New set types, decomposition of continuity and Γ-$\mathcal{J}$-continuity via local closure function

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Abstract

New set types are defined by using the local closure function. The properties and necessary comparisons of these set types are given. With the help of these new set types, new types of continuity are introduced. Well-known continuity and $\mathcal{J}$-continuity are generalized. Moreover, the decompositions of well-known continuity and Γ-$\mathcal{J}$-continuity are obtained by new continuity types.

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1 Introduction

The concepts of localization was examined [10, 11] by Kuratowski. The local function defined using a set ideal and a topology on the same set. Vaidyanathaswamy studied [17] on the local function and some special ideals. The Cantor-Bendixson Theorem was generalized [5] by Freud using the concept of the ideal. Jankovic and Hamlet combined [9] important results on ideal topological spaces and obtained new results.

The concept of $\mathcal{J}$-open is defined in [8]. The basic properties of $\mathcal{J}$-open sets were examined in [1]. Moreover, $\mathcal{J}$-continuity was defined by using $\mathcal{J}$-open sets. Many characterizations of $\mathcal{J}$-continuity were given. Later, $\mathcal{J}$-openness and openness were generalized [4] by the concept of pre-$\mathcal{J}$-open. Moreover, the decompositions of $\mathcal{J}$-continuity and well-known countinuity were obtained.

θ-open and θ-closed set concepts were defined [19] by Veličko in 1946. The family of θ-open sets has an important place since it forms a topology.

Many new types of local function have been described in the literature. Some of these are semi-closure local function [7], local closure [2] and weak semi-local function [22] . The properties of local closure function were examined by Al-Omari and Noiri in [2]. Moreover, they defined the operator Ψ_Γ and gave its basic properties. Thanks to the operator Ψ_Γ, they defined two new topologies finer than the topology created by θ-open sets. It has been shown by Pavlović [14] that these topologies are different of each other. Pavlović obtained results on the local and local closure function. Moreover, idempotency is discussed by considering the local closure function and many unusual examples are given in [13].

In [21], the authors gave results regarding the local closure function in extremally disconnected and hyperconnected spaces. Moreover, they gave [20] new types of connected sets using the local closure function, and restated the intermediate value theorem in ideal topological spaces. Moreover, these new types of connectedness have been shown to provide properties similar to the well-known connectedness.

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In [23], a local function is defined from a different perspective than those in the literature. They obtained the ζ_Γ^*-local function using Kuratowski's local function. In other words, a new local function is obtained from Kuratowski's local function.

In this study, we first define the concepts of Γ-$\mathcal{J}$-open set and pre-Γ-$\mathcal{J}$-open set with the help of local closure function. We compare these newly defined sets and some set types in the literature. We define and characterize the concept of Γ-$\mathcal{J}$-continuity. Later, we obtain the decomposition of Γ-$\mathcal{J}$-continuity and well-known continuity by defining new set types and new continuity concepts.

2 Preliminaries

Definition 2.1. [11] An ideal on $V \neq \varnothing$ is a family $\mathcal{J} \subseteq \mathcal{P}(V)$ having the following properties:

1. $\varnothing \in \mathcal{J}$.
2. If $T \in \mathcal{J}$ and $K \subseteq T$, then $K \in \mathcal{J}$.
3. If $T, K \in \mathcal{J}$, then $T \cup K \in \mathcal{J}$.

If (V, τ) is a topological space with the ideal $\mathcal{J}$, the triplet $(V, \tau, \mathcal{J})$ is called an ideal topological space or $\mathcal{J}$-space (briefly $\mathcal{JS}$). The ideal of finite (resp. countable, meager, nowhere dense, relatively compact, closed-discrete) subsets of V is denoted by $\mathcal{J}_f$ (resp. $\mathcal{J}_c$, $\mathcal{J}_{mg}$, $\mathcal{J}_{nw}$, $\mathcal{J}_K$, $\mathcal{J}_{cd}$). Throughout this work, the family of open neighborhoods of x is denoted by $\tau(x)$, the interior and the closure of the subset T are denoted by $i(T)$ and $c(T)$, respectively. ω is the set of natural numbers containing 0.

Definition 2.2. ([11, 2]) Let $(V, \tau, \mathcal{J})$ be an $\mathcal{JS}$ and $T \subseteq V$. The operators $(.)^* : \mathcal{P}(V) \to \mathcal{P}(V)$ and $\Gamma : \mathcal{P}(V) \to \mathcal{P}(V)$ are defined by

$$T^*(\mathcal{J}, \tau) = \{x \in V : (K \cap T) \notin \mathcal{J} \text{ for every } K \in \tau(x)\}$$

$$\Gamma(T)(\mathcal{J}, \tau) = \{x \in V : (c(K) \cap T) \notin \mathcal{J} \text{ for every } K \in \tau(x)\}$$

These operators are called the local function of T and the local closure function of T , respectively. Sometimes, $T^*(\mathcal{J})$ or T^* are used instead of the notation $T^*(\mathcal{J}, \tau)$, and $\Gamma(T)(\mathcal{J})$ or $\Gamma(T)$ are used instead of the notation $\Gamma(T)(\mathcal{J}, \tau)$.

The $*$-closure of the subset T is defined as $c^*(T) = T \cup T^*$. Moreover it is Kuratowski closure operator [11].

Definition 2.3. [2] Let $(V, \tau, \mathcal{J})$ be an $\mathcal{JS}$ and $T \subseteq V$. An operator $\Psi_\Gamma(T) : \mathcal{P}(V) \to \mathcal{P}(V)$ is defined as:

$$\begin{aligned}\Psi_\Gamma(T) &= \{x \in V : \text{there exists } K \in \tau(x) \text{ such that } (c(K) \setminus T) \in \mathcal{J}\} \\ &= V \setminus \Gamma(V \setminus T)\end{aligned}$$

.

Theorem 2.4. [2] Let $(V, \tau, \mathcal{J})$ be an $\mathcal{JS}$. The families

$$\sigma = \{T \subseteq V : T \subseteq \Psi_\Gamma(T)\}$$

$$\sigma_0 = \{T \subseteq V : T \subseteq i(c(\Psi_\Gamma(T)))\}$$

are topologies on V. Their elements is called σ-open and σ_0-open, respectively.

