Can we compute denominators of Eisenstein classes?

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Abstract

We discuss the general conjecture that the denominators of Eisenstein classes should be related to the prime factorisation of certain special values of L-functions. We propose an experimental procedure to verify (or falsify) this conjecture in a given special case. We also discuss an interesting special case, where this experimental approach could be tested.

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1 The general problem

We start from a semi simple (or reductive) group $G/\mathbb{Q}$, an arithmetic subgroup $\Gamma \subset G(\mathbb{R})$ and a Γ module $\mathcal{M}$ which should be finitely generated as $\mathbb{Z}$ module. The group Γ acts on the symmetric space $X = G(\mathbb{R})/K_\infty$. The Γ- module provides a sheaf $\tilde{\mathcal{M}}$ on the locally symmetric space $\Gamma\backslash X$. If $\pi : X \to \Gamma\backslash X$ is the natural projection, then for any open subset $V \subset \Gamma\backslash X$

$$\tilde{\mathcal{M}}(V) = \{f : \pi^{-1}(V) \to \mathcal{M} \mid f \text{ locally constant and } f(\gamma u) = \gamma f(u)\}. \tag{1}$$

In this note we consider the cohomology groups $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$. A general theorem by Raghunathan asserts that the cohomology groups are finitely generated $\mathbb{Z}$-modules.

In general the quotient $\Gamma\backslash X$ is not compact, hence we can define the cohomology with compact supports $H^\bullet_c(\Gamma\backslash X, \tilde{\mathcal{M}})$. We also can compactify and add the Borel-Serre boundary $\partial(\Gamma\backslash X)$ at infinity and get the fundamental long exact sequence ([10],sec. 1.2.8, Thm. 6.2.1)

$$\to H^{q-1}(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \to H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}}) \to H^q(\Gamma\backslash \bar{X}, \tilde{\mathcal{M}}) \xrightarrow{r} H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \to \dots \tag{2}$$

In this note we discuss some problems and conjectures, which can be investigated experimentally (see 1.5). To achieve this goal we have to write an algorithm which does the following

Task A) Compute all the modules in this exact sequence explicitly and compute the arrows between these modules.

Let us assume that $\mathcal{M} = \mathcal{M}_\lambda$ is a highest weight module over $\mathbb{Z}$. In [10], Chap. 3, and Chap. 6 we define the action of the Hecke algebra on the above cohomology groups. More specifically for any prime p and any cocharacter $\chi : \mathbb{G}_m \to T$ we define endomorphisms $T^{\mathrm{coh},\lambda}_{p,\chi}$ on all the cohomology groups $H^q_?(\Gamma\backslash X, \tilde{\mathcal{M}}_\lambda)$. These endomorphisms commute with the arrows in the long exact sequence above.

Task B) Give explicit expressions for the $T^{\mathrm{coh},\lambda}_{p,\chi}$ under the assumptions that A) is done.

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In [10] Chap. 2 , Sec. 3 and 4 we discuss a general strategy which allows us to tackle these tasks (in principle). The cohomology is computed from the Čech complex of an orbiconvex covering [10] Chap. 2 , loc.cit. and then we also can write a procedure which computes Hecke operators. I have no idea whether this strategy is effective or optimal. In any case it is clear that in principle we can solve A) and B) in any concrete situation.

One goal of such computation would be to get some data about the denominator of Eisenstein classes and to verify the conjectural relations to some specific values of L-functions. (See further down.)

The first example

In a joint effort with Herbert Gangl we investigated a "baby" case : The group $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$, the symmetric space is the upper half plane $\mathbb{H} = \mathrm{Sl}_2(\mathbb{R})/SO(2)$ and $\mathcal{M} = \mathcal{M}_n = \{\sum_{\nu=0}^n a_\nu X^\nu Y^{n-\nu} \mid a_\nu \in \mathbb{Z}\}$ where $n > 0$ is even, this is the highest weight module with highest weight $\lambda = n\gamma$. In this case the cohomology in degree one is of interest. Task A) is relatively easy (see [10] Chap. 2 , 2.1.8). If we divide by the torsion and observe that $H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}})/\mathrm{tors} = \mathbb{Z}\omega_n$ and break the exact sequence (2) then we get

$$0 \to H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \to H^1(\Gamma\backslash\bar{\mathbb{H}}, \tilde{\mathcal{M}})/\mathrm{tors} \xrightarrow{r} \mathbb{Z}\omega_n \to 0, \tag{3}$$

here $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ is simply defined as the kernel of r. If we tensor by the rationals then $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Q})$ is the so called inner cohomology, i.e. the image of the compactly supported cohomology in the cohomology.

We have the Hecke operator $T_p = T^{\mathrm{coh},\lambda}_{p,\chi}$, where $\chi(p) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, (see [10], Chap. 3) . We know that ([10], 3.3.1)

$$T_p\omega_n = (p^{n+1} + 1)\omega_n.$$

On the other hand we know that the eigenvalues of T_p on $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \otimes \mathbb{C}$ are equal to the eigenvalues of T_p on the space of holomorphic cusp forms $S_{n+2}(\Gamma)$ (Eichler-Shimura isomorphism). These latter eigenvalues satisfy the famous estimate $|a_p| \leq 2p^{\frac{n+1}{2}}$, hence the they are definitely smaller than $p^{n+1} + 1$, provided p is not too small. But in our concrete situation we may assume $n \geq 10$, because otherwise there are no cusp forms. Hence we get for any prime p that

$$T_p\omega_n = (p^{n+1} + 1)\omega_n \text{ and } \det((p^{n+1} + 1)\mathrm{Id} - T_p)|_{H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})} \neq 0.$$

Since the T_p commute we get a canonical splitting

$$H^1(\Gamma\backslash\bar{\mathbb{H}}, \tilde{\mathcal{M}} \otimes \mathbb{Q}) = H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Q}) \oplus \mathbb{Q}\tilde{\omega}_n \tag{4}$$

where $r(\tilde{\omega}_n) = \omega_n$ and $T_p\tilde{\omega}_n = (p^{n+1} + 1)\tilde{\omega}_n$. (Manin-Drinfeld principle).

The class $\tilde{\omega}_n$ is called the Eisenstein class, We are interested to find the smallest positive integer $\Delta(n)$ such that $\Delta(n)\tilde{\omega}_n \in H^1(\Gamma\backslash\bar{\mathbb{H}}, \tilde{\mathcal{M}})/\mathrm{tors}$. This number $\Delta(n)$ is called the *Denominator of the Eisenstein class.*

Any $y \in H^1(\Gamma\backslash\bar{\mathbb{H}},\tilde{\mathcal{M}})/\text{tors}$ we can be write $\Delta(n)y = y_! + n(y)\Delta(n)\tilde{\omega}_n$ with $y_! \in H^1_!(\Gamma\backslash\bar{\mathbb{H}},\tilde{\mathcal{M}})/\text{tors}$. Then the induced map $y \mapsto y_! \mod \Delta(n)H^1_!(\Gamma\backslash\bar{\mathbb{H}},\tilde{\mathcal{M}})/\text{tors}$ yields an inclusion

$$\mathbb{Z}/\Delta(n)\,\omega_n \hookrightarrow H^1_!(\Gamma\backslash\bar{\mathbb{H}},\tilde{\mathcal{M}})/\text{tors} \otimes \mathbb{Z}/\Delta(n) \tag{5}$$

This denominator can be computed from any of the Hecke operators. With Gangl we wrote a program which accomplishes task A) and also task B) for T_2. A rough description of the algorithm is given in [10], Chapter 3. We could compute the matrix of T_2 for a large number of n (I think $n \leq 150$) and we found experimentally

$$\Delta(n) = \text{Numerator of}(\zeta(-1-n)). \tag{6}$$

This assertion is actually a theorem and is proved in [10] Thm. 5.1.1.

It is well known and very easy to see that we get congruences from the denominator. To be more precise: The Hecke algebra acts semi-simply on $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{Q})$, we can find a finite normal extension $F/\mathbb{Q}$ such that we get a decomposition into eigenspaces

$$H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes F) = \bigoplus_{\pi_f} H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes F)(\pi_f), \tag{7}$$

where π_f is a homomorphism from the Hecke algebra to the ring of integers $\mathcal{O}_F$, we write $\pi_f(T_p) = a_p = a_p(\pi_f)$. This implies that for the integral cohomology we can find a filtration

$$\{0\} \subset \mathcal{F}^1H^1 \subset \mathcal{F}^2H^1 \subset \dots H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathcal{O}_F)/\text{tors} \tag{8}$$

such the the successive subquotients are locally free and eigenmodules for the Hecke algebra with the above eigenvalues. This implies: If a prime ℓ divides $\Delta(n)$ then we have the inclusion $\mathcal{O}_F/\ell\omega_n \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathcal{O}_F)/\text{tors}\otimes\mathcal{O}_\mathrm{F}/\ell$ then we can find a prime $\mathfrak{l} \subset \mathcal{O}_F$ which divides ℓ, and a π_f such that

$$\pi_f(T_p) \equiv p^{n+1} + 1 \mod \mathfrak{l} \tag{9}$$

for all p. For $n = 10$ we get the the famous Ramanujan congruences $\tau(p) \equiv p^{11} + 1 \mod 691$.

Here we have to be aware of the following fact: We have the implication

$$\text{Non trivial denominator} \implies \text{Non trivial congruences} \tag{10}$$

but it is clear that there is no way to reverse the implication arrow.

(Of course, if V is a two dimensional vector space over the finite field $\mathbb{F}_\ell$ and if $T : V \to V$ is an endomorphism with $T^2 = 0$ then we have $T \neq 0$ with probability $(l^2-1)/l^2$. But this kind of a dead end we meet very often in number theory.)

This is of importance once we want to verify this kind of assertions experimentally. If we want to compute the denominator we only have to write an algorithm which achieves task A) and B). (See later in section 1.5)

If we want to verify only congruences experimentally we can check (9) for a finite number of primes and then we hope that this is not an accident. For this we also refer to the next example.

