x_1, x_2)$,we have

$$\begin{aligned}&\widetilde{\rho}(x_1, x_2) \left|\left(S_w^{\zeta,\zeta} f\right)(x_1, x_2) - f(x_1, x_2)\right| \\ &\leq \frac{\|f\|_{\widetilde{\rho}}}{w^2} \left[M_0(\zeta) \widetilde{M}_2(\zeta) + 4M_1(\zeta) \widetilde{M}_1(\zeta) + 2wM_0(\zeta) \widetilde{M}_1(\zeta) + 2M_2(\zeta) \widetilde{M}_0(\zeta) + 2wM_1(\zeta) \widetilde{M}_0(\zeta) \right] \\ &+ \varepsilon \left[M_0(\zeta) \widetilde{M}_0(\zeta) + 2 \|f\|_{\widetilde{\rho}} \left(M_0(\zeta) + \widetilde{M}_0(\zeta) \right) \right]\end{aligned}$$

and taking supremum over $(x_1, x_2) \in \mathbb{R}^2$ we obtain (4.2) for $w \to +\infty$. Q.E.D.

5 Rate of convergence of the operators $S_w^{\zeta,\zeta}$

This section deals with the determine rate of convergence the operators $S_w^{\zeta,\zeta}$ in the weighted spaces of bivariate functions via weighted modulus of continuity given in (2.1).

Theorem 5.1. Let ζ be kernel with $\alpha = \beta = 6$. Then, for $f \in C_{\widetilde{\rho}}\left(\mathbb{R}^2\right)$ the inequality

$$\begin{aligned}\left\|S_w^{\zeta,\zeta} f - f\right\|_{\widetilde{w}} \leq & 256 \Omega\left(f; w^{-1}, w^{-1}\right) \\ & \times \Big\{ M_0(\zeta) \widetilde{M}_0(\zeta) + 16\sqrt{2} \left(M_0(\zeta) \widetilde{M}_3(\zeta) + M_3(\zeta) \widetilde{M}_0(\zeta) \right) \\ & + 16 \left[M_0(\zeta) \widetilde{M}_6(\zeta) + 2M_3(\zeta) \widetilde{M}_3(\zeta) + M_6(\zeta) \widetilde{M}_0(\zeta) \right] \Big\}\end{aligned}$$

holds for $w \geq 1$.

Proof. By using definition of $S_w^{\zeta,\zeta}$ and the inequality (2.4) we have

$$\begin{aligned}&\left|\left(S_w^{\zeta,\zeta} f\right)(x_1, x_2) - f(x_1, x_2)\right| \\ &\leq 256 \left(1 + x_1^2 + x_2^2\right) \Omega(f; \delta_1, \delta_2) \sum_{(k_1,k_2) \in \mathbb{Z}^2} |\zeta(w(x_1, x_2) - (k_1, k_2))| \, w^2 \int_{\mathbb{R}^2} |\zeta(w(u_1, u_2) - (k_1, k_2))| \\ &\times \left\{ 1 + \frac{|u_1 - x_1|^3}{\delta_1^3} + \frac{|u_2 - x_2|^3}{\delta_2^3} + \frac{|u_1 - x_1|^3}{\delta_1^3} \frac{|u_2 - x_2|^3}{\delta_2^3} \right\} du_1 du_2\end{aligned}$$

$$
\begin{aligned}
&\leq 256\left(1+x_1^2+x_2^2\right)\Omega\left(f;\delta_1,\delta_2\right)\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\zeta\left(w\left(x_1,x_2\right)-(k_1,k_2)\right)\right|w^2\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-(k_1,k_2)\right)\right| \\
&\times\left\{1+\frac{4}{w^3\delta_1^3}\left(\left|wu_1-k_1\right|^3+\left|wx_1-k_1\right|^3\right)+\frac{4}{w^3\delta_2^3}\left(\left|wu_2-k_2\right|^3+\left|wx_2-k_2\right|^3\right)\right. \\
&\left.+\frac{16}{\delta_1^3\delta_2^3w^6}\left[\left|wu_1-k_1\right|^3+\left|wx_1-k_1\right|^3\right]\left[\left|wu_2-k_2\right|^3+\left|wx_2-k_2\right|^3\right]\right\}du_1du_2 \\
&=I_1+I_2.
\end{aligned}
$$