Throughout this work, we denote the closure according to the topology σ by c_σ.

Definition 2.5. [16] The subset T is said to be $\mathcal{J}_\Gamma$-perfect (resp.Γ-dense-in-itself) if $T = \Gamma(T)$ (resp. $T \subseteq \Gamma(T)$).

Definition 2.6. [19] A subset T of V is called θ-open in any topological space (V, τ) if each point of T has an $K \in \tau(x)$ such that $c(K) \subseteq T$.

The family of all θ-open sets in (V, τ) is denoted by τ_θ and it is a topology on V. θ-closure and θ-interior of the subset T are defined as $c_\theta(T) = \{x \in V : (c(K) \cap T) \neq \varnothing \text{ for every } K \in \tau(x)\}$ and $i_\theta(T) = \{x \in V : \text{there exists } K \in \tau(x) \text{ such that } c(K) \subseteq T\}$, respectively. More detailed information on θ-closure and θ-interior can be found in [3]. The subset T is called pre-open if $T \subseteq i(c(T))$ [12].

Proposition 2.7. [2] In any $\mathcal{J}$-space $(V, \tau, \mathcal{J})$,

1. If $T \subseteq K$, then $\Gamma(T) \subseteq \Gamma(K)$.
2. $\Gamma(T) \subseteq c_\theta(T)$.
3. If $c(\tau) \cap \mathcal{J} = \{\varnothing\}$, $\Gamma(V) = V$. In here $c(\tau) = \{c(K) : K \in \tau\}$.
4. If $T \in \mathcal{J}$, then $\Gamma(T) = \varnothing$.

Lemma 2.8. [2] For any subset T in $\mathcal{JS}$, $T^* \subseteq \Gamma(T)$.

Definition 2.9. Let $(V, \tau, \mathcal{J})$ be an $\mathcal{JS}$. The subset T is called

1. [8] $\mathcal{J}$-open if $T \subseteq i(T^*)$.
2. [4] pre-$\mathcal{J}$-open if $T \subseteq i(c^*(T))$.

Theorem 2.10. [14] If $(V, \tau, \mathcal{J})$ satisfies at least one of the conditions

1. τ has a clopen base.
2. τ is a T_3-space on V.
3. $\mathcal{J} = \mathcal{J}_{cd}$
4. $\mathcal{J} = \mathcal{J}_K$
5. $\mathcal{J}_{nw} \subseteq \mathcal{J}$
6. $\mathcal{J} = \mathcal{J}_{mg}$

then $T^* = \Gamma(T)$ for any subset T of V.

Definition 2.11. A topological space is hyperconnected [15] (resp. extremally disconnected [18]) if every nonempty open set (resp. the closure of every open) is dense (resp. open).

3 Γ-$\mathcal{J}$-open and pre-Γ-$\mathcal{J}$-open sets

We now give new set types using local closure function.

Definition 3.1. A subset T in any $\mathcal{J}$-space $(V, \tau, \mathcal{J})$ is called

1. Γ-$\mathcal{J}$-open if $T \subseteq i(\Gamma(T))$
2. pre-Γ-$\mathcal{J}$-open if $T \subseteq i(T \cup \Gamma(T))$.

The family of all Γ-$\mathcal{J}$-open (resp. pre-Γ-$\mathcal{J}$-open) subsets is denoted by $\Gamma\mathcal{J}\mathcal{O}(V)$ (resp. $\mathcal{P}\Gamma\mathcal{J}\mathcal{O}(V)$).

Theorem 3.2. In any $\mathcal{J}\mathcal{S}$,

1. Every $\mathcal{J}$-open subset is Γ-$\mathcal{J}$-open.
2. Every Γ-$\mathcal{J}$-open subset is pre-Γ-$\mathcal{J}$-open.
3. Every pre-$\mathcal{J}$-open subset is pre-Γ-$\mathcal{J}$-open.

Proof. 1. Let T be $\mathcal{J}$-open. Since T is $\mathcal{J}$-open and Lemma 2.8, $T \subseteq i(T^*) \subseteq i(\Gamma(T))$. Therefore it is Γ-$\mathcal{J}$-open.

2. From Definition 3.1, it is obvious.

3. Let T be pre-$\mathcal{J}$-open. Then, using Lemma 2.8,

$$\begin{aligned} T &\subseteq i(c^*(T)) \\ &= i(T \cup T^*) \\ &\subseteq i(T \cup \Gamma(T)) \end{aligned}$$

.

Q.E.D.

From Theorem 3.2 and Diagram in [6], the Diagram I is obtained:

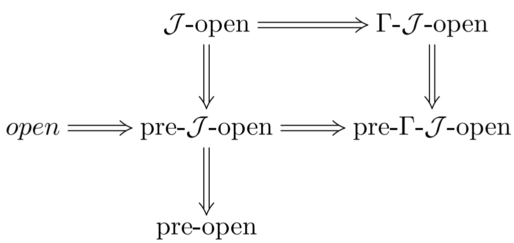

Diagram I

Corollary 3.3. In any $\mathcal{J}$-space, Γ-$\mathcal{J}$-openness and openness, Γ-$\mathcal{J}$-openness and pre-$\mathcal{J}$-openness, Γ-$\mathcal{J}$-openness and pre-openness, pre-Γ-$\mathcal{J}$-openness and pre-openness are independent of each other.

The necessary examples for the Diagram I and Corollary 3.3 are given below.

Example 3.4. Let $\tau_L = \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$ be left-ray topology on real numbers set $\mathbb{R}$ and $\mathcal{J}_f$ be finite sets ideal of $\mathbb{R}$. For the subset $T = [a, b]$, $T^*(\mathcal{J}_f) = [a, \infty)$ and $\Gamma(T)(\mathcal{J}_f) = \mathbb{R}$. Therefore $T \nsubseteq i(T^*(\mathcal{J}_f)) = i(c(T)) = \varnothing$ and $T \subseteq i(\Gamma(T)(\mathcal{J}_f)) = \mathbb{R}$. Consequently, the subset T is Γ-$\mathcal{J}$-open. However, T is neither pre-$\mathcal{J}$-open nor $\mathcal{J}$-open.