1.1 Interlude: The q-expansion denominator

In our special situation we can discuss another notion of the denominator of the Eisenstein class. We consider the space $M_{n+2}(\Gamma)$ of holomorphic modular forms of weight $n+2$. This space is the direct sum of the space of cusp forms $S_{n+2}(\Gamma)$ and a one dimensional space generated by the Eisenstein series $E_{n+2}(z)$. These modular forms have a q-expansion and we can define the space of modular forms with rational coefficients in their expansion. We get a decomposition

$$M_{n+2}(\Gamma)(\mathbb{Q}) = \mathbb{Q}E_{n+2} \oplus S_{n+2}(\Gamma)(\mathbb{Q}) \tag{11}$$

where the Eisenstein series has the q-expansion

$$E_{n+2}(z) = 1 + (-1)^{\frac{n+2}{2}} \frac{2}{\zeta(-1-n)} \sum_{\nu=1}^{\infty} \sigma_{n+1}(\nu) q^{\nu} \ ; q = e^{2\pi i z}$$

and $\sigma_{n+1}(\nu) = \sum_{d|\nu} d^{n+1}$ (See [21], Chap. VII.) Now we consider the space $M_{n+2}(\Gamma)(\mathbb{Z})$, this is the space of modular forms of weight $n+2$ with integral coefficients in the q expansion. Then we can use the formulas in [21] , Chap. VII, §3 to show that we can find a modular form f in $M_{n+2}(\Gamma)(\mathbb{Z})$ which starts $f(z) = 1 + \sum a_m q^m$.

We have the action of the Hecke operators on $M_{n+2}(\Gamma)(\mathbb{Z})$, the Eisenstein series satisfies $T_p(E_{n+2}) = (p^{n+1} + 1)E_{n+2}$. We intersect the decomposition (11) with $M_{n+2}(\Gamma)(\mathbb{Z})$ and get *a decomposition up to isogeny*

$$M_{n+2}(\Gamma)(\mathbb{Z}) \supset \mathbb{Z}\zeta(-1-n)E_{n+2} \oplus S_{n+2}(\Gamma)(\mathbb{Z}). \tag{12}$$

The finite quotient $M_{n+2}(\Gamma)(\mathbb{Z})/(\mathbb{Z}\zeta(-1-n)E_{n+2} \oplus S_{n+2}(\Gamma)(\mathbb{Z}))$ is a cyclic group generated by the modular form f, hence we see that the denominator of E_{n+2} is the numerator of $\zeta(-1-n)$,

We know that for an eigenform $f \in S_{n+2}(\Gamma)$ the coefficients in the q-expansion are equal to the eigenvalues of the respective Hecke operator and hence we see that we get essentially the same sort of congruences as in (9). But it seems to be difficult to relate the (p-adic) integral structure on the space of modular forms and the (p-adic) integral structure on the Betti cohomology. Hence it is not clear how to relate the denominator in Betti cohomology to the q -expansion denominator. The referee suggests that one could try to use the Fontaine-Lafaille functor [7] in the case $p > n$.

1.2 A second example

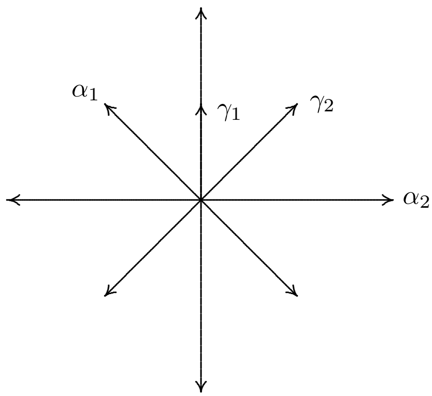
The denominator of Eisenstein classes is ubiquitous in the cohomology of arithmetic groups and we expect that there should be some a strong linkage between primes dividing certain special values of L-functions and primes dividing the denominator. This presumption has been formulated in [8], Kapitel III.

We briefly discuss the example, which was the first one, where we found some experimental evidence for this general expectation.

We start from $\mathcal{G}/\mathbb{Z} = \mathrm{Sp}_2/\mathbb{Z}$, we realise it as the automorphism group of the lattice $\mathcal{L} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}f_2 \oplus \mathbb{Z}f_1$ equipped with the alternating form $< e_1, f_1 >=< e_2, f_2 >= 1$ and and all other not yet defined values $= 0$. (See also 1.3).Let $\mathcal{T}$ be the standard maximal torus

$$\mathcal{T} := \left\{ \underline{t} = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \right\} \tag{13}$$

and $\mathcal{B}/\mathbb{Z}$ the standard Borel subgroup, it is the stabiliser of the flag $\{e_1\} \subset \{e_1, e_2\} \subset \{e_1, e_2, f_2\} \subset \mathcal{L}$. The corresponding generic fibers are G, T, B. The drawing shows its system of roots and the dominant fundamental weights.



the character module is $X^*(T) = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2, \gamma_1(\underline{t}) = t_1, \gamma_2(\underline{t}) = t_1 t_2$. Here γ_1, γ_2 are the two fundamental weights. The cocharacter module is $X_*(T) = \mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_2$ where $< \chi_i, \gamma_j >= \delta_{i,j}$. The simple roots are α_1, α_2. We have two maximal parabolic subgroups $P_1, P_2 \supset B$, the reductive quotients of these two parabolics are denoted by M_1, M_2. Our convention is that the root system of M_i is $\{\alpha_i, -\alpha_i\}$. The group P_1 is also called the Siegel parabolic and P_2 is the Klingen parabolic. For any root α we denote by U_α the corresponding one-parameter subgroup. The two fundamental weights extend to characters $\gamma_2 : P_1 \to \mathbb{G}_m, \gamma_1 : P_2 \to \mathbb{G}_m$, these extensions are also called the fundamental weights.

Our arithmetic group will be $\Gamma = \mathrm{Sp}_2(\mathbb{Z})$, Our coefficient system will be $\mathcal{M} = \mathcal{M}_\lambda$ where $\lambda = n_1\gamma_1 + n_2\gamma_2, n_2 \equiv 0(2)$. In this case the cohomology of the boundary becomes much more complicated. In the description of the boundary cohomology some genus one modular cusp forms f of weight $2n_2 + n_1 + 4$ enter the stage. These modular forms are eigenforms for the Hecke operators, the eigenvalues are algebraic integers, which generate a finite extension $\mathbb{Q}(f)/\mathbb{Q}$ of $\mathbb{Q}$. To such an eigenform we can attach a Hecke L-function. For these L-functions $L(f, s)$ we can find two carefully chosen periods $\Omega(f)_\pm$ which are real numbers-well defined up to elements in in $\mathbb{Q}(f)^\times$ - such that for $\nu = 1, \ldots, 2n_2 + n_1 + 3$ the numbers

$$\frac{L(f, \nu)}{\Omega(f)_{\varepsilon(\nu)}} \in \mathbb{Q}(f)$$

These numbers ν are the so called *critical arguments* and the values $L(f, \nu)$ are the *critical values* . There is some kind of canonical choice for the periods $\Omega(f)_\pm$ such that the values

$$\Delta(n)\frac{L(f,1)}{\Omega(f)_+}, \frac{L(f,3)}{\Omega(f)_+}, \frac{L(f,5)}{\Omega(f)_+}, \ldots, \frac{L(f, 2n_2 + n_1 + 1)}{\Omega(f)_+}, \Delta(n)\frac{L(f, 2n_2 + n_1 + 3)}{\Omega(f)_+}$$

as well as

$$\frac{L(f,2)}{\Omega(f)_-}, \frac{L(f,4)}{\Omega(f)_-}, \dots, \frac{L(f,2n_2+n_1+2)}{\Omega(f)_-}$$

form an array of coprime integers in $\mathbb{Q}(f)$. Here we assume that the class number of $\mathbb{Q}(f)$, is one. If this is not the case we need a slightly more sophisticated formulation and define the periods locally for a covering by Zariski open subsets.

Then we expect that some (large ?) primes $\mathfrak{l}$, *which divide certain critical values* $\frac{L(f,\nu)}{\Omega(f)_{\varepsilon(\nu)}}$ *should also divide the denominator* $\Delta(f)$ *of the Eisenstein class* $\mathrm{Eis}(f)$.

We want to check this in examples, to do so we assume that our cusp form is defined over $\mathbb{Q}$, i.e. $\mathbb{Q}(f) = \mathbb{Q}$. We have exactly one such form f_k for each of the weights $k = 12, 16, 18, 20, 22, 26$. The first case where we see a divisibility by a "large" prime is the case $k = 22$, we have $41|\frac{L(f_{22},14)}{\Omega_+}$. In [9] we explain that we have to take $\lambda = 4\gamma_1 + 7\gamma_2$, and then we see an Eisenstein class

$$\mathrm{Eis}(f_{22}) \in H^3(\Gamma\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}),$$

and our expectation is that 41 divides its denominator $\Delta(f_{22})$.

Hence the challenge is to carry out task A) and B) in this special case for one Hecke operator we would get a verification of $41|\Delta(f)$. To my best knowledge so far we do not yet have an effective algorithm for this case.

It is also not clear to me whether in this case we are already beyond the limit of capability of existing computers.

But the the resulting congruence for the Hecke eigenvalues of $T^{\mathrm{coh},\lambda}_{p,\chi_2}$ have been verified by Faber and van der Geer for $p \leq 37$ (See [6] and [9]). Here
$\chi_2 : \mathbb{G}_m \to T^{\mathrm{ad}}$ is the cocharacter which satisfies $<\chi_2, \alpha_2>= 1$ and
$<\chi_2, \alpha_1>= 0$.

In the meanwhile congruences of this type have been verified in many more cases by many different people, always for a finite number of operators $T^{\mathrm{coh},\lambda}_{p,\chi}$ (see for instance [1],[2],[17]). The experimental evidence is very convincing. To get the necessary data one has to compute many traces of Hecke operators $T^{\mathrm{coh},\lambda}_{p,\chi}$ for many p and some specific choices of χ.