Now, first we estimate I_1. By simple calculations we have

$$
\begin{aligned}
I_1 &\leq 256\left(1+x_1^2+x_2^2\right)\Omega\left(f;\delta_1,\delta_2\right)\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\zeta\left(w\left(x_1,x_2\right)-(k_1,k_2)\right)\right|w^2\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-(k_1,k_2)\right)\right| \\
&\times\left\{1+\frac{4}{w^3}\left(\frac{1}{\delta_1^3}+\frac{1}{\delta_2^3}\right)\left[2\left(\left|wu_1-k_1\right|^2+\left|wu_2-k_2\right|^2\right)\right]^{\frac{3}{2}}\right. \\
&\left.+\frac{4}{w^3}\left(\frac{1}{\delta_1^3}+\frac{1}{\delta_2^3}\right)\left[2\left(\left|wx_1-k_1\right|^2+\left|wx_2-k_2\right|^2\right)\right]^{\frac{3}{2}}\right\}du_1du_2 \\
&=256\left(1+x_1^2+x_2^2\right)\Omega\left(f;\delta_1,\delta_2\right)\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\zeta\left(w\left(x_1,x_2\right)-(k_1,k_2)\right)\right|w^2\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-(k_1,k_2)\right)\right| \\
&\times\left\{1+\frac{8\sqrt{2}}{w^3}\left(\frac{1}{\delta_1^3}+\frac{1}{\delta_2^3}\right)\left[\left(\left|wu_1-k_1\right|^2+\left|wu_2-k_2\right|^2\right)^{\frac{3}{2}}+\left(\left|wx_1-k_1\right|^2+\left|wx_2-k_2\right|^2\right)^{\frac{3}{2}}\right]\right\}du_1du_2 \\
&\leq 256\left(1+x_1^2+x_2^2\right)\Omega\left(f;\delta_1,\delta_2\right) \\
&\times\left\{M_0(\zeta)\widetilde{M}_0(\zeta)+\frac{8\sqrt{2}}{w^3}\left(\frac{1}{\delta_1^3}+\frac{1}{\delta_2^3}\right)\left(M_0(\zeta)\widetilde{M}_3(\zeta)+M_3(\zeta)\widetilde{M}_0(\zeta)\right)\right\}.
\end{aligned}
$$

For I_2, from the facts that $a^3b^3\leq\left(a^2+b^2\right)^3$ and $a^3\leq\left(a^2+b^2\right)^{3/2}$ for $a,b>0$ we obtain

$$
\begin{aligned}
I_2 &= 256\left(1+x_1^2+x_2^2\right)\Omega\left(f;\delta_1,\delta_2\right)\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\zeta\left(w\left(x_1,x_2\right)-(k_1,k_2)\right)\right|w^2\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-(k_1,k_2)\right)\right| \\
&\times\frac{16}{\delta_1^3\delta_2^3w^6}\left[\left|wu_1-k_1\right|^3+\left|wx_1-k_1\right|^3\right]\left[\left|wu_2-k_2\right|^3+\left|wx_2-k_2\right|^3\right]du_1du_2 \\
&\leq 256\left(1+x_1^2+x_2^2\right)\Omega\left(f;\delta_1,\delta_2\right)\sum_{(k_1,k_2)\in\mathbb{Z}^2}\left|\zeta\left(w\left(x_1,x_2\right)-(k_1,k_2)\right)\right|w^2\int_{\mathbb{R}^2}\left|\zeta\left(w\left(u_1,u_2\right)-(k_1,k_2)\right)\right| \\
&\times\frac{16}{\delta_1^3\delta_2^3w^6}\left[\left|wu_1-k_1\right|^3\left|wu_2-k_2\right|^3+\left|wu_1-k_1\right|^3\left|wx_2-k_2\right|^3\right. \\
&\left.+\left|wx_1-k_1\right|^3\left|wu_2-k_2\right|^3+\left|wx_1-k_1\right|^3\left|wx_2-k_2\right|^3\right]du_1du_2 \\
&\leq 256\left(1+x_1^2+x_2^2\right)\Omega\left(f;\delta_1,\delta_2\right)\frac{16}{\delta_1^3\delta_2^3w^6}\left[M_0(\zeta)\widetilde{M}_6(\zeta)+2M_3(\zeta)\widetilde{M}_3(\zeta)+M_6(\zeta)\widetilde{M}_0(\zeta)\right].
\end{aligned}
$$