Example 3.5. Let τ_u be usual topology on $\mathbb{R}$, $\mathcal{J}_c$ be countable subsets ideal of $\mathbb{R}$ and $T = \mathbb{Q}$. Since $\mathbb{Q} \in \mathcal{J}_c$, $\Gamma(\mathbb{Q}) = \varnothing$. Therefore $T = \mathbb{Q}$ is not Γ-$\mathcal{J}$-open. However, $T = \mathbb{Q}$ is pre-open.

Example 3.6. For $T = [a, b]$ in the ideal space in Example 3.4, $T \nsubseteq i(c^*(T)) = \varnothing$ and $T \subseteq i(T \cup \Gamma(T)) = \mathbb{R}$. T is pre-Γ-$\mathcal{J}$-open. However, T is not pre-$\mathcal{J}$-open.

Example 3.7. Consider the set $V = \omega + 1 = \omega \cup \{\omega\}$ with $\tau = P(\omega) \cup \{\{\omega\} \cup (\omega \setminus K) : K \subseteq \omega$ and $K \in \mathcal{J}_f\}$ and let (V, τ) be $\mathcal{J}_f$-space. For $N = \omega$, $\Gamma(\omega) = \{\omega\}$ [14]. Since $i(\Gamma(\omega)) = \varnothing$, the subset N is not Γ-$\mathcal{J}$-open. Since $\omega \subseteq i(\omega \cup \Gamma(\omega)) = V$, $N = \omega$ is pre-Γ-$\mathcal{J}$-open.

Example 3.8. In Example 3.6, $T = [a, b]$ is Γ-$\mathcal{J}$-open but it is not pre-$\mathcal{J}$-open. In Example 3.7, $N = \omega$ is pre-$\mathcal{J}$-open. However, $N = \omega$ is not Γ-$\mathcal{J}$-open. Because $\omega \nsubseteq i(\Gamma(\omega)) = \varnothing$ and $\omega \subseteq i(c^*(\omega)) = V$.

Example 3.9. In Example 3.4, the subset T is Γ-$\mathcal{J}$-open. However, T is not open. In Example 3.7, N is open. However, N is not Γ-$\mathcal{J}$-open.

Theorem 3.10. Let any of the conditions in Theorem 2.10 be satisfied. Then,

1. The subset T is $\mathcal{J}$-open iff T is Γ-$\mathcal{J}$-open.

2. The subset T is pre-$\mathcal{J}$-open iff T is pre-Γ-$\mathcal{J}$-open.

Proof. From Theorem 2.10, it is obtained. Q.E.D.

Theorem 3.11. [2] Let $(V, \tau, \mathcal{J})$ be $\mathcal{JS}$, $K \in \tau_\theta$ and $T \subseteq V$. Then, $(K \cap \Gamma(T)) = K \cap \Gamma(K \cap T) \subseteq \Gamma(K \cap T)$.

For any subset T in (V, τ), although $i_\theta(T)$ is always open set, it need to not be θ-open. Therefore, we now generalize the theorem above.

Theorem 3.12. Let $(V, \tau, \mathcal{J})$ be an $\mathcal{JS}$ and $T, K \subseteq V$. Then, $(i_\theta(K) \cap \Gamma(T)) = i_\theta(K) \cap \Gamma(K \cap T) \subseteq \Gamma(K \cap T)$

Proof. Let $x \in (i_\theta(K) \cap \Gamma(T))$. So, $x \in i_\theta(K)$ and $x \in \Gamma(T)$. Since $x \in i_\theta(K)$, there exists $S \in \tau(x)$ such that $x \in S \subseteq c(S) \subseteq K$. For any $D \in \tau(x)$, $(D \cap S) \in \tau(x)$. Since $x \in \Gamma(T)$, $[c(D \cap S) \cap T] \notin \mathcal{J}$. From the definition of ideal, $c(D \cap S) \cap T \subseteq c(D) \cap (c(S) \cap T) \subseteq [c(D) \cap (K \cap T)] \notin \mathcal{J}$. From the definition of local closure function, $x \in \Gamma(K \cap T)$. That is, $(i_\theta(K) \cap \Gamma(T)) \subseteq \Gamma(K \cap T)$. Since $\Gamma(K \cap T) \subseteq \Gamma(T)$, $i_\theta(K) \cap \Gamma(K \cap T) \subseteq (i_\theta(K) \cap \Gamma(T))$. Therefore $(i_\theta(K) \cap \Gamma(T)) = i_\theta(K) \cap \Gamma(K \cap T)$. Q.E.D.

Indeed, Theorem 3.11 is obtained from Theorem 3.12 if $K \in \tau_\theta$.

Theorem 3.13. Let $(V,\tau,\mathcal{J})$ be an $\mathcal{JS}$, $T\in\Gamma\mathcal{JO}(V)$ and $K\subseteq V$. Then, $(i_\theta(K)\cap T)\subseteq i(\Gamma(K\cap T))$.

Proof. Using Theorem 3.12,

$$\begin{aligned}(i_\theta(K)\cap T) &\subseteq (i_\theta(K)\cap i(\Gamma(T)) \\ &= i(i_\theta(K)\cap\Gamma(T)) \\ &\subseteq i(\Gamma(K\cap T))\end{aligned}$$

Q.E.D.

Theorem 3.14. Let $(V,\tau,\mathcal{J})$ be an $\mathcal{JS}$, $T\in\Gamma\mathcal{JO}(V)$ and $K\in\tau_\theta$. Then, $K\cap T\in\Gamma\mathcal{JO}(V)$.

Proof. Since $K=i_\theta(K)$ and Theorem 3.13, $(K\cap T)\subseteq i(\Gamma(K\cap T))$. Q.E.D.

Theorem 3.15. If $(V,\tau,\mathcal{J})$ is hyperconnected $\mathcal{JS}$, then $\mathcal{P}(V)\setminus\mathcal{J}=\Gamma\mathcal{JO}(V)\setminus\{\varnothing\}$.