T. Ibukiyama proved a half integral version of these congruences in [15].

Chenevier and Lannes proved that the congruences mod 41 holds for all primes p, Lannes reported on this at the Mini-conference in Oberwolfach. In the meanwhile the proof appeared in the book [5]. T. Mégarbané extended their method in [18], und proved many new congruences. If I understand correctly these authors actually use the denominator argument to prove the congruences, but they use different groups. They start from semi simple groups $G/\mathbb{Z}$ for which $G(\mathbb{R})$ is compact. Then the symmetric space is simply a point $*$ and the locally symmetric space is replaced by $\mathcal{S}^G_{K_f} = G(\mathbb{Q})\backslash(* \times G(\mathbb{A}_f)/K_f)$ where $\mathbb{A}_f$ is the ring of finite adeles and $K_f = G(\hat{\mathbb{Z}})$ where $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. Then $\mathcal{S}^G_{K_f}$ is a finite set. They consider the cohomology $H^0(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ and the action of the Hecke algebra on it. This is certainly easier than the case above where the locally symmetric space is an "honest" space. But they have to pay a price, they have to pass to much larger groups, namely

SO(24) in [5] or to SO(23), SO(25) in [18]. The cohomology $H^0(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q})$ is semi simple and decomposes into irreducibles. This induces a decomposition *up to isogeny*

$$H^0(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \supset \bigoplus_{\Pi_f} H^0(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\Pi_f) \tag{14}$$

here $H^0(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)/H^0(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\Pi_f)$ is torsion free. Then the finite -hopefully non zero- quotient $H^0(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \supset \bigoplus_{\Pi_f} H^0(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\Pi_f)$ yields denominators and hence congruences. Now the authors use the very deep results of Arthur on the trace formula and show that some of the Π_f are endoscopic lifts from f_{22} or from a Siegel modular form and they see the congruences.

In [9] I give a heuristic argument why we should have the denominators. The argument was based on some speculations about mixed Tate-motives. These speculations become much more concrete if these mixed Tate-motives are mixed Kummer-motives (See section 1.6). We get these mixed Kummer motives, if our coefficient system is trivial, this means if $\lambda = 0$. But then we have to allow ramification. We discuss this issue for the rest of this paper.

1.3 Another promising case

During the conference in Oberwolfach some people discussed the case of the subgroup $\Gamma_0(p) \subset \Gamma = \mathrm{Sp}_2(\mathbb{Z})$ where $\Gamma_0(p) \subset \Gamma = \mathrm{Sp}_2(\mathbb{Z})$ is the inverse image of $P_2(\mathbb{F}_p) \subset \mathrm{Sp}_2(\mathbb{F}_p)$. The symmetric space is $\mathbb{H}_2$ the Siegel upper halfspace. From now on our coefficient system will be trivial.

We are mainly interested in the cohomology in degree 3. We look at our fundamental exact sequence

$$\to H^3_c(\Gamma_0(p)\backslash\mathbb{H}_2, \mathbb{Z}) \xrightarrow{j} H^3(\Gamma_0(p)\backslash\mathbb{H}_2, \mathbb{Z}) \xrightarrow{r} H^3(\partial(\Gamma_0(p)\backslash\mathbb{H}_2), \mathbb{Z}) \xrightarrow{\delta^{\mathfrak{q}}} H^4_c(\dots). \tag{15}$$

As usual we denote by $H^3_!(\Gamma_0(p)\backslash\mathbb{H}_2, \mathbb{Z})$ the image of j which is equal to the kernel of r.

Now we ask for an algorithm which in this special case -at least for some small values of p, *and a few "small" Hecke operators- solves task A) and B) for all the modules and the arrows in this sequence.*

In the following we collect some information we have about the above cohomology and the action of the Hecke algebra. To be a little bit more precise, we exhibit some explicit Hecke modules, which occur in these cohomology groups, and of course the algorithm must see these pieces.

But the main problem will be to check the conjectures about denominators of Eisenstein classes (see (93)), which can be verified or falsified by this algorithm.

1.3.1 The Borel-Serre boundary

Our first goal is to understand $\partial(\Gamma_0(p)\backslash\mathbb{H}_2)$. We know that it is the union of strata

$$\partial(\Gamma_0(p)\backslash\mathbb{H}_2) = \partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2) \cup \partial_{[P_2]}(\Gamma_0(p)\backslash\mathbb{H}_2) \cup \partial_{[B]}(\Gamma_0(p)\backslash\mathbb{H}_2) \tag{16}$$

where these strata correspond to conjugacy class of the Siegel parabolic P_1 the Klingen parabolic P_2 and the Borel subgroup B. We know that these strata have connected components and we have an explicit understanding of these connected components.

The connected components of the boundary strata are in 1-1 correspondence with orbits of $\Gamma_0(p)$ acting on the rational points of the flag variety $\mathcal{X}_{P_1}, \mathcal{X}_{P_2}, \mathcal{X}_B$ of parabolic subgroups of type P_1, P_2, B respectively. If P any of the three parabolic subgroups then we get for the set of connected components of $\partial_{[P]}$

$$\pi_0(\partial_{[P]}(\Gamma_0(p)\backslash\mathbb{H}_2)) = \Gamma_0(p)\backslash\Gamma/\Gamma_P \text{ where } \Gamma_P = P(\mathbb{Z}) = P(\mathbb{Q}) \cap \Gamma.$$

Since $\Gamma_0(p)$ contains the full congruence subgroup mod p (the kernel of $\Gamma \to \mathrm{Sp}_2(\mathbb{F}_p)$) this double coset is also equal to

$$P_2(\mathbb{F}_p)\backslash\mathrm{Sp}_2(\mathbb{F}_p)/P(\mathbb{F}_p) = W_{P_2}\backslash W/W_P :$$

where W_P= Weyl group of M_P, the reductive quotient of P.

The Weyl group W is generated by the reflections s_2, s_1. The quotient

$$W_{P_2}\backslash W = W^{P_2} = \{e, s_1, s_1s_2, s_1s_2s_1\} = \text{ set of Kostant representatives. .} \tag{17}$$

We have to compute the orbits of W_P on $\{e, s_1, s_1s_2, s_1s_2s_1\}$

i) If $P = B$ then $W_P = \{e\}$ and we have four orbits.

ii) If $P = P_2$ then $W_{P_2} = \{e, s_2\}$ and $\{e, s_1, s_1s_2, s_1s_2s_1\}/\{e, s_2\} = \{e, \{s_1, s_1s_2\}, s_1s_2s_1\}$ i.e. we have three orbits, two of length one and one of length 2.

iii) If $P = P_1$ then $W_{P_1} = \{e, s_1\}$ and $\{\{e\}, s_1, s_1s_2, \{s_1s_2s_1\}\}/\{e, s_1\} = \{\{e, s_1\}, \{s_1s_2s_1, s_1s_2\}\}$ we have two orbits of length two.

The description of the boundary strata: We apply reduction theory. The parabolic subgroup P has a reductive quotient $M_P = P/U_P$. This reductive quotient is

i) The maximal torus $T/\mathbb{Z}$ if $P = B$.

ii) If $P = P_2$ then

$$M_2 = M_{P_2} = \left\{ \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} ; \det(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = 1 \right\}.$$

iii) If $P = P_1$ then

$$M_1 = \left\{ M_{P_1} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \end{pmatrix} ; \text{ where } \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \det(\begin{pmatrix} a & b \\ c & d \end{pmatrix})^{-1} \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \right\}.$$

We see that $\partial_{[P]}(\Gamma\backslash X) = \bigcup_{\xi \in W_{P_2}\backslash W/W_P} \partial_{P^\xi}(\Gamma\backslash X)$, here $P^\xi = \xi P \xi^{-1}$ and $\partial_{P^\xi}(\Gamma\backslash X)$ is the connected component of the stratum corresponding to P^ξ. We describe $\partial_{P^\xi}(\Gamma\backslash X)$. We start from the well known fact that

$$P^\xi(\mathbb{R}) \times K_\infty \to G(\mathbb{R})$$

is surjective. If $P \supset B$ is maximal then we have the fundamental weight character $\gamma_\xi : P^\xi \to \mathbb{G}_m$. If $P = B$ then $\gamma_\xi = (\gamma_1^\xi, \gamma_2^\xi) : B \to \mathbb{G}_m^2$ is the pair of dominant fundamental weights. Now γ_ξ induces a surjective homomorphism

$$|\gamma_\xi| : P^\xi(\mathbb{R}) \to (\mathbb{R}_{>0}^\times)^{d_P}.$$

This homomorphism $|\gamma_\xi|$ is trivial on $\Gamma_0(p) \cap P^\xi(\mathbb{R}) = \Gamma_{P^\xi}$ and therefore we can pick any $t_0 \in (\mathbb{R}_{>0}^\times)^{d_P}$ and then we know (geodesic action [10])

$$\partial_{P^\xi}(\Gamma\backslash X) \xrightarrow{\sim} \Gamma_{P^\xi}\backslash\gamma_\xi^{-1}(t_0)$$

Let $P^{(1,\xi)}(\mathbb{R})$ be the kernel of γ_ξ. This kernel acts transitively on $\gamma_\xi^{-1}(t_0)$ hence we can say

$$\underline{\gamma}_\xi^{-1}(1) = P^{(1,\xi)}(\mathbb{R})/K_\infty^P \text{ where } K_\infty^P = P^\xi(\mathbb{R}) \cap K_\infty.$$

We have the projection $\pi_P^\xi : P^\xi \to P^\xi/U_{P^\xi} = M_{P^\xi}$, the fundamental dominant weights γ_ν are trivial on U_{P^ξ} hence they are characters on $M_{P.\xi}$ The image of K_∞^P under the projection is $K_\infty^{M_P}$ it is of finite index in a maximal compact subgroup of $M^{(1,\xi)}(\mathbb{R})$. Then we put

$$X^{M_{P^\xi}} = M^{(1,\xi)}(\mathbb{R})/K_\infty^{M_{P^\xi}}$$

and π_P induces a map

$$P^{(1,\xi)}(\mathbb{R})/K_P \to X^{M_{P^\xi}}.$$

This is a fibration with fiber $U_{P^\xi}(\mathbb{R})$. Let $\Gamma_{M_{P^\xi}}$ be the image of Γ_{P^ξ} under π_P then we get a fibration

$$\partial_{P^\xi}(\Gamma\backslash X) = \Gamma_0(p) \cap \Gamma_{P^\xi}\backslash\gamma_\xi^{-1}(t_0) \xrightarrow{\bar{\pi}_P} \Gamma_{M_{P^\xi}}\backslash X^{M_{P^\xi}} \tag{18}$$

where the fiber is $\Gamma_{U_{P^\xi}}\backslash U_{P^\xi}(\mathbb{R})$.