Combining the above estimates, we have

$$\begin{aligned}\left|\left(S_w^{\zeta,\zeta} f\right)(x_1,x_2) - f(x_1,x_2)\right| &\le 256\left(1+x_1^2+x_2^2\right)\Omega\left(f;\delta_1,\delta_2\right)\\ &\times\Bigg\{M_0(\zeta)\widetilde{M}_0(\zeta)+\frac{8\sqrt{2}}{w^3}\left(\frac{1}{\delta_1^3}+\frac{1}{\delta_2^3}\right)\left(M_0(\zeta)\widetilde{M}_3(\zeta)+M_3(\zeta)\widetilde{M}_0(\zeta)\right)\\ &+\frac{16}{\delta_1^3\delta_2^3 w^6}\left[M_0(\zeta)\widetilde{M}_6(\zeta)+2M_3(\zeta)\widetilde{M}_3(\zeta)+M_6(\zeta)\widetilde{M}_0(\zeta)\right]\Bigg\}.\end{aligned}$$

Finally, by choosing $\delta_1 = w^{-1}, \delta_2 = w^{-1}$ and taking the supremum for $(x_1, x_2) \in \mathbb{R}^2$ we get desired result. Q.E.D.

6 Voronovskaja type theorem for the operators $S_w^{\zeta,\zeta}$

In this section, we present Voronovskaja type theorem in quantitative form for the operators $S_w^{\zeta,\zeta}$ in the weighted spaces of bivariate functions.

Let $\underline{x} = (x_1, x_2), \underline{k} = (k_1, k_2) \in \mathbb{R}_+^2, |\underline{k}| = r$, for a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$D^r f := \frac{\partial^r}{\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}} f$$

we denote the r-th order derivatives of f. For $r \in \mathbb{N}$, by $C^{(r)}(K)$ we denote the subspace of $C^0(K)$ which consist of all functions f with the derivatives up to the order r in $C^0(K)$. By the Taylor expansion of f (see [39]),

$$\begin{aligned} f(s_1,s_2) = f(x_1,x_2) + (s_1-x_1)\frac{\partial f}{\partial x_1}(x_1,x_2) + (s_2-x_2)\frac{\partial f}{\partial x_2}(x_1,x_2)\\ + R_1(s_1,s_2), \end{aligned} \tag{6.1}$$

where

$$R_1(s_1,s_2) = \left\{(s_1-x_1)\left[\frac{\partial f}{\partial x_1}(\eta_{x_1},\eta_{x_2}) - \frac{\partial f}{\partial x_1}(x_1,x_2)\right] + (s_2-x_2)\left[\frac{\partial f}{\partial x_2}(\eta_{x_1},\eta_{x_2}) - \frac{\partial f}{\partial x_2}(x_1,x_2)\right]\right\} \tag{6.2}$$

such that $\eta_{x_1} = x_1 + \theta(s_1 - x_1), \eta_{x_2} = x_2 + \theta(s_2 - x_2)$ and $0 < \theta < 1$.

In view of the inequality (2.4), with similar method presented in [1] and [8] we conclude that

$$\begin{aligned} &|R_1(s_1,s_2)|\\ &\le 256\left(1+x_1^2+x_2^2\right)\\ &\times\Bigg\{\Omega\left(f_{x_1};\delta_1,\delta_2\right)\left[|s_1-x_1| + \frac{1}{\delta_2^3}|s_2-x_2|^3|s_1-x_1| + \frac{1}{\delta_1^3}|s_1-x_1|^4 + \frac{1}{\delta_1^3\delta_2^3}|s_1-x_1|^4|s_2-x_2|^3\right]\\ &+\Omega\left(f_{x_2};\delta_1,\delta_2\right)\left[|s_2-x_2| + \frac{1}{\delta_2^3}|s_2-x_2|^4 + \frac{1}{\delta_1^3}|s_1-x|^3|s_2-x_2| + \frac{1}{\delta_1^3\delta_2^3}|s_2-x_2|^4|s_1-x_1|^3\right]\Bigg\}. \end{aligned} \tag{6.3}$$