Proof. Let $T\in\mathcal{P}(V)\setminus\mathcal{J}$. From Theorem 3.6 in [21], $\Gamma(T)=V$. So, $T\in\Gamma\mathcal{JO}(V)\setminus\{\varnothing\}$. Conversely, let $T\in\Gamma\mathcal{JO}(V)\setminus\{\varnothing\}$. Then, $T\subseteq i(\Gamma(T))\subseteq\Gamma(T)$. Since $\Gamma(T)\neq\varnothing$, $T\notin\mathcal{J}$. That is, $T\in\mathcal{P}(V)\setminus\mathcal{J}$. Q.E.D.

Theorem 3.16. Let $(V,\tau,\mathcal{J})$ be extremally disconnected $\mathcal{JS}$. Then, local closure space is idempotent i.e. $\Gamma(\Gamma(T))\subseteq\Gamma(T)$ for every $T\subseteq V$.

Proof. Let $x\in\Gamma(\Gamma(T))$. Therefore $c(K)\cap\Gamma(T)\notin\mathcal{J}$ for every $K\in\tau(x)$. There exists $y\in V$ such that $y\in c(K)\cap\Gamma(T)$. That is, $y\in c(K)$ and $y\in\Gamma(T)$. Since this space is extremally disconnected, $c(K)\in\tau(y)$. Since $y\in\Gamma(T)$, $c(K)\cap T\notin\mathcal{J}$. Therefore $x\in\Gamma(T)$. Consequently $\Gamma(\Gamma(T))\subseteq\Gamma(T)$. Q.E.D.

Example 3.17 shows that $\Gamma(\Gamma(T))\subseteq\Gamma(T)$ strictly holds in extremally disconnected spaces.

Example 3.17. For the subset $N=\omega$ of the $\mathcal{J}_f$-space in Example 3.7, $\Gamma(\Gamma(\omega))=\varnothing\subsetneq\{\omega\}=\Gamma(\omega)$.

The counterexample for Theorem 3.16 is as follows:

Example 3.18. Consider the set $\mathbb{R}$ with usual topology and any ideal on $\mathbb{R}$. From Theorem 2.10, $T^*=\Gamma(T)$ for every $T\subseteq\mathbb{R}$ in this space. Therefore the local closure function Γ is idempotent. But this space is not extremally disconnected.

Theorem 3.19. If $(V,\tau,\mathcal{J})$ is extremally disconnected $\mathcal{JS}$ and $T\in\Gamma\mathcal{JO}(V)$, then $\Gamma(\Gamma(T))=\Gamma(T)$.

Proof. Let $T\in\Gamma\mathcal{JO}(V)$. Since $T\subseteq i(\Gamma(T))$, $\Gamma(T)\subseteq\Gamma(i(\Gamma(T)))\subseteq\Gamma(\Gamma(T))$. Moreover, from Theorem 3.16, $\Gamma(\Gamma(T))\subseteq\Gamma(T)$. Consequently, $\Gamma(\Gamma(T))=\Gamma(T)$. Q.E.D.

Theorem 3.20. [13] If $\Gamma(\Gamma(T))\subseteq\Gamma(T)$ for every $T\subseteq V$, then $c_\sigma(T)=T\cup\Gamma(T)$. That is, the closure of T in (V,σ) equals $T\cup\Gamma(T)$.

Theorem 3.21. If (V,τ) is extremally disconnected, then $c_\sigma(T)=T\cup\Gamma(T)$.

Proof. From Theorem 3.16 and Theorem 3.20, it is obtained. Q.E.D.

Theorem 3.22. [13] Let $(V, \tau, \mathcal{J})$ be $\mathcal{JS}$ and $T \subseteq V$. The subset T is closed in (V, σ) iff $\Gamma(T) \subseteq T$.

Theorem 3.23. Let $(V, \tau, \mathcal{J})$ be $\mathcal{JS}$ and $T \subseteq V$. The subset T is closed in (V, σ_0) iff $c(i(\Gamma(T))) \subseteq T$.

Proof. Let T is closed in (V, σ_0). Then

$$\begin{aligned} T \text{ is closed in } (V, \sigma_0) &\Leftrightarrow V \setminus T \subseteq i(c(\Psi_\Gamma(V \setminus T))) \\ &\Leftrightarrow V \setminus T \subseteq i(c(V \setminus \Gamma(T))) \\ &\Leftrightarrow V \setminus T \subseteq V \setminus c(i(\Gamma(T))) \\ &\Leftrightarrow c(i(\Gamma(T))) \subseteq T \end{aligned}$$

Q.E.D.

Theorem 3.24. Let $(V, \tau, \mathcal{J})$ be any $\mathcal{JS}$. If the subset T is both σ_0-closed and Γ-$\mathcal{J}$-open, then T is open and $T = i(\Gamma(T))$.

Proof. Let T be both σ_0-closed and Γ-$\mathcal{J}$-open. From Theorem 3.23, $i(\Gamma(T)) \subseteq c(i(\Gamma(T))) \subseteq T$. Since T is Γ-$\mathcal{J}$-open, $T \subseteq i(\Gamma(T))$. Consequently, $i(\Gamma(T)) = T$. Q.E.D.

Since $\tau_\theta \subseteq \sigma \subseteq \sigma_0$ [2] and the above theorem, the following result is obtained.

Corollary 3.25. Let $(V, \tau, \mathcal{J})$ be any $\mathcal{JS}$. If the subset T is both σ-closed (or θ-closed) and Γ-$\mathcal{J}$-open, then T is open and $T = i(\Gamma(T))$.

Definition 3.26. Let $(V, \tau, \mathcal{J})$ be any $\mathcal{JS}$. If $(V \setminus T) \in \Gamma\mathcal{J}\mathcal{O}(V)$, then the subset T is called Γ-$\mathcal{J}$-closed. The family of all Γ-$\mathcal{J}$-closed subsets is denoted by $\Gamma\mathcal{J}\mathcal{C}(V)$.