1.3.2 The cohomology of the boundary strata:

This fibration provides a spectral sequence with E_1-term

$$H^{\mathrm{p}}(\Gamma_{M_{P^\xi}}\backslash X^{M_{P^\xi}}, \widetilde{H^{\mathrm{q}}(\Gamma_{U_{P^\xi}}\backslash U_{P^\xi}}(\mathbb{R}), \mathbb{Z})) \Rightarrow H^{\mathrm{p+q}}(\partial_{P^\xi}(\Gamma\backslash X), \mathbb{Z})$$

here $H^{\mathrm{q}}(\Gamma_{U_{P,\xi}}\backslash U_{P,\xi}(\mathbb{R}), \mathbb{Z})$ is a module for $\Gamma_{M_{P^\xi}}$ and the sheaf in the above formula is obtained by the usual process module to sheaf).

We compute these E_1 terms for our three cases of parabolic subgroups.

i) $P = B$. In this case $M = T$ the split maximal torus. In this case $T(\mathbb{Z}) = B(\mathbb{R}) \cap K_\infty = K_\infty^B$. Hence we see that $X^T = \{\mathrm{pt}\}$. Clearly $\Gamma_{M_{P^\xi}} = T(\mathbb{Z})$, we have to compute

$$H^0(T(\mathbb{Z})\backslash\{\mathrm{pt}\}, \widetilde{H^{\mathrm{q}}(\Gamma_{U_{P,\xi}}\backslash U_{P,\xi}}(\mathbb{R}), \mathbb{Z})).$$

The cohomology $H^\bullet(\Gamma_{U_{P^\xi}}\backslash U_{P^\xi}(\mathbb{R}),\mathbb{Z})$ is a module for the torus T and we know its structure by the theorem of Kostant (perhaps it is better to invert p in the coefficients)

$$H^\bullet(\Gamma_{U_{P^\xi}}\backslash U_{P^\xi}(\mathbb{R}),\mathbb{Z}\Big[\frac{1}{p}\Big])) = \bigoplus_{w\in W}\mathbb{Z}\Big[\frac{1}{p}\Big]e(w\cdot 0)$$

where $e(w\cdot 0)$ is a generator sitting in degree $l(w)$ and $w\cdot 0=\rho^w-\rho$. Here $\rho=\gamma_1+\gamma_2$ and this is the half sum of positive roots.

Then

$$H^0(T(\mathbb{Z})\backslash\{\mathrm{pt}\},\mathbb{Z}\Big[\frac{1}{p}\Big]\tilde{e}(w\cdot 0))) = \mathbb{Z}\Big[\frac{1}{p}\Big] \iff w\cdot 0 \text{ is trivial on } T(\mathbb{Z})$$

and otherwise it is zero. Therefore

$$H^\bullet(\partial_{[B]}(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Z}\Big[\frac{1}{p}\Big]) = \bigoplus_{\xi,w:w\cdot 0 \text{ trivial on } T(\mathbb{Z})}\mathbb{Z}\Big[\frac{1}{p}\Big]e(w\cdot 0).$$

Now we consider a maximal parabolic subgroup P. We replace $\mathbb{Z}$ by the larger ring $\mathbb{Z}_S$ where we we have inverted p and the denominators of Eisenstein classes for the occurring congruence subgroups of $\mathrm{Gl}_2(\mathbb{Z})$. Then have a decomposition into the inner part and the Eisenstein part

$$\begin{aligned} &H^{\mathrm{p}}(\Gamma_{M_{P^\xi}}\backslash X^{M_{P^\xi}},H^{\mathrm{q}}(U_{P^\xi}\widetilde{(\mathbb{Z})\backslash U}_{P^\xi}(\mathbb{R}),\mathbb{Z}_S) = \\ &H^{\mathrm{p}}_!(\Gamma_{M_{P^\xi}}\backslash X^{M_{P^\xi}},H^{\mathrm{q}}(U_{P^\xi}\widetilde{(\mathbb{Z})\backslash U}_{P^\xi}(\mathbb{R}),\mathbb{Z}_S)\oplus \\ &H^{\mathrm{p}}_{\mathrm{Eis}}(\Gamma_{M_{P^\xi}}\backslash X^{M_{P^\xi}},H^{\mathrm{q}}(U_{P^\xi}\widetilde{(\mathbb{Z})\backslash U}_{P^\xi}(\mathbb{R}),\mathbb{Z}_S). \end{aligned} \tag{19}$$

On the summands $H^{\mathrm{p}}_!(\Gamma_{M_{P^\xi}}\backslash X^{M_{P^\xi}},H^{\mathrm{q}}(U_{P^\xi}\widetilde{(\mathbb{Z})\backslash U}_{P^\xi}(\mathbb{R}))$ all differentials $d^1_{\mathrm{p,q}}$ and also the higher differentials vanish and hence the direct sum over the $H_!$ terms inject into the cohomology of the boundary. We get

$$H^\bullet(\partial(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Z}_S) = H^\bullet_!(\partial(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Z}_S)\oplus H^\bullet_{\mathrm{Eis}}(\partial(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Z}_S). \tag{20}$$

We look at the $_!$ summand first. We compute

$$H^{\mathrm{q}+1}_!(\partial(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Z}_S) = \bigoplus_{P=P_1,P_2,\xi,\mathrm{q}} H^1_!(\Gamma_{M_{P^\xi}}\backslash X^{M_{P^\xi}},H^{\mathrm{q}}(\Gamma_{U_{P^\xi}}\widetilde{\backslash U_{P^\xi}}(\mathbb{R}),\mathbb{Z}_S) \tag{21}$$

ii) We consider the case $P=P_2$. Then $M(\mathbb{R})=\mathrm{Sl}_2(\mathbb{R})\times\mathbb{R}^\times$ hence $M^{(1)}(\mathbb{R})=\mathrm{Sl}_2(\mathbb{R})\times\{\pm1\}$ and $M^{(1)}(\mathbb{R})\cap K_\infty=\mathrm{SO}(2)\times\{\pm 1\}$. Therefore

$$M^{(1)}(\mathbb{R})/K^M_\infty = \mathrm{Sl}_2(\mathbb{R})/\mathrm{SO}(2)=\mathbb{H}.$$

Hence we have to compute

$$H^1_!(\Gamma_{M_{P^\xi}}\backslash\mathbb{H},H^{\mathrm{q}}(\Gamma_{U_{P^\xi}}\widetilde{\backslash U_{P^\xi}}(\mathbb{R}),\mathbb{Z}_S)).$$

In this case $\xi \in \{e, \{s_1, s_1 s_2\}, s_1 s_2 s_1\}$, if $\xi = e$ or $\xi = s_1 s_2 s_1$ then $\Gamma_{M_{P^\xi}} = \mathrm{Sl}_2(\mathbb{Z})$ and if $\xi = s_1$ then

$$\Gamma_{M_{P^\xi}} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z}) | b \equiv 0 \mod p\}. \tag{22}$$

The coefficients are obtained from a $M_2 = \mathrm{Sl}_2 \times \mathbb{G}_m-$ module these are the modules of highest weight

$$\{\dots, w\cdot 0, \dots\}_{w \in W^{P_2}} = \{0, -\gamma_1 + \bar{\gamma}_2, -3\gamma_1 + \bar{\gamma}_2, -4\gamma_1\} \tag{23}$$

where $\bar{\gamma}_2$ is the fundamental weight of the group M_2 (see [10] 9.1.3). These modules contribute to cohomology in degree 1,2,3,4. Since $-\mathrm{Id} \in \Gamma_{M_{P^\xi}}$ the contributions in degree 2 and 3 vanish, we get

$$H^1_! \partial_{[P_2]}(\Gamma_0(p)\backslash \mathbb{H}_2), \mathbb{Z}_S) = \bigoplus\nolimits_\xi H^1_!(\Gamma_{M_{P^\xi}} \backslash X^{M_{P^\xi}}, \mathbb{Z}_S \omega(e\cdot 0))$$

$$H^4_!(\partial_{[P_2]}(\Gamma_0(p)\backslash \mathbb{H}_2), \mathbb{Z}_S) = \bigoplus\nolimits_\xi H^1_!(\Gamma_{M_{P^\xi}} \backslash X^{M_{P^\xi}}, \mathbb{Z}_S \omega(s_1 s_2 s_3 \cdot 0)),$$

where the $\omega(w\cdot 0)$ are generators of the rank one modules $H^{l(w)}(\Gamma_{U_{P^\xi}} \backslash U_{P^\xi}(\mathbb{R}), \mathbb{Z}\left[\frac{1}{p}\right]))$, here $l(w) = 0, 3$.