Now, let $\underline{\mathrm{h}} = (h_1, h_2) \in \mathbb{N}_0^2$ and let $v = |\underline{\mathrm{h}}|$. For $\underline{\mathrm{u}} = (u_1, u_2) \in \mathbb{R}_+^2$ we define the discrete and continuous algebraic moments of order $\underline{\mathrm{h}}$ of ζ as

$$\begin{aligned} m_{\underline{\mathrm{h}}}^{v}(\zeta, \underline{\mathrm{u}}) &:= \sum_{(k_1,k_2)\in\mathbb{Z}^2} \zeta\left((u_1, u_2) - (k_1, k_2)\right) \langle (k_1, k_2) - (u_1, u_2)\rangle^{\underline{\mathrm{h}}} \\ &= \sum_{(k_1,k_2)\in\mathbb{Z}^2} \zeta\left((u_1, u_2) - (k_1, k_2)\right) (k_1 - u_1)^{h_1} (k_2 - u_2)^{h_2} \end{aligned}$$

and

$$\begin{aligned} \widetilde{m}_{\underline{\mathrm{h}}}^{v}(\zeta, \underline{\mathrm{u}}) &:= \int_{\mathbb{R}^2} \zeta\left((u_1, u_2)\right) \langle (u_1, u_2)\rangle^{\underline{\mathrm{h}}} du_1 du_2 \\ &= \int_{\mathbb{R}^2} \zeta\left((u_1, u_2)\right) u_1^{h_1} u_2^{h_2} du_1 du_2. \end{aligned}$$

We need one more assumption on the kernel function ζ to estimate the order of approximation under a local regularity assumption on function f. In particular, there exists a natural number $l \in \mathbb{N}$ such that $\underline{\mathrm{h}} \in \mathbb{N}_0^2$ and $|\underline{\mathrm{h}}| \leq l$

$$\begin{aligned} m_{\underline{\mathrm{h}}}^{|\underline{\mathrm{h}}|}(\zeta, \underline{\mathrm{u}}) &:= m_{\underline{\mathrm{h}}}^{|\underline{\mathrm{h}}|}(\zeta) \text{ is independent of } \underline{\mathrm{u}}. \\ \widetilde{m}_{\underline{\mathrm{h}}}^{|\underline{\mathrm{h}}|}(\zeta, \underline{\mathrm{u}}) &:= \widetilde{m}_{\underline{\mathrm{h}}}^{|\underline{\mathrm{h}}|}(\zeta) \text{ is independent of } \underline{\mathrm{u}}. \end{aligned} \tag{$\star$}$$

Theorem 6.1. Let ζ be kernel with $\alpha = \beta = 7$ and $(\star)$ for $l = 1$ such that

$$\left(m_0(\zeta)\, \widetilde{m}_{(1,0)}^{1}(\zeta) + m_{(1,0)}^{1}(\zeta)\, \widetilde{m}_0(\zeta)\right) \neq 0$$

or

$$\left(m_0(\zeta)\, \widetilde{m}_{(0,1)}^{1}(\zeta) + m_{(0,1)}^{1}(\zeta)\, \widetilde{m}_0(\zeta)\right) \neq 0.$$