Theorem 3.27. Let $(V, \tau, \mathcal{J})$ be any $\mathcal{JS}$. If T is Γ-$\mathcal{J}$-closed, then $c(i_\theta(T)) \subseteq T$.

Proof. Let T is Γ-$\mathcal{J}$-closed. Since $V \setminus T$ is Γ-$\mathcal{J}$-open,

$$\begin{aligned} V \setminus T &\subseteq i(\Gamma(V \setminus T)) \\ &\subseteq i(c_\theta(V \setminus T)) \\ &= V \setminus (c(i_\theta(T))). \end{aligned}$$

Therefore $c(i_\theta(T)) \subseteq T$. Q.E.D.

The counterexample for Theorem 3.27 is as follows:

Example 3.28. Let τ_u be usual topology on $\mathbb{R}$ with the ideal $\mathcal{J}_c$ on $\mathbb{R}$. For the subset $T = \mathbb{R} \setminus \mathbb{Q}$, $c(i_\theta(T)) = \varnothing \subseteq T$. Since $\Gamma(\mathbb{Q}) = \varnothing$, $\mathbb{Q}$ is not Γ-$\mathcal{J}$-open. Therefore the subset T is not Γ-$\mathcal{J}$-closed.

The following result is obtained from Theorem 3.14.

Corollary 3.29. Let $(V, \tau, \mathcal{J})$ be any $\mathcal{JS}$. If $T \in \Gamma\mathcal{J}\mathcal{C}(V)$ and K is θ-closed, then $(T \cup K) \in \Gamma\mathcal{J}\mathcal{C}$.

4 Γ-$\mathcal{J}$-continuity

Definition 4.1. Let $f:(V,\tau_1,\mathcal{J})\to(Z,\tau_2)$ be function. The function f is called Γ-$\mathcal{J}$-continuous if $f^{-1}(D)\in\Gamma\mathcal{J}\mathcal{O}(V)$ for every $D\in\tau_2$.

Definition 4.2. [1] Let $f:(V,\tau_1,\mathcal{J})\to(Z,\tau_2)$ be function. The function f is called $\mathcal{J}$-continuous if $f^{-1}(D)$ is $\mathcal{J}$-open for every $D\in\tau_2$.

Theorem 4.3. Every $\mathcal{J}$-continuous function is Γ-$\mathcal{J}$-continuous.

Proof. Since every $\mathcal{J}$-open set is Γ-$\mathcal{J}$-open, it is obtained. Q.E.D.

The counterexample for Theorem 4.3 is as follows:

Example 4.4. Let τ_L be left-ray topology and τ_u be usual topology on $\mathbb{R}$ and $\mathcal{J}=\{\varnothing\}$. Let $f:(\mathbb{R},\tau_L,\mathcal{J})\to(\mathbb{R},\tau_u)$ be the function such that $f(x)=x$. For the subset $(a,b)\in\tau_u$, $i[(f^{-1}(a,b))^*]=i[(a,b)^*]=\varnothing$. Therefore (a,b) is not $\mathcal{J}$-open in $(\mathbb{R},\tau_L,\mathcal{J})$. That is, the function f is not $\mathcal{J}$-continuous but it is Γ-$\mathcal{J}$-continuous. Because, for every $K\in\tau_u\setminus\{\varnothing\}$, $f^{-1}(K)=K\subseteq i(\Gamma(f^{-1}(K)))=\mathbb{R}$.

Theorem 4.5. Let $f:(V,\tau_1,\mathcal{J})\to(Z,\tau_2)$ be function. The following conditions are equivalent:

1. The function f is Γ-$\mathcal{J}$-continuous.
2. There exists a $S\in\Gamma\mathcal{J}\mathcal{O}(V)_{(x)}$ such that $f(S)\subseteq D$ for every $x\in V$ and for every $D\in\tau_2(f(x))$. (where $\Gamma\mathcal{J}\mathcal{O}(V)_{(x)}=\{S\subseteq V: x\in S \text{ and } S\in\Gamma\mathcal{J}\mathcal{O}(V)\}$)
3. For every $x\in V$ and for every $D\in\tau_2(f(x))$, $\Gamma(f^{-1}(D))$ is a neighborhood of x.
4. For every closed subset F in (Z,τ_2), $f^{-1}(F)\in\Gamma\mathcal{J}\mathcal{C}(V)$.

Proof. (1)⇒(2) Let any $x\in V$. Since f is Γ-$\mathcal{J}$-continuous, $S=f^{-1}(D)\in\Gamma\mathcal{J}\mathcal{O}(V)$ for every $D\in\tau_2(f(x))$. Therefore $f(S)=f(f^{-1}(D))\subseteq D$.

(2)⇒(3) Let us suppose that there exists $S\in\Gamma\mathcal{J}\mathcal{O}(V)_{(x)}$ such that $f(S)\subseteq D$ for every $x\in V$ and for every $D\in\tau_2(f(x))$. Then $x\in S\subseteq i(\Gamma(S))\subseteq i(\Gamma(f^{-1}(D)))\subseteq\Gamma(f^{-1}(D))$.

(3)⇒(1) Let $D\in\tau_2$. If $f^{-1}(D)=\varnothing$, then $f^{-1}(D)\in\Gamma\mathcal{J}\mathcal{O}(V)$. Let $f^{-1}(D)\neq\varnothing$ and $x\in f^{-1}(D)$. Then $D\in\tau_2(f(x))$. From 3), $x\in i(\Gamma(f^{-1}(D)))$. Therefore $f^{-1}(D)\subseteq i(\Gamma(f^{-1}(D)))$ and $f^{-1}(D)\in\Gamma\mathcal{J}\mathcal{O}(V)$.

(1)⇒(4) Let the subset F be closed in (Z,τ_2). Therefore $Z\setminus F$ is open subset in (Z,τ_2). From 1), $f^{-1}(Z\setminus F)=V\setminus f^{-1}(F)\in\Gamma\mathcal{J}\mathcal{O}(V)$. Consequently $f^{-1}(F)\in\Gamma\mathcal{J}\mathcal{C}(V)$.