For $\xi = e$ or $\xi = s_1 s_2 s_1$ the cohomology $H^1_!(\Gamma_{M_{P^\xi}} \backslash X^{M_{P^\xi}}, \mathbb{Z}\left[\frac{1}{p}\right]) = 0$ (There are no cusp forms of weight 2 for $\Gamma_{M_{P^\xi}} = \mathrm{Sl}_2(\mathbb{Z})$). But for $\xi = s_1$ the cohomology

$$H^1_!(\Gamma_{M_{P^\xi}} \backslash X^{M_{P^\xi}}, \mathbb{Z}_S \omega(e\cdot 0)) \oplus H^1_!(\Gamma_{M_{P^\xi}} \backslash X^{M_{P^\xi}}, \mathbb{Z}_S \omega(s_1 s_2 s_3 \cdot 0))) \tag{24}$$

is not necessarily zero, In both degrees we get two copies of a Hecke module, which is isomorphic to the space of cusp forms of weight 2 for $\Gamma_{M_{P^\xi}}$.

iii) We consider the case $P = P_1$. Under our present assumptions this is the most interesting case. In this case $M_P = \mathrm{Gl}_2$ and $M^{(1)}(\mathbb{R}) = \mathrm{Gl}_2^{\pm 1}(\mathbb{R}) = \{g \in \mathrm{Gl}_2(\mathbb{R}) | \det(g) = \pm 1\}$. The group $K^M_\infty = \mathrm{O}(2)$ and hence we get again

$$X^M = \mathrm{Gl}_2^{\pm 1}(\mathbb{R})/\mathrm{O}(2) = \mathrm{Sl}_2(\mathbb{R})/\mathrm{SO}(2) = \mathbb{H}.$$

For both values of ξ the group

$$\Gamma_{M_{P^\xi}} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2(\mathbb{Z}) | b \equiv 0 \mod p \text{ or (depending on } \xi) \ c \equiv 0 \mod p\}.$$

This group contains the group $\Gamma^{(1)}_{M_{P^\xi}} = \Gamma_{M_{P^\xi}} \cap \mathrm{Sl}_2(\mathbb{Z})$ as a subgroup of index 2. The coefficients are highest weight modules of weight

$$\{\dots, w\cdot 0, \dots\}_{w \in W^{P_1}} = \{0, -\gamma_2 + 2\bar{\gamma}_1, -2\gamma_2 + 2\bar{\gamma}_1, -3\gamma_2\}. \tag{25}$$

For $w = s_2$ or $w = s_2 s_1$ the module $\mathbb{Z}_S w\cdot 0$ is the M_{P^ξ} module of rank 3 with highest weight $w\cdot 0$ (here it suffices that $2 \in S$.)

On $H^1_!(\Gamma^{(1)}_{M_{P\xi}}\backslash X^M, \widetilde{\mathbb{Z}_S w\cdot 0})$ we have the action of $\pi_0(\mathrm{Gl}_2(\mathbb{R}))=\mathrm{O}(2)/\mathrm{SO}(2)$ and this cohomology decomposes into a $+$ and a $-$ eigenspace. Then we get

$$H^1_!(\Gamma_{M_{P\xi}}\backslash X^{M_{P\xi}}, \mathbb{Z}_S w\cdot 0) = H^1_!(\Gamma^{(1)}_{M_{P\xi}}\backslash X^{M_{P\xi}}, \mathbb{Z}_S w\cdot 0)(+). \tag{26}$$

The tensor product of this module by $\mathbb{C}$ is -as a Hecke module- isomorphic to the space of cusp forms for $\Gamma^{(1)}_{M_{P\xi}}$, of weight $2,4,4,2$ respectively the cohomology groups sit in degrees $1,2,3,4$ respectively.

We rewrite (21)

$$H^\bullet_!(\partial(\Gamma_0(p)\backslash \mathbb{H}_2), \mathbb{Z}) = \bigoplus_{P\in\{P_1,P_2\},\xi,w\in W^P} H^1_!(\Gamma_{M_{P\xi}}\backslash X^{M_{P\xi}}, \widetilde{\mathbb{Z}(w\cdot 0)}) \tag{27}$$

To compute the contribution of the Borel stratum we have to compute the differentials or in other words we have to compute the $E_2^{\bullet,\bullet}$. This means we have to compute the extremal terms in the exact sequence

$$0\to E_2^{0,\mathrm{q}} \to H^{\mathrm{q}}_{\mathrm{Eis}}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Q}) \oplus H^{\mathrm{q}}_{\mathrm{Eis}}(\partial_{[P_2]}(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Q}) \to$$

$$H^{\mathrm{q}}_{\mathrm{Eis}}(\partial_{[B]}(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Q}) \to E_2^{1,\mathrm{q}} \to 0$$

and then we get

$$0\to E_2^{1,\mathrm{q}-1} \to H^q_{\mathrm{Eis}}(\partial(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Q}) \to E_2^{0,\mathrm{q}}\to 0.$$

A somewhat tedious computation gives us

$$E_2^{0,0}=\mathbb{Q};\ E_2^{1,0}=\mathbb{Q}^4;\ E_2^{1,1}=\mathbb{Q};\ E_2^{0,3}=\mathbb{Q};\ E_2^{0,4}=\mathbb{Q}^4;\ E_2^{1,4}=\mathbb{Q} \text{ and all others} = 0, \tag{28}$$

the exponent 4 in $\mathbb{Q}^4$ comes from the four connected components of $\pi_0(\partial_{[B]}(\Gamma_0(p)\backslash\mathbb{H}_2))$.

We summarize: We have complete understanding of the cohomology of the boundary as Hecke module in terms of elliptic modular forms of low weight. Hence we can ask the next question: What is the image Im*(r) ?*

To achieve this goal we need some information on local intertwining operators between Hecke-Iwahori modules.

1.3.3 Interlude: Iwahori -Hecke modules.

We consider $G/\mathbb{Z}=\mathrm{GSp}_2/\mathbb{Z}$ as a Chevalley scheme over $\mathbb{Z}$, then $K_p^{(0)}=\mathrm{GSp}_2(\mathbb{Z}_p))\subset \mathrm{GSp}_2(\mathbb{Q}_p))$ is a maximal compact subgroup. We have the standard reduction map $G(\mathbb{Z}_p)\to G(\mathbb{F}_p)$, let $\mathcal{I}\subset K_p^{(0)}$ be the Iwahori subgroup, it is the inverse image of the standard Borel $B(\mathbb{F}_p)\subset G(\mathbb{F}_p)$. We define the Iwahori-Hecke algebra

$$\begin{gathered}\mathcal{H}_{p,\mathcal{I}} := \mathcal{C}_c(\mathcal{I}\backslash G(\mathbb{Q}_p)/\mathcal{I},\mathbb{Z})) = \{f: G(\mathbb{Q}_p)\to\mathbb{Z} \mid \\ f \text{ has compact support and is biinvariant under } \mathcal{I}\}.\end{gathered} \tag{29}$$

This algebra of functions is an algebra under convolution, the elements $f \in \mathcal{H}_{\mathcal{I}}$ act upon the cohomology $H^\bullet(\Gamma_{00}(p)\backslash\mathbb{H}_2,\mathbb{Z})$ where $\Gamma_{00}(p)$ is the inverse image of $B(\mathbb{F}_p)$.

Of course we can also define the parahoric subgroups $\mathcal{I}_1,\mathcal{I}_2$ which are the inverse images of the groups $P_1(\mathbb{F}_p),P_2(\mathbb{F}_p)$ respectively. The Iwahori subgroup corresponds to a fundamental simplex Σ in the Bruhat-Tits building, our maximal compact subgroup $K_p^{(0)}$ is one of the vertices, the parahoric subgroups $\mathcal{I}_1,\mathcal{I}_2$ are the two faces of Σ, which meet in $K_p^{(0)}$. The other vertices of $-$term resp. $\mathcal{I}_2$ correspond to maximal compact subgroups $K_p^{(1)}$(resp. $K_p^{(2)}$.) The subgroup $K_p^{(1)}$ is conjugate to $K_p^{(0)}$ by an element of $G(\mathbb{Q}_p)$ (these are the hyper special maximal compact subgroups) and $K_p^{(2)}$ is not conjugate to $K_p^{(0)}$.

It K_p is one of these open compact subgroups then we can define the open compact subgroup $K_f(p)=\prod_{\ell\neq p}G(\mathbb{Z}_\ell)\times K_p\subset G(\mathbb{A}_f)$, here $G(\mathbb{A}_f)$ is the group of finite adeles. We define a congruence subgroup

$$\Gamma=G(\mathbb{Q})\cap(G(\mathbb{R})\times K_f(p))\subset G(\mathbb{Q}).$$

If $K_p=K_p^{(2)}$ then Γ is the so called paramodular subgroup in the symplectic group. If $K_p=\mathcal{I}_2$ then $\Gamma=\Gamma_0(p)$.

Later we will consider cohomology groups $H_c^\bullet(\Gamma\backslash\mathbb{H}_2,\mathbb{Z}),H^\bullet(\Gamma\backslash\mathbb{H}_2,\mathbb{Z})\dots$ as modules for the Hecke algebra $\mathcal{H}_{K_p}\times\mathcal{H}^{(p)}$ where the second factor is the commutative unramified Hecke algebra $\bigotimes_{\ell\neq p}\mathcal{C}_c(G(\mathbb{Q}_\ell)//G(\mathbb{Z}_\ell),\mathbb{Z}))$, where the // means the we consider functions which right and left invariant under the action of $G(\mathbb{Z}_\ell)$. The absolutely irreducible modules for $\mathcal{H}^{(p)}$ are simply homomorphisms $h(\sigma_f):\mathcal{H}^{(p)}\to F$, where $F/\mathbb{Q}$ is a number field.

In this subsection we are interested in the structure of irreducible (or indecomposable) modules for the Hecke algebra $\mathcal{H}_{p,\mathcal{I}}$ more precisely we want to study finitely generated free $\mathbb{Z}$ -modules with an action of $\mathcal{H}_{p,\mathcal{I}}$, (For this see also C. Jantzen "Degenerate principal Series for Symplectic Groups", work of Casselman, Borel and P. Garrett.)

For any field L of characteristic zero, we consider the module of unramified characters $\mathrm{Hom}(T(\mathbb{Q}_p)/T(\mathbb{Z}_p),L^\times)$. We have the isomorphism $X_*(T)\xrightarrow{\sim}T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ which is given by $\chi\mapsto\chi(p)$ and since $X_*(T)$ and $X^*(T)$ are duals of each other we get

$$\mathrm{Hom}(T(\mathbb{Q}_p)/T(\mathbb{Z}_p),L^\times)=X^*(T)\otimes L^\times.$$

For $\underline{u}=\{u_1,u_2\}\in L^\times\times L^\times$ we define $\chi_{\underline{u}}=\gamma_1\otimes u_1+\gamma_2\otimes u_2$. Since we have the homomorphism $B(\mathbb{Q}_p)\to T(\mathbb{Q}_p)$ every character $\chi_{\underline{u}}$ extends canonically to a character on $B(\mathbb{Q}_p)\to L^\times$ which is also called $\chi_{\underline{u}}$.