Then, for $f' \in C_{\widetilde{\rho}}^{1}\left(\mathbb{R}^2\right)$ we have

$$\begin{aligned} &\Bigg| w\left[\left(S_w^{\zeta,\zeta} f\right)(x_1, x_2) - f(x_1, x_2)\right] - \left(m_0(\zeta)\, \widetilde{m}_{(1,0)}^{1}(\zeta) + m_{(1,0)}^{1}(\zeta)\, \widetilde{m}_0(\zeta)\right) \frac{\partial f}{\partial x_1}(x_1, x_2) \\ &+ \left(m_0(\zeta)\, \widetilde{m}_{(0,1)}^{1}(\zeta) + m_{(0,1)}^{1}(\zeta)\, \widetilde{m}_0(\zeta)\right) \frac{\partial f}{\partial x_2}(x_1, x_2) \Bigg| \\ &\leq 256\left(1 + x_1^2 + x_2^2\right)\left[\Omega\left(f_{x_1}; w^{-1}, w^{-1}\right) + \Omega\left(f_{x_2}; w^{-1}, w^{-1}\right)\right] \\ &\times \Bigg\{ \sqrt{2}\left(M_0(\zeta)\, \widetilde{M}_1(\zeta) + M_1(\zeta)\, \widetilde{M}_0(\zeta)\right) + 40\left(M_0(\zeta)\, \widetilde{M}_4(\zeta) + M_4(\zeta)\, \widetilde{M}_0(\zeta)\right) \\ &+ 512\sqrt{2}\left(M_0(\zeta)\, \widetilde{M}_7(\zeta) + M_7(\zeta)\, \widetilde{M}_0(\zeta)\right) \Bigg\}. \end{aligned}$$

Proof. Using definition of the operator $S_w^{\zeta,\zeta}$ and consider the Taylor expansion given in (6.1) we can write

$$\begin{aligned}\left(S_w^{\zeta,\zeta}f\right)(x_1,x_2) &= \sum_{(k_1,k_2)\in\mathbb{Z}^2}\zeta\left(w\left(x_1,x_2\right)-(k_1,k_2)\right)w^2\int_{\mathbb{R}^2}\zeta\left(w\left(u_1,u_2\right)-(k_1,k_2)\right)f\left(u_1,u_2\right)du_1du_2\\ &= \sum_{(k_1,k_2)\in\mathbb{Z}^2}\zeta\left(w\left(x_1,x_2\right)-(k_1,k_2)\right)w^2\int_{\mathbb{R}^2}\zeta\left(w\left(u_1,u_2\right)-(k_1,k_2)\right)\\ &\times\left[f\left(x_1,x_2\right)+\left(u_1-x_1\right)\frac{\partial f}{\partial x_1}\left(x_1,x_2\right)+\left(u_2-x_2\right)\frac{\partial f}{\partial x_2}\left(x_1,x_2\right)\right]du_1du_2\\ &+\sum_{(k_1,k_2)\in\mathbb{Z}^2}\zeta\left(w\left(x_1,x_2\right)-(k_1,k_2)\right)w^2\int_{\mathbb{R}^2}\zeta\left(w\left(u_1,u_2\right)-(k_1,k_2)\right)R_1\left(u_1,u_2\right)du_1du_2\\ &:= I_1+I_2\end{aligned}$$

where $R_1(u_1,u_2)$ is the remainder as in (6.2). Let us first estimate I_1. Using definition of the discrete algebraic moments and the operators we get

$$\begin{aligned}I_1 &= f\left(x_1,x_2\right)+\frac{1}{w}\left(m_0\left(\zeta\right)\widetilde{m}^1_{(1,0)}\left(\zeta\right)+m^1_{(1,0)}\left(\zeta\right)\widetilde{m}_0\left(\zeta\right)\right)\frac{\partial f}{\partial x_1}\left(x_1,x_2\right)\\ &+\frac{1}{w}\left(m_0\left(\zeta\right)\widetilde{m}^1_{(0,1)}\left(\zeta\right)+m^1_{(0,1)}\left(\zeta\right)\widetilde{m}_0\left(\zeta\right)\right)\frac{\partial f}{\partial x_2}\left(x_1,x_2\right).\end{aligned}$$

Now, we take into account that I_2. By using the inequality (6.3), we have

$$\begin{aligned}|I_2| &\leq 256\left(1+x_1^2+x_2^2\right)\sum_{(k_1,k_2)\in\mathbb{Z}^2}\zeta\left(w\left(x_1,x_2\right)-(k_1,k_2)\right)w^2\int_{\mathbb{R}^2}\zeta\left(w\left(u_1,u_2\right)-(k_1,k_2)\right)\\ &\times\Bigg\{\Omega\left(f_{x_1};\delta_1,\delta_2\right)\left[\left|u_1-x_1\right|+\frac{1}{\delta_2^3}\left|u_2-x_2\right|^3\left|u_1-x_1\right|+\frac{1}{\delta_1^3}\left|u_1-x_1\right|^4+\frac{1}{\delta_1^3\delta_2^3}\left|u_2-x_2\right|^3\left|u_1-x_1\right|^4\right]\\ &+\Omega\left(f_{x_2};\delta_1,\delta_2\right)\left[\left|u_2-x_2\right|+\frac{1}{\delta_2^3}\left|u_2-x_2\right|^4+\frac{1}{\delta_1^3}\left|u_1-x_1\right|^3\left|u_2-x_2\right|^1+\frac{1}{\delta_1^3\delta_2^3}\left|u_2-x_2\right|^4\left|u_1-x_1\right|^3\right]\Bigg\}du_1du_2\end{aligned}$$