(4)⇒(1) Let $D\in\tau_2$. $Z\setminus D$ is closed in (Z,τ_2). From 4), $f^{-1}(Z\setminus D)=V\setminus f^{-1}(D)\in\Gamma\mathcal{J}\mathcal{C}(V)$. Therefore $f^{-1}(D)\in\Gamma\mathcal{J}\mathcal{O}(V)$. Q.E.D.

Theorem 4.6. Let $f:(V,\tau_1,\mathcal{J})\to(Z,\tau_2)$ be Γ-$\mathcal{J}$-continuous function and $K\in\tau_{1\theta}$. The restriction $f|K$ is Γ-$\mathcal{J}$-continuous function.

Proof. Let $D \in \tau_2$. Since $f^{-1}(D) \in \Gamma\mathcal{J}\mathcal{O}(V)$, $f^{-1}(D) \subseteq i(\Gamma(f^{-1}(D)))$.Therefore $(f|K)^{-1}(D) = K \cap f^{-1}(D) \subseteq K \cap i(\Gamma(f^{-1}(D)))$. Since $K \in \tau_{1\theta}$ and Theorem 3.11),

$$\begin{aligned}(f|K)^{-1}(D) &\subseteq i(K \cap \Gamma(f^{-1}(D))) \\ &\subseteq i(\Gamma(K \cap f^{-1}(D))) \\ &\subseteq i(\Gamma((f|K)^{-1}(D))).\end{aligned}$$

Consequently the restriction $f|K$ is Γ-$\mathcal{J}$-continuous function. Q.E.D.

Theorem 4.7. Every Γ-$\mathcal{J}$-open set is a Γ-dense-in-itself.

Proof. Let T be Γ-$\mathcal{J}$-open. Since $T \subseteq i(\Gamma(T)) \subseteq \Gamma(T)$, the subset T is Γ-dense-in-itself. Q.E.D.

Example 4.8 is a counterexample for Theorem 4.7.

Example 4.8. For the interval $K = [a, b)$ in $(\mathbb{R}, \tau_u, \{\varnothing\})$, $\Gamma(K) = [a, b]$ and $i(\Gamma(K)) = (a, b)$. Therefore, K is Γ-dense-in-itself but it is not Γ-$\mathcal{J}$-open.

Theorem 4.9. Let $(V, \tau, \mathcal{J})$ be an $\mathcal{JS}$ and $T \subseteq V$. The following statements are equivalent:

1. The subset T is Γ-$\mathcal{J}$-open.
2. The subset T is pre-Γ-$\mathcal{J}$-open and Γ-dense-in-itself.

Proof. (1)⇒(2) Let T be Γ-$\mathcal{J}$-open. From Diagram I, the subset T is pre-Γ-$\mathcal{J}$-open. Moreover, from Theorem 4.7, the subset T is Γ-dense-in-itself.
(2)⇒(1) Let T be both pre-Γ-$\mathcal{J}$-open and Γ-dense-in-itself. Since $T \subseteq i(T \cup \Gamma(T)) = i(\Gamma(T))$. Therefore T is Γ-$\mathcal{J}$-open. Q.E.D.

5 Γ_Γ-open, m_Γ-open and almost Γ-$\mathcal{J}$-open sets

Definition 5.1. Let $(V, \tau, \mathcal{J})$ be an $\mathcal{JS}$ and $T \subseteq V$.

1. T is called Γ_Γ-open set if $i_\theta(T) = c(i(\Gamma(T)))$.
2. T is called m_Γ-open if $T = K \cap D$ where $K \in \Gamma\mathcal{J}\mathcal{O}(V)$ and D is Γ_Γ-open.
3. T is called almost Γ-$\mathcal{J}$-open if $T \subseteq c(i(\Gamma(T)))$.

Theorem 5.2. Let $(V, \tau, \mathcal{J})$ be an $\mathcal{JS}$ and $c(\tau) \cap \mathcal{J} = \{\varnothing\}$ (where $c(\tau) = \{c(K) : K \in \tau\}$).

1. Every Γ-$\mathcal{J}$-open subset is m_Γ-open.
2. Every Γ_Γ-open subset is m_Γ-open.

Proof. Let T be Γ-$\mathcal{J}$-open. Since $c(\tau) \cap \mathcal{J} = \{\varnothing\}$, $\Gamma(V) = V$. Therefore V is Γ_Γ-open. Since $T = T \cap V$, T is m_Γ-open. Similarly, let T be Γ_Γ-open. Since V is Γ-$\mathcal{J}$-open, $T = T \cap V$ is m_Γ-open. Q.E.D.

Theorem 5.3. In any $\mathcal{J}$-space, every Γ-$\mathcal{J}$-open subset is almost Γ-$\mathcal{J}$-open.

Proof. Let T be Γ-$\mathcal{J}$-open. Since $T \subseteq i(\Gamma(T)) \subseteq c(i(\Gamma(T)))$, T is almost Γ-$\mathcal{J}$-open. Q.E.D.

The converses of Theorem 5.2 and Theorem 5.3 is not true.

Example 5.4. Consider the ideal space $(\mathbb{R}, \tau_u, \mathcal{J}_c)$. In this ideal space, $c(\tau_u) \cap \mathcal{J}_c = \{\varnothing\}$. For $T = \mathbb{Q}$, $i_\theta(\mathbb{Q}) = c(i(\Gamma(\mathbb{Q}))) = \varnothing$. Therefore $T = \mathbb{Q}$ is Γ_Γ-open set. Since $\mathbb{R} \subseteq i(\Gamma(\mathbb{R})) = \mathbb{R}$, $\mathbb{R}$ is Γ-$\mathcal{J}$-open subset. Consequently, $\mathbb{R} \cap \mathbb{Q} = \mathbb{Q}$ is m_Γ-open but $\mathbb{Q}$ is not Γ-$\mathcal{J}$-open. Consider this space with $\mathcal{J} = \{\varnothing\}$. Then, for the interval $N = [1,2)$, $i(\Gamma(N)) = (1,2)$. Therefore $N = [1,2)$ is not Γ-$\mathcal{J}$-open. Since $c(i(\Gamma(N))) = [1,2]$, it is almost Γ-$\mathcal{J}$-open.