We have a standard embedding $X^*(T)\hookrightarrow\mathrm{Hom}(T(\mathbb{Q}_p)/T(\mathbb{Z}_p),L^\times)$ we simply send γ to $|\gamma|_p$: $t\mapsto|\gamma(t)|_p$. It is clear that this map is trivial on $T(\mathbb{Z}_p)$. With respect to the above identification this means that $\gamma\mapsto\gamma\otimes p^{-1}$.

We define the induced module

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_{\underline{u}}=\{f:G(\mathbb{Q}_p)\to L\mid f(bg)=\chi_{\underline{u}}(b)f(g),\forall b\in B(\mathbb{Q}_p),g\in G(\mathbb{Q}_p)\}\tag{30}$$

the group $G(\mathbb{Q}_p)$ acts on this L-vector space by $R_g(f)(x) = f(xg)$. It has invariants under the action of the Iwahori subgroup $\mathcal{I}$ and it is easy to see that

$$(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_{\underline{u}})^{\mathcal{I}} = \mathcal{C}(B(\mathbb{F}_p)\backslash G(\mathbb{F}_p)/B(\mathbb{F}_p), L) = \mathcal{C}(W, L), \tag{31}$$

where $\mathcal{C}(?, ??)$ means functions on ? with values in ??. This module of invariants under the group $\mathcal{I}$ is a module for the Hecke algebra $\mathcal{H}_{\mathcal{I}}$.

The theory of induced representations provides intertwining operators between these induced modules. Let us assume for a moment that u_1, u_2 are algebraically independent over $\mathbb{Q}$. For any element $w \in W$ there is an intertwining operator

$$T^{\mathrm{st}}(w, \chi_{\underline{u}}) : \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_{\underline{u}} \to \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} w \cdot \chi_{\underline{u}}, \tag{32}$$

which is defined by the integral

$$T^{\mathrm{st}}(w, \chi_{\underline{u}}) : (g \mapsto f(g)) \mapsto (g \mapsto \int_{U^{(w)}(\mathbb{Q}_p)} f(w^{-1}vg)dv)). \tag{33}$$

Here of course $\chi_{\underline{u}} \to w \cdot \chi_{\underline{u}}$ denotes the twisted action of the Weyl group, i.e. $w \cdot \chi_{\underline{u}} = w(\chi_{\underline{u}}) + w(\rho) - \rho$ and $U^{(w)} = \prod U_\alpha$ where the product is taken over the positive roots α for which $w^{-1}(\alpha)$ is negative. Forming the integral is not problematic, it is an infinite sum, but up to a finite sum it is a sum of nested geometric series.

This intertwining operator is an isomorphism, remember that we assumed that u_1, u_2 are algebraically independent. Since under this condition the induced modules are irreducible, the intertwining operators are unique up to a scalar. Let $\varphi_{\underline{u}} \in \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_{\underline{u}}$ be the spherical function, i.e. $\varphi_{\underline{u}}(g) = \varphi_{\underline{u}}(bk) = \chi_{\underline{u}}(b), b \in B(\mathbb{Q}_p), k \in K_p^{(0)}$ then $T^{\mathrm{st}}(w, \chi_{\underline{u}})(\varphi_{\underline{u}}) = c(w, \chi_{\underline{u}})\varphi_{w \cdot \underline{u}}$ where $c(w, \underline{u})$ is a non zero element in $\mathbb{Q}(u_1, u_2) \subset L$. We also define the local intertwining operator

$$T^{\mathrm{loc}}(w, \chi_{\underline{u}}) : \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_{\underline{u}} \to \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} w \cdot \chi_{\underline{u}} \tag{34}$$

by requiring $T^{\mathrm{loc}}(w, \chi_{\underline{u}})(\varphi_{\underline{u}}) = \varphi_{w \cdot \underline{u}}$.

We want to understand these intertwining operators. We apply the usual approach and write them as composition of simpler intertwining operators. If our element $w = s_i$ is a reflection at a simple root α_i then it is in the Weylgroup of the group the reductive group M_i and we can write our induced module as a two step induction

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_{\underline{u}} = \mathrm{Ind}_{P_i(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\mathrm{Ind}_{\bar{B}_i(\mathbb{Q}_p)}^{M_i(\mathbb{Q}_p)}\chi_{\underline{u}}). \tag{35}$$

Now we have the intertwining operator

$$T^{i,\mathrm{st}}(s_i, \chi_{\underline{u}}) : \mathrm{Ind}_{\bar{B}_i(\mathbb{Q}_p)}^{M_i(\mathbb{Q}_p)}\chi_{\underline{u}} \to \mathrm{Ind}_{\bar{B}_i(\mathbb{Q}_p)}^{M_i(\mathbb{Q}_p)} s_i \cdot \chi_{\underline{u}} \tag{36}$$

and $T^{\mathrm{st}}(s_i, \chi_{\underline{u}}) = \mathrm{Ind}_{P_i(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} T^{i,\mathrm{st}}(s_i, \chi_{\underline{u}})$ is the induced intertwining operator. In this case a well known and rather elementary calculation gives us (see [3], section 3)

$$c(s_i, \chi_{\underline{u}}) = \frac{1 - u_i}{1 - pu_i}. \tag{37}$$

.

If now $w = s_{i_1} s_{i_2} \ldots s_{i_r}$ then the intertwining operator can be given as an iterated integral and we get

$$T^{\mathrm{st}}(w, \chi_{\underline{u}}) = T^{\mathrm{st}}(s_{i_r}, s_{i_r} w^{-1} \cdot \chi_{\underline{u}}) \circ \cdots \circ T^{\mathrm{st}}(s_{i_2}, s_{i_1} \cdot \chi_{\underline{u}}) \circ T^{\mathrm{st}}(s_{i_1}, \chi_{\underline{u}}). \tag{38}$$

this gives us a simple expression for $c(w, \chi_{\underline{u}})$. If $w_0 \in W$ is the longest element then $c(w_0, \chi_{\underline{u}})$ is a meromorphic functions in the variables u_1, u_2. It has a pole-divisor D_∞ and zero divisor D_0, On the complement $\{\underline{u} = (u_1, u_2) | \underline{u} \notin D_\infty \cup D_0\}$ we can evaluate and $c(w, \chi_{\underline{u}}) \neq 0, \infty$, then $T^{\mathrm{st}}(w, \chi_{\underline{u}})$ is an isomorphism.

At the singular points $\underline{u}_0 \in D_\infty \cup D_0$ we can regularise $T(w, \chi_{\underline{u}})$: We can find a simple expression $P_{\underline{u}_0}(u_1, u_2)$ so that

$$T^{\mathrm{reg}}(w, \chi_{\underline{u}}) = P_{\underline{u}_0}(u_1, u_2) T^{\mathrm{st}}(w, \chi_{\underline{u}})|_{\underline{u} = \underline{u}_0} : \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{\underline{u}_0} \to \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} w \cdot \chi_{\underline{u}_0} \tag{39}$$

becomes a non zero intertwining operator. If for instance $w = s_i$ then we can define $T^{\mathrm{reg}}(s_i, \chi_{\underline{u}}) = (1 - p u_i) T^{\mathrm{loc}}(w, \chi_{\underline{u}})$, this operator can be evaluated at any value of $\underline{u}$, it is an isomorphism, unless we have $u_i = 1$ (See below). Then we can define $T^{\mathrm{reg}}(w, \chi_{\underline{u}})$ using (38).

We want to study its restriction to the Hecke-module

$$T^{\mathrm{reg}}(w, \chi_{\underline{u}}) : (\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{\underline{u}})^{\mathcal{I}} \to (\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} w \cdot \chi_{\underline{u}})^{\mathcal{I}}, \tag{40}$$

it is an 8×8 matrix with coefficients in L. We need to study this matrix in the neighbourhood of certain singular points.

We consider the same situation for the two reductive Levi-subgroups M_i. Let $\mathcal{I}_i$ their standard Iwahori subgroup. We consider induced representations and the intertwining operator

$$T^{i,\mathrm{st}}(s_i, \chi_{\underline{u}}) : \mathrm{Ind}_{\bar{B}_i(\mathbb{Q}_p)}^{M_i(\mathbb{Q}_p)} \chi_{\underline{u}} \to \mathrm{Ind}_{\bar{B}_i(\mathbb{Q}_p)}^{M_i(\mathbb{Q}_p)} s_i \cdot \chi_{\underline{u}} \tag{41}$$

The composition is

$$T^{i,\mathrm{st}}(s_i, s_i \cdot \chi_{\underline{u}}) \circ T^{i,\mathrm{st}}(s_i, \chi_{\underline{u}}) = \frac{(1 - u_i)(1 - p^{-2} u_i^{-1})}{(1 - p^{-1} u_i)(1 - p^{-1} u_i^{-1})}$$

and this composition vanishes for $u_i = 1$ and $u_i = p^{-2}$. It is easy to see that in case $u_i = 1$ the kernel of $T^{i,\mathrm{st}}(s_i, \chi_{\underline{u}})$ is the one dimensional subspace generated by $\varphi_{\underline{u}}$. Then $T^{i,\mathrm{st}}(s_i, \chi_{\underline{u}})$ provides an isomorphism between the irreducible quotient $\mathrm{Ind}_{\bar{B}_i(\mathbb{Q}_p)}^{M_i(\mathbb{Q}_p)} \chi_{\underline{u}} / L\varphi_{\underline{u}}$ and a submodule $\mathrm{St}(\underline{u}; i) \subset \mathrm{Ind}_{\bar{B}_i(\mathbb{Q}_p)}^{M_i(\mathbb{Q}_p)} s_i \cdot \chi_{\underline{u}}$ which has codimension 1. Remember this notation means that $u_i = p^{-2}$ and the value of the other coordinate $u_{i'}$ is arbitrary.

This module $\mathrm{St}(\underline{u}; i)$ is the Steinberg module. We will be mainly interested in the case $i = 1$ then we define $\mathrm{St}(1, \tau) := \mathrm{St}(\underline{u}; 1)$ where $u_1 = 1$ and $u_2 = \tau$.