and by considering the inequalities

$$\begin{array}{rlcrl}\left|u_1-x_1\right| & \leq\left|u_1-x_1\right|+\left|u_2-x_2\right| & , & \left|u_2-x_2\right| & \leq\left|u_1-x_1\right|+\left|u_2-x_2\right|\\ \left|u_1-x_1\right|^4 & \leq\left(\left|u_1-x_1\right|^2+\left|u_2-x_2\right|^2\right)^2 & , & \left|u_2-x_2\right|^4 & \leq\left(\left|u_1-x_1\right|^2+\left|u_2-x_2\right|^2\right)^2\\ \left|u_1-x_1\right|\left|u_2-x_2\right|^3 & \leq 2^2\left(\left|u_1-x_1\right|^2+\left|u_2-x_2\right|^2\right)^2 & , & \left|u_1-x_1\right|^3\left|u_2-x_2\right| & \leq 2^2\left(\left|u_1-x_1\right|^2+\left|u_2-x_2\right|^2\right)^2\\ \left|u_1-x_1\right|^3\left|u_2-x_2\right|^4 & \leq 2^{\frac{7}{2}}\left(\left|u_1-x_1\right|^2+\left|u_2-x_2\right|^2\right)^{\frac{7}{2}} & , & \left|u_1-x_1\right|^4\left|u_2-x_2\right|^3 & \leq 2^{\frac{7}{2}}\left(\left|u_1-x_1\right|^2+\left|u_2-x_2\right|^2\right)^{\frac{7}{2}}\end{array}$$

we get

$$\left|I_2\right| \leq 256\left(1+x_1^2+x_2^2\right) \sum_{(k_1,k_2)\in\mathbb{Z}^2} \left|\zeta\left(w\left(x_1,x_2\right)-\left(k_1,k_2\right)\right)\right| w^2 \int_{\mathbb{R}^2} \left|\zeta\left(w\left(u_1,u_2\right)-\left(k_1,k_2\right)\right)\right|$$

$$\times\left\{\Omega\left(f_{x_1};\delta_1,\delta_2\right)\left[\left|u_1-x_1\right|+\left|u_2-x_2\right|+\frac{2^2}{\delta_2^3}\left(\left|u_1-x_1\right|^2+\left|u_2-x_2\right|^2\right)^2\right.\right.$$

$$\left.+\frac{1}{\delta_1^3}\left(\left|u_1-x_1\right|^2+\left|u_2-x_2\right|^2\right)^2+\frac{2^{\frac{7}{2}}}{\delta_1^3\delta_2^3}\left(\left|u_1-x_1\right|^2+\left|u_2-x_2\right|^2\right)^{\frac{7}{2}}\right]$$

$$+\Omega\left(f_{x_2};\delta_1,\delta_2\right)\left[\left|u_1-x_1\right|+\left|u_2-x_2\right|+\frac{1}{\delta_2^3}\left(\left|u_1-x_1\right|^2+\left|u_2-x_2\right|^2\right)^2\right.$$

$$\left.\left.+\ \frac{2^2}{\delta_1^3}\left(\left|u_1-x_1\right|^2+\left|u_2-x_2\right|^2\right)^2+\frac{2^{\frac{7}{2}}}{\delta_1^3\delta_2^3}\left(\left|u_1-x_1\right|^2+\left|u_2-x_2\right|^2\right)^{\frac{7}{2}}\right]\right\} du_1du_2.$$