Example 5.5. Let τ_L be left-ray topology on $\mathbb{R}$ and $\mathcal{J} = \{\varnothing\}$. For $K = (-\infty, 1)$, $i(\Gamma(K)) = \mathbb{R}$. Therefore $K = (-\infty, 1)$ is Γ-$\mathcal{J}$-open. Since $i_\theta(\mathbb{R}) = c(i(\Gamma(\mathbb{R}))) = \mathbb{R}$, $\mathbb{R}$ is Γ_Γ-open. Therefore $K = K \cap \mathbb{R}$. That is, $K = (-\infty, 1)$ is m_Γ-open. Since $\varnothing = i_\theta(K) \neq c(i(\Gamma(K))) = \mathbb{R}$, $K = (-\infty, 1)$ is not Γ_Γ-open.

In these examples, the following result is obtained.

Corollary 5.6. The concepts of Γ_Γ-openness and Γ-$\mathcal{J}$-openness are independent of each other.

Theorem 5.7. Let $(V, \tau, \mathcal{J})$ be an $\mathcal{JS}$ and $c(\tau) \cap \mathcal{J} = \{\varnothing\}$. The following statement is equivalent:

1. The subset T is Γ-$\mathcal{J}$-open.
2. The subset T is almost Γ-$\mathcal{J}$-open and m_Γ-open.

Proof. (1)⇒(2) Let T be Γ-$\mathcal{J}$-open. Then, from Theorem 5.2-1) and Theorem 5.3, T is almost Γ-$\mathcal{J}$-open and m_Γ-open.
(2)⇒(1) Let T be both almost Γ-$\mathcal{J}$-open and m_Γ-open. Since T is m_Γ-open, there exist $K \in \Gamma\mathcal{JO}(V)$ and D is Γ_Γ-open subsets such that $T = K \cap D$. Since T is almost Γ-$\mathcal{J}$-open and D is Γ_Γ-open,

$$\begin{aligned} T &\subseteq c(i(\Gamma(T))) \\ &= c(i(\Gamma(K \cap D))) \\ &\subseteq c(i(\Gamma(K) \cap \Gamma(D))) \\ &= c((i(\Gamma(K)) \cap i(\Gamma(D))) \\ &\subseteq c(i(\Gamma(K))) \cap c(i(\Gamma(D))) \\ &= c(i(\Gamma(K))) \cap i_\theta(D). \end{aligned}$$

Since $K \in \Gamma\mathcal{JO}(V)$ and Theorem 3.12,

$$\begin{aligned} T = K \cap T &\subseteq K \cap [c(i(\Gamma(K))) \cap i_\theta(D)] \\ &= [K \cap c(i(\Gamma(K)))] \cap i_\theta(D) \\ &= K \cap i_\theta(D) \\ &\subseteq i(\Gamma(K)) \cap i_\theta(D) \\ &= i(\Gamma(K) \cap i_\theta(D)) \\ &\subseteq i(\Gamma(K \cap D)) \\ &= i(\Gamma(T)). \end{aligned}$$

That is, the subset T is Γ-$\mathcal{J}$-open. Q.E.D.

6 Γ-locally closed sets

Definition 6.1. Let $(V, \tau, \mathcal{J})$ be an $\mathcal{JS}$ and $T \subseteq V$. The subset T is called Γ-locally closed if $T = K \cap D$ where $K \in \tau$ and D is $\mathcal{J}_\Gamma$-perfect.

Theorem 6.2. Let $(V, \tau, \mathcal{J})$ be an $\mathcal{JS}$ and $c(\tau) \cap \mathcal{J} = \{\varnothing\}$. Every open is a Γ-locally closed.

Proof. Let T be an open subset. Since $c(\tau) \cap \mathcal{J} = \{\varnothing\}$, $\Gamma(V) = V$. Since $T \cap V = T$, T is Γ-locally closed. Q.E.D.

Under the condition $c(\tau) \cap \mathcal{J} = \{\varnothing\}$, Example 6.3 is a counterexample for Theorem 6.2.

Example 6.3. Let τ_u be usual topology on $\mathbb{R}$ with the ideal $\mathcal{J} = \{\varnothing\}$. Consider the intervals $D = [1, 3]$, $K = (2, 4)$. Then, $D = \Gamma(D) = [1, 3]$ and $K \in \tau_u$. Therefore $T = K \cap D = (2, 3]$ is Γ-locally closed but it is not open.

Theorem 6.4. Let $(V, \tau, \mathcal{J})$ be an $\mathcal{JS}$ and $c(\tau) \cap \mathcal{J} = \{\varnothing\}$. The following is equivalent:

1. The subset T is open.
2. The subset T is pre-Γ-$\mathcal{J}$-open and Γ-locally closed.

Proof. (1)⇒(2) Let T be an open subset. From Theorem 6.2, T is Γ-locally closed. Moreover, from Diagram I, T is pre-Γ-$\mathcal{J}$-open.
(2)⇒(1) Let T be both pre-Γ-$\mathcal{J}$-open and Γ-locally closed. Then, there exist K, D such that $K \in \tau$, D is $\mathcal{J}_\Gamma$-perfect and $T = K \cap D$. Since T is pre-Γ-$\mathcal{J}$-open, $T \subseteq i(T \cup \Gamma(T)) = i((K \cap D) \cup \Gamma(K \cap D))$. Then,

$$\begin{aligned} T = K \cap T &\subseteq K \cap i((K \cap D) \cup \Gamma(K \cap D)) \\ &\subseteq K \cap i((K \cap D) \cup (\Gamma(K) \cap \Gamma(D))) \\ &= i((K \cap (K \cap D)) \cup (K \cap (\Gamma(K) \cap \Gamma(D)))) \\ &= i((K \cap D) \cup ((K \cap D) \cap \Gamma(K))) \\ &= i(K \cap D) \\ &= i(T). \end{aligned}$$

Therefore $T = i(T)$. So T is open subset. Q.E.D.

7 Decompositions of Γ-$\mathcal{J}$-continuity and continuity

In this section, we obtain the decompositions of Γ-$\mathcal{J}$-continuity and well-known continuity.