We come to the case, which is especially relevant for our problem. We specialise our variable $\underline{u}$ to values $\underline{u} = (1, \tau)$. We consider the Steinberg module $\mathrm{St}(1,\tau) \subset \mathrm{Ind}_{\bar{B}_i(\mathbb{Q}_p)}^{M_i(\mathbb{Q}_p)} \chi_{\underline{u}}$ and the induced modules

$$\mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathrm{St}(1,\tau) \subset \mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{\underline{u}}. \tag{42}$$

If we restrict to the the invariants under the Iwahori subgroup we get a rank four submodule

$$(\mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathrm{St}(1,\tau))^{\mathcal{I}} \subset (\mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_{\underline{u}})^{\mathcal{I}} = \bigoplus_{w \in W} R\, \delta_w. \tag{43}$$

This submodule can be described explicity. We numerate the elements in W, we write

$$W = \{1, s_2, s_2 s_1, s_2 s_1 s_2, s_1, s_1 s_2, s_1 s_2 s_1, s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1\} \tag{44}$$

and we define $\delta_i = \delta_w$ if w is at place i. Then it is easy to see ([11], 2.4.1) that

$$(\mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathrm{St}(1,\tau))^{\mathcal{I}} = \{x_1(\delta_1 - \frac{1}{p}\delta_5) + x_2(\delta_2 - \frac{1}{p}\delta_6) + x_3(\delta_3 - \frac{1}{p}\delta_7) + x_4(\delta_4 - \frac{1}{p}\delta_8)\}. \tag{45}$$

We define another intertwining operator

$$T_{P_1}^{\mathrm{st}}(\mathrm{St}(1,\tau)) : \mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathrm{St}(1,\tau) \to \mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathrm{St}(1, p^{-3}\tau^{-1}) \tag{46}$$

which again is defined as an integral

$$T_{P_1}^{\mathrm{st}}(\mathrm{St}(1,\tau)) : (g \mapsto f(g)) \mapsto (g \mapsto \int_{U_1(\mathbb{Q}_p)} f(s_2 s_1 s_2 v g) dv)).$$

We can extend this to the operator

$$T(\chi_{\underline{u}}, s_2 s_1 s_2) : \mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathrm{Ind}_{\bar{B}_i(\mathbb{Q}_p)}^{M_i(\mathbb{Q}_p)} s_1 \cdot \chi_{\underline{u}} \to \mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \mathrm{Ind}_{\bar{B}_i(\mathbb{Q}_p)}^{M_i(\mathbb{Q}_p)} s_2 s_1 s_2 \cdot (s_1 \cdot \chi_{\underline{u}}) \tag{47}$$

For a moment we drop the assumption that $\underline{u} = (1,\tau)$. We also drop the assumption that the prime p is the one fixed at the beginning, it may be any prime. We compute the the value of the operator at the spherical function, applying (38) gives us

$$T(\chi_{\underline{u}}, s_2 s_1 s_2)(\varphi_{s_1 \cdot \underline{u}}) = \frac{1 - p u_1 u_2}{1 - p^2 u_1 u_2} \frac{1 - p^2 u_1 u_2^2}{1 - p^3 u_1 u_2^2} \frac{1 - u_2}{1 - p u_2} \varphi_{s_2 s_1 s_2 \cdot s_1 \cdot \underline{u}} \tag{48}$$

We come back to this formula in the next section.

Now p will be again our prime fixed at the beginning. We also assume that $\underline{u} = (1,\tau)$. Then $s_1 \cdot \underline{u} = (p^{-2}, p\tau)$ and (48) yields

$$T(\chi_{\underline{u}}, s_2 s_1 s_2)(\varphi_{\underline{u}}) = \frac{1-\tau}{1-p^2\tau} \frac{1-p^2\tau^2}{1-p^3\tau^2} \varphi_{s_2 s_1 s_2 \cdot \underline{u}} \tag{49}$$

We restrict the intertwining operator to the induced Steinberg module we take invariants under the Iwahori subgroup and consider

$$T^{\mathrm{st}}_{P_1}(\mathrm{St}(1,p^{-2}\tau)) : \mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\mathrm{St}(1,p^{-2}\tau)^{\mathcal{I}} \to \mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\mathrm{St}(1,p^{-1}\tau^{-1})^{\mathcal{I}}, \tag{50}$$

this operator is holomorphic at $\tau = 1$. The following calculation is done with the help of Mathematica and hopefully correct. We evaluate at $\tau = 1$ then

$$T^{\mathrm{st}}_{P_1}(\mathrm{St}(1,p^{-2})) : \mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\mathrm{St}(1,p^{-2})^{\mathcal{I}} \to \mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\mathrm{St}(1,p^{-1})^{\mathcal{I}}$$

has a kernel, which equal to

$$\ker(T^{\mathrm{st}}_{P_1}) = \{x(\delta_1 - \frac{1}{p}\delta_5 - \frac{1}{p}\delta_2 + \frac{1}{p^2}\delta_6)) + y(\delta_2 - \frac{1}{p}\delta_6 - \frac{1}{p}\delta_4 + \frac{1}{p^2}\delta_8)\}.$$

The intersection

$$(\mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\mathrm{St}(1,p^{-2}))^{\mathcal{I}_2} \cap \ker(T^{\mathrm{st}}_{P_1})) = R(\delta_1 - \frac{1}{p}\delta_5 - \frac{1}{p}\delta_2 + \frac{1}{p^2}\delta_6) = Rh_p^{(0)}. \tag{51}$$

If we expand around $\tau = 1$ then we get

$$T^{\mathrm{st}}_{P_1}(\mathrm{St}(1,\tau p^{-2}))(h_p^{(0)}) = \frac{p+1}{p^3(p^3-1)}(\tau - 1)h_p^{(0)} + O((\tau-1)^2). \tag{52}$$

This formula will become essential. We modify the factor in front slightly. The local Euler factor attached to the representation $\mathrm{St}(1,p^{-2})$ is

$$L(\mathrm{St}(1,p^{-2}),z) = \frac{1}{1-p^3p^{-z}} \tag{53}$$

Eventually we will put $p^{-z} = \tau$ define the local intertwining operator

$$T^{\mathrm{loc}}_{P_1}(\mathrm{St}(1,p^{-2}\tau)) = \frac{L(\mathrm{St}(1,p^{-2}),z+3)}{L(\mathrm{St}(1,p^{-2}),z+2)}T^{\mathrm{st}}_{P_1}(\mathrm{St}(1,p^{-2}\tau)) \tag{54}$$

and now our formula above becomes

$$T^{\mathrm{loc}}_{P_1}(\mathrm{St}(1,p^{-2}\tau))(h_p^{(0)}) = \frac{(1+p^2)^2}{(-1+p)(1+p+p^2)^2}(\tau-1)(h_p^{(0)}) + O((\tau-1)^2). \tag{55}$$

We abbreviate

$$c(p) := \frac{(1+p^2)^2}{(-1+p)(1+p+p^2)^2} \tag{56}$$

Notice that we have written $c(p)$ as a product of almost coprime factors.

This is the end of the interlude, we return to our study of the Eisenstein cohomology and the image $\mathrm{Im}(r)$.

1.4 Eisenstein cohomology

This image splits accordingly to (27)

$$\mathrm{Im}(r) = \mathrm{Im}_!(r) \oplus \mathrm{Im}_{\mathrm{Eis}}(r) \tag{57}$$

At the Oberwolfach meeting I distributed some handwritten notes where I claimed

$$\mathrm{Im}_!(r) \subset \bigoplus_{P \in \{P_1, P_2\}} \bigoplus_{\xi \in W^{P_2}/W_P} \bigoplus_{w \in W^P : l(w) \geq 2} H^1_!(\Gamma_{M_{P\xi}} \backslash X^{M_{P\xi}}, \widetilde{\mathbb{Z}(w \cdot 0)}) \tag{58}$$

and the left hand side is of finite index in the right hand side.

THIS IS WRONG!

But it is only wrong in the case where we have $P = P_1$ and $l(w) = 2$ i.e. $w = s_2 s_1$.

To formulate the correct result we concentrate on this case, we return to our formulas (21) (27) and tensor by $\mathbb{Q}$. We start from our parabolic subgroup $P = P_1$ and the element $w = s_2 s_1 \in W^{P_1}$ and consider (see 1.3.1 , iii))

$$H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p) \backslash \mathbb{H}_2), \mathbb{Q}) = \bigoplus_{\xi \in W^{P_2}/W_{P_1}} H^1_!(\Gamma_{M_{P\xi}} \backslash X^{M_{P\xi}}, \widetilde{\mathbb{Q}(w \cdot 0)}), \tag{59}$$

this is a direct summand and a semi simple module under the action of the Hecke algebra. If we tensor by a suitable finite extension $F/\mathbb{Q}$ we get a decomposition into absolutely irreducible modules

$$H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p) \backslash \mathbb{H}_2), F) = \bigoplus_{\sigma_f} \bigoplus_{\xi} H^1_!(\Gamma_{M_{P\xi}} \backslash X^{M_{P\xi}}, \widetilde{F(w \cdot 0)})(\sigma_f) \tag{60}$$

where here σ_f is the isomorphism type of an irreducible Hecke module for the Hecke algebra $\mathcal{H}^{M_{P\xi}}_{\mathcal{I}^M}$ of $M_{P\xi}$. This Hecke algebra contains the Hecke algebra $\mathcal{H}_{\mathcal{I}} = \mathcal{H}_{p,\mathcal{I}} \times \mathcal{H}^{(p)}$ as a subalgebra, hence each summand $H^1_!(\Gamma_{M_{P\xi}} \backslash X^{M_{P\xi}}, \widetilde{\mathbb{Q}(w \cdot 0)})(\sigma_f)$ is a $\mathcal{H}_{\mathcal{I}}$ module. Moreover we know that each summand

$$H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p) \backslash \mathbb{H}_2), F)(\sigma_f) = \bigoplus_{\xi} H^1_!(\Gamma_{M_{P\xi}} \backslash X^{M_{P\xi}}, \widetilde{F(w \cdot 0)})(\sigma_f) \tag{61}$$

is an irreducible $\mathcal{H}_{\mathcal{I}}$ module, i.e. comes with multiplicity one (see [13], [14]).