Finally, by definition of absolute moments we conclude

$$\leq 256\left(1+x_1^2+x_2^2\right)\left\{\Omega\left(f_{x_1};\delta_1,\delta_2\right)\left[\frac{\sqrt{2}}{w}\left(M_0\left(\zeta\right)\widetilde{M}_1\left(\zeta\right)+M_1\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)\right)\right.\right.$$

$$+\frac{32}{\delta_2^3w^4}\left(M_0\left(\zeta\right)\widetilde{M}_4\left(\zeta\right)+M_4\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)\right)+\frac{8}{\delta_1^3w^4}\left(M_0\left(\zeta\right)\widetilde{M}_4\left(\zeta\right)+M_4\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)\right)$$

$$\left.+\frac{512\sqrt{2}}{\delta_1^3\delta_2^3w^7}\left(M_0\left(\zeta\right)\widetilde{M}_7\left(\zeta\right)+M_7\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)\right)\right]$$

$$+\Omega\left(f_{x_2};\delta_1,\delta_2\right)\left[\frac{\sqrt{2}}{w}\left(M_0\left(\zeta\right)\widetilde{M}_1\left(\zeta\right)+M_1\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)\right)+\frac{8}{\delta_2^3w^4}\left(M_0\left(\zeta\right)\widetilde{M}_4\left(\zeta\right)+M_4\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)\right)\right.$$

$$\left.\left.+\ \frac{32}{\delta_1^3w^4}\left(M_0\left(\zeta\right)\widetilde{M}_4\left(\zeta\right)+M_4\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)\right)+\frac{512\sqrt{2}}{\delta_1^3\delta_2^3w^7}\left(M_0\left(\zeta\right)\widetilde{M}_7\left(\zeta\right)+M_7\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)\right)\right]\right\}.$$

Hence, choosing $\delta_1=\delta_2=w^{-1}$ we get

$$\left|w\left[\left(S_w^{\zeta,\zeta}f\right)\left(x_1,x_2\right)-f\left(x_1,x_2\right)\right]-\left(m_0\left(\zeta\right)\widetilde{m}^{1}_{(1,0)}\left(\zeta\right)+m^1_{(1,0)}\left(\zeta\right)\widetilde{m}_0\left(\zeta\right)\right)\frac{\partial f}{\partial x_1}\left(x_1,x_2\right)\right.$$

$$\left.+\left(m_0\left(\zeta\right)\widetilde{m}^{1}_{(0,1)}\left(\zeta\right)+m^1_{(0,1)}\left(\zeta\right)\widetilde{m}_0\left(\zeta\right)\right)\frac{\partial f}{\partial x_2}\left(x_1,x_2\right)\right|$$

$$\leq 256\left(1+x_1^2+x_2^2\right)\left[\Omega\left(f_{x_1};w^{-1},w^{-1}\right)+\Omega\left(f_{x_2};w^{-1},w^{-1}\right)\right]$$

$$\times\left\{\sqrt{2}\left(M_0\left(\zeta\right)\widetilde{M}_1\left(\zeta\right)+M_1\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)\right)+40\left(M_0\left(\zeta\right)\widetilde{M}_4\left(\zeta\right)+M_4\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)\right)\right.$$

$$\left.+512\sqrt{2}\left(M_0\left(\zeta\right)\widetilde{M}_7\left(\zeta\right)+M_7\left(\zeta\right)\widetilde{M}_0\left(\zeta\right)\right)\right\}$$

which is desired result. Q.E.D.

Corollary 6.2. Under the assumption of Theorem 6.1, in view of (2.2), we have a qualitative form of the asymptotic formula for $S_w^{\zeta;\zeta}$, i.e.,

$$\lim_{w\to+\infty} w\left[\left(S_w^{\zeta;\zeta}f\right)(x,y)-f(x,y)\right]=\left(m_0(\zeta)\,\widetilde{m}^1_{(1,0)}(\zeta)+m^1_{(1,0)}(\zeta)\,\widetilde{m}_0(\zeta)\right)\frac{\partial f}{\partial x_1}(x_1,x_2)$$
$$+\left(m_0(\zeta)\,\widetilde{m}^1_{(0,1)}(\zeta)+m^1_{(0,1)}(\zeta)\,\widetilde{m}_0(\zeta)\right)\frac{\partial f}{\partial x_2}(x_1,x_2).$$