Definition 7.1. A function $f : (V, \tau_1, \mathcal{J}) \to (Z, \tau_2)$ is called m_Γ-continuous (resp. almost Γ-$\mathcal{J}$-continuous, pre-Γ-$\mathcal{J}$-continuous, Γ-$\mathcal{LC}$-continuous, Γ-dense-continuous) if for every $D \in \tau_2$, $f^{-1}(D)$ is m_Γ-open (almost Γ-$\mathcal{J}$-open, pre-Γ-$\mathcal{J}$-open, Γ-locally closed, Γ-dense-in-itself).

Theorem 7.2. If f is Γ-$\mathcal{J}$-continuous, then f is pre-Γ-$\mathcal{J}$-continuous.

Proof. From Theorem 3.2-2), the desired result is obtained. Q.E.D.

A pre-Γ-$\mathcal{J}$-continuous function need not be Γ-$\mathcal{J}$-continuous.

Example 7.3. The identity function $f : (\mathbb{R}, \tau_u, \mathcal{P}(\mathbb{R})) \to (\mathbb{R}, \tau_u)$ is pre-Γ-$\mathcal{J}$-continuous. However, f is not Γ-$\mathcal{J}$-continuous. Because, for every $K \in \tau_u$, $f^{-1}(K) = K \not\subset i(\Gamma(K)) = \varnothing$.

Theorem 7.4. If f is Γ-$\mathcal{J}$-continuous, then f is Γ-dense-continuous.

Proof. From Theorem 4.7, the desired result is obtained. Q.E.D.

A Γ-dense-continuous function need not be Γ-$\mathcal{J}$-continuous.

Example 7.5. Let $\mathbb{R}_l$ be the lower limit topology on $\mathbb{R}$. The identity function $f : (\mathbb{R}, \tau_u, \{\varnothing\}) \to (\mathbb{R}, \mathbb{R}_l)$ is Γ-dense-continuous. However, f is not Γ-$\mathcal{J}$-continuous. Because, for $[a, b) \in \mathbb{R}_l$, $f^{-1}([a, b)) = [a, b) \not\subset i(\Gamma([a, b))) = (a, b)$.

Theorem 7.6. Let $(V, \tau_1, \mathcal{J})$ be an $\mathcal{JS}$. For $f : (V, \tau_1, \mathcal{J}) \to (Z, \tau_2)$, the following statements is equivalent:

1. f is Γ-$\mathcal{J}$-continuous.
2. f is pre-Γ-$\mathcal{J}$-continuous and Γ-dense-continuous.

Proof. From Theorem 4.9 and Definition 7.1, it is obtained. Q.E.D.

Theorem 7.7. Every Γ-$\mathcal{J}$-continuous function is almost Γ-$\mathcal{J}$-continuous.

Proof. From Theorem 5.3, it is obtained. Q.E.D.

An almost Γ-$\mathcal{J}$-continuous function need not be Γ-$\mathcal{J}$-continuous. Example 7.5 is also an example for this.

Theorem 7.8. Under the condition $c(\tau_1) \cap \mathcal{J} = \{\varnothing\}$, every Γ-$\mathcal{J}$-continuous function is m_Γ-continuous.

Proof. From Theorem 5.2-1), it is obtained. Q.E.D.

m_Γ-continuous function need not be Γ-$\mathcal{J}$-continuous.

Example 7.9. For the identity function $f : (\mathbb{R}, \tau_u, \mathcal{J}_c) \to (\mathbb{R}, \tau = \{\mathbb{R}, \varnothing, \mathbb{Q}\})$, $f^{-1}(\mathbb{Q}) = \mathbb{Q} \not\subset i(\Gamma(\mathbb{Q})) = \varnothing$. That is, f is not Γ-$\mathcal{J}$-continuous. It is m_Γ-continuous.

Theorem 7.10. Let $(V, \tau_1, \mathcal{J})$ be an $\mathcal{JS}$ and $c(\tau_1) \cap \mathcal{J} = \{\varnothing\}$. For $f : (V, \tau_1, \mathcal{J}) \to (Z, \tau_2)$, the following is equivalent:

1. f is Γ-$\mathcal{J}$-continuous.
2. f is almost Γ-$\mathcal{J}$-continuous and m_Γ-continuous.

Proof. From Theorem 5.7 and Definition 7.1, it is obtained. Q.E.D.

Theorem 7.11. If f is continuous, then f is pre-Γ-$\mathcal{J}$-continuous.

Proof. From Diagram I, the desired result is obtained, respectively. Q.E.D.

A pre-Γ-$\mathcal{J}$-continuous function need not be continuous.

Example 7.12. The identity function $f : (\mathbb{R}, \tau_L, \{\varnothing\}) \to (\mathbb{R}, \tau_u)$ is not continuous. However, it is pre-Γ-$\mathcal{J}$-continuous.

Theorem 7.13. Under the condition $c(\tau_1) \cap \mathcal{J} = \{\varnothing\}$, every continuous function is Γ-$\mathcal{LC}$-continuous.

Proof. From Theorem 6.2, it is obtained. Q.E.D.

A Γ-$\mathcal{LC}$-continuous function need not be continuous.

Example 7.14. The identity function $f : (\mathbb{R}, \tau_u, \{\varnothing\}) \to (\mathbb{R}, \tau = \{\mathbb{R}, \varnothing, [1, 2]\})$ is not continuous since $f^{-1}([1, 2]) = [1, 2] \notin \tau_u$. It is Γ-$\mathcal{LC}$-continuous.

Theorem 7.15. Let $(V, \tau_1, \mathcal{J})$ be an $\mathcal{JS}$ and $c(\tau_1) \cap \mathcal{J} = \{\varnothing\}$. For $f : (V, \tau_1, \mathcal{J}) \to (Z, \tau_2)$, the following is equivalent:

1. f is continuous.
2. f is pre-Γ-$\mathcal{J}$-continuous and Γ-$\mathcal{LC}$-continuous.

Proof. From Theorem 6.4 and Definition 7.1, it is obtained. Q.E.D.

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