We may choose our splitting field $F/\mathbb{Q}$ to be a subfield of $\mathbb{C}$, then there is a minimal splitting field, it is a normal extension of $\mathbb{Q}$. The Galois group $\mathrm{Gal}(F/\mathbb{Q})$ acts on the set of σ_f which occur, because the cohomology is defined over $\mathbb{Q}$. We denote this action by $\sigma_f \mapsto \tau\sigma_f$. We have a τ-semi linear isomorphism

$$\Psi_\tau : H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p) \backslash \mathbb{H}_2), F)(\sigma_f) \xrightarrow{\sim} H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p) \backslash \mathbb{H}_2), F)(\tau\sigma_f) \tag{62}$$

which is induced by the action of the Galois group on the coefficient system. Then we define the field $\mathbb{Q}(\sigma_f) \subset F$ by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\sigma_f)) = \{\tau | \tau\sigma_f = \sigma_f\}$ We consider the orbit of $[\sigma_f] = \{\ldots, \tau\sigma_f, \ldots\}$

and let $\mathbb{Q}([\sigma_f])$ be the normal closure of $\mathbb{Q}(\sigma_f)$ in F. The direct sum over the isotypical spaces in the orbit is defined over $\mathbb{Q}$

$$H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Q})[\sigma_f]\otimes F=\bigoplus_{\tau} H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),F)(\tau\sigma_f) \tag{63}$$

We look at the corresponding contribution for the element $w'=s_2$. We get

$$H^{1+l(w')}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),F)=\bigoplus_{\sigma'_f}\bigoplus_{\xi} H^1_!(\Gamma_{M_{P^\xi}}\backslash X^{M_{P^\xi}},\widetilde{F(w'\cdot 0)})(\sigma'_f) \tag{64}$$

and again we know that $H^{1+l(w')}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),F)(\sigma'_f)$ is isotypical.

Now it follows from known facts in representation theory of ℓ-adic groups that the two $\mathcal{H}_{\mathcal{I}}$ modules

$$H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),F)(\sigma_f) \text{ and } H^{1+l(w')}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),F)(\sigma'_f)$$

are isomorphic if and only if $\sigma'_f=\sigma_f^\vee\otimes|\gamma_2|^3$. Here $\sigma_f^\vee$ is the dual module of σ_f in our special situation $\sigma_f^\vee=\sigma_f\otimes|\gamma_2|^2$ (see [4]).

Our general principle tell us that the image $\mathrm{Im}_!(r)\otimes F$ is compatible with this decomposition into isotypicals and hence our problem to compute the individual terms

$$\begin{gathered}\mathrm{Im}_!(r)(\sigma_f)\otimes F\subset\\ H^{1+l(w')}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),F)(\sigma_f^\vee\otimes|\gamma_2|^3)\oplus H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Z})(\sigma_f).\end{gathered} \tag{65}$$

Now the general expectation is that the image should be just the second summand, but this is not always the case.

We know that σ_f is the finite part of an automorphic representation

$$V_\sigma=\mathcal{D}_{k,\nu}\otimes V_{\sigma_f}\subset\mathcal{A}_{\mathrm{cusp}}(M_1(\mathbb{Q})\backslash M_1(\mathbb{A})).$$

Here $\mathcal{D}_{k,\nu}$ is a discrete series representation of $M_1(\mathbb{R})$. Recall that $\mathbb{C}w\cdot 0$ is a finite dimensional representation of M_1 of a highest weight from the list (25), and $k-2$ is the coefficient of $\bar\gamma_1$, ν is the coefficient of γ_2, hence $k=4,\ \nu=-2$. The parameter 2ν is the central character of the representation, only the parity of ν plays a role.

Let $\mathfrak{m}$ be the Lie algebra of M_1, then

$$\begin{gathered}H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),F)(\sigma_f)\otimes_F\mathbb{C}=\\ H^1(\mathfrak{m},K_\infty^{M_1},\mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})}\mathcal{D}_{k,\nu}\otimes\mathbb{C}(w\cdot 0))\otimes\mathrm{Ind}_{P_1(\mathbb{A}_f)}^{G(\mathbb{A}_f)}V_{\sigma_f}.\end{gathered} \tag{66}$$

Then it is known that $H^1(\mathfrak{m},K_\infty^{M_1},\mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})}\mathcal{D}_{k,\nu}\otimes\mathbb{C}(w\cdot 0))$ is one dimensional. (See for instance [10], Chapter 4., the group $K_\infty^{M_1}$ is not connected. See also (26).)

We apply Langlands theory of Eisenstein series and perform the procedure which is outlined in [10], 9.3. We have the Eisenstein intertwining operator, it maps $h \in (\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} V_\sigma \otimes |\gamma_P|^z)$ to

$$\mathrm{Eis(z)} : \{\underline{g} \mapsto h(\underline{g})\} \mapsto \{\underline{g} \mapsto \sum_{\gamma \in P_1(\mathbb{Q})\backslash G(\mathbb{Q})} h(\gamma \underline{g})\}, \tag{67}$$

this infinite sum converges for $\Re(z) >> 0$, and defines a holomorphic function in a suitable half space. It has meromorphic continuation into the entire complex plane.

We want to evaluate at $z = 0$, hence we ask whether the Eisenstein intertwining operator is holomorphic at $z = 0$. This depends on the constant term (see [10] Chapter 9). For $h \in (\mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})} V_\sigma \otimes |\gamma_P|^z)^{K_{0,f}}$

$$\begin{gathered} \mathcal{F}^P \circ \mathrm{Eis}(z)(h) = h + \mathcal{L}(\sigma, z) T^{\mathrm{loc}}(z)(h) \in \\ \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} V_\sigma \otimes |\gamma_P|^z \oplus \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} V_{\sigma^\vee} \otimes |\gamma_P|^{3-z} \end{gathered} \tag{68}$$

We remember that $\sigma^\vee \otimes |\gamma_2|^{3-z} = \sigma \otimes |\gamma_2|^{1-z}$. Here the local intertwining operator $T^{\mathrm{loc}}(z)$ is a restricted tensor product of local operators

$$T_v^{\mathrm{loc}}(z) : \mathrm{Ind}_{P_i(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)} V_{\sigma_v} \otimes |\gamma_P|_v^z) \to \mathrm{Ind}_{P_i(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)} V_{\sigma_v} \otimes |\gamma_P|_v^{1-z}) \tag{69}$$

the tensor product is taken over all places:

At the unramified finite places $v = \ell$ the local operator defined by sending the spherical function to the spherical function. It does not depend on z.

At the (ramified) place $\ell = p$ the representation σ_p must be a Steinberg module because there are no unramified cusp forms of weight 2 or 4. In this case $\sigma_p = \mathrm{St}(1, p^{-2})$ and we defined the local operator by the formula (50). It depends on z and it is holomorphic at $z = 0$ (this corresponds to $\tau = 1$).

At the place $v = \infty$ we also have a canonical choice of T_∞^{loc}, we require that induces the identity on a certain specific K_∞ type.

In the product of the local operators the only local operator depending on z is $T_p^{\mathrm{loc}}(z)$ and this operator is holomorphic at $z = 0$.

We come to the factor $\mathcal{L}(\sigma, z)$. In [10] Chapter 7 , 7.1.2 we attach cohomological L-functions $L^{\mathrm{coh}}(\pi_f, r, z)$ to an irreducible Hecke module H_{π_f} which occurs in the cohomology of an arithmetic group, the datum r is an (irreducible) representation of the dual group. These L-functions are Euler products of local L-functions, i.e.

$$L^{\mathrm{coh}}(\sigma_f, r, z) = \prod_\ell L_\ell^{\mathrm{coh}}(\sigma_f, r, z)$$

We apply this to our σ_f and put

$$L(\sigma_f, z) := L^{\mathrm{coh}}(\sigma_f, r_1, z) \tag{70}$$

here r_1 is the tautological representation. In this special case the Eichler-Shimura isomorphism yields a modular cusp form F of weight 4 to σ_f. This modular form has has a q expansion

$$F(z) = e^{2\pi i\tau} + a_2 e^{2\pi i 2\tau} + a_3 e^{2\pi i 3\tau} \dots \tag{71}$$

and then the above L-function

$$L(\sigma_f,z):=1+\frac{a_2}{2^z}+\frac{a_3}{3^z}\cdots=\frac{1}{1-p^{3-2s}}\prod_{\ell:\ell\neq p}\frac{1}{1-a_\ell\ell^{-z}+\ell^{3-2z}} \tag{72}$$

is the classical Hecke L-function defined by Hecke.

We also define the cohomological Euler factor at infinity, it is given by $L_\infty^{\mathrm{coh}}(\sigma_\infty,z)=\frac{\Gamma(z)}{(2\pi)^z}$ and then the complete cohomological L-function will be

$$\Lambda(\sigma,z)=L_\infty(\sigma_\infty,z)\cdot L(\sigma_f,z). \tag{73}$$

and find

$$\mathcal{L}(\sigma,z)=\frac{\Lambda(\sigma,z+2)}{\Lambda(\sigma,z+3)}\frac{\zeta(2z+1)}{\zeta(2z+2)} \tag{74}$$

Hence we see that the constant term is holomorphic at $z=0$ if the factor $\mathcal{L}(\sigma,z)$ is holomorphic at $z=0$. The argument $z_0=2$ is the central point for the functional equation of $\Lambda^{\mathrm{coh}}(\sigma,z)$ and we know that $\Lambda(\sigma,z)$ is holomorphic at $z_0=2$. On the other hand we know that $\zeta(2z+1)$ has a first order pole at $z=0$ and hence we see that

$$\mathcal{L}(\sigma,z) \text{ has a first order pole at } z=0 \iff \Lambda(\sigma,2)\neq 0. \tag{75}$$

We recall that the cohomological L function satisfies a functional equation

$$\Lambda(\sigma,z)=p^{z-2}\varepsilon(\sigma)\cdot\Lambda(\sigma,4-z) \tag{76}$$

where the root number $\varepsilon(\sigma)=\pm1$, the point $z=2$ is the central point in this functional equation. Hence we that necessarily $\Lambda(\sigma,2)=0$ if the root number is $\varepsilon(\sigma)=-1$. If the root number is $+1$ we still may have $\Lambda(\sigma,2)=0$, the zero is a second order zero.

Theorem