At the end of the paper, we give examples of kernel functions which can be used for bivariate sampling Durrmeyer series. In bivariate case, it is not easy to verify a function satisfies the conditions of being kernel. According to [23], a a bivariate kernel can be obtained by producting two univariate kernel. So, we will follow this process to obtain a bivariate kernel. Suppose $\zeta_1,\zeta_2\in L^1(\mathbb{R})$, such that both bounded in a neighborhood of the origin,

$$\sum_{k\in\mathbb{Z}}\zeta_i(u-k)=1,\quad i=1,2$$

for every $u\in\mathbb{R}$ and

$$\int\limits_{\mathbb{R}}\zeta_i(u)\,du,\quad i=1,2.$$

Now we set $\zeta(x_1,x_2):=\zeta_1(x_1)\,\zeta_2(x_2)$. ζ is a bounded function in a neighborhood of the origin and also we have

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2}\zeta((u_1,u_2)-(k_1,k_2))=\sum_{k_1\in\mathbb{Z}}\zeta_1(u_1-k_1)\sum_{k_2\in\mathbb{Z}}\zeta_2(u_2-k_2)=1$$

and

$$\int\limits_{\mathbb{R}^2}\zeta(u_1,u_2)\,du_1du_2=\int\limits_{\mathbb{R}}\zeta(u_1)\,du_1\int\limits_{\mathbb{R}}\zeta(u_2)\,du_2=1$$

which means ζ is a kernel function in bivariate setting.

Let B_n is a central B-spline of order $n\in\mathbb{N}$, and it is defined by

$$B_n(t):=\frac{1}{(n-1)!}\sum_{j=0}^{n}(-1)^i\binom{n}{j}\left(\frac{n}{2}+t-j\right)_+^{n-1},\qquad t\in\mathbb{R},$$

where $(t)_+:=max\{t,0\}$, $t\in\mathbb{R}$. Since B-spline is a univariate kernel, see [22, 16], we can use it to obtain bivariate kernel. For simplicity, we take into account the case $n=2$, i.e., we will use $B_2(t)=(1-|t|)_{\chi_{[-1,1]}(t)}$ as a univariate kernel where χ is a characteristic function. Putting, $\zeta_1=\zeta_2=B_2$ we construct a bivariate kernel

$$\begin{aligned}\zeta(t_1,t_2)&=\zeta_1(t_1)\,\zeta_2(t_2)\\&:=B_{2,2}(t_1,t_2)\\&=\begin{cases}(t_1-1)(t_2-1) & 0\le t_1\le 1 \text{ and } 0\le t_2\le 1\\ -(t_1+1)(t_2-1) & -1\le t_1<0 \text{ and } 0\le t_2\le 1\\ -(t_1-1)(t_2+1) & 0\le t_1\le 1 \text{ and } -1\le t_2<0\\ (t_1+1)(t_2+1) & -1\le t_1<0 \text{ and } -1\le t_2<0\end{cases}.\end{aligned}\tag{6.4}$$

Since $B_{n,n}$ are bounded on $\mathbb{R}^2$ for all $n \in \mathbb{N}$ with compact support $\left[-\frac{n}{2}, \frac{n}{2}\right]$, see [26, 23], we have $B_{n,n} \in L^1\left(\mathbb{R}^2\right)$ and the absolute moment condition $M_r\left(B_{n,n}\right) < +\infty$ satisfied for all $r > 0$.

Therefore, putting as a kernel $\zeta = B_{2,2}$ bivariate sampling Durrmeyer series comes out

$$\begin{aligned}&\left(S_w^{B_{2,2},B_{2,2}} f\right)(x_1, x_2)\\ &= \sum_{(k_1,k_2)\in\mathbb{Z}^2} B_{2,2}\left(w\left(x_1, x_2\right) - \left(k_1, k_2\right)\right) w^2 \int_{\mathbb{R}^2} B_{2,2}\left(w\left(u_1, u_2\right) - \left(k_1, k_2\right)\right) f\left(u_1, u_2\right) du_1 u_2\end{aligned}$$

and since $M_r\left(B_{n,n}\right) < +\infty$ for all $r > 0$, all results obtained in this paper can be used with this kernel.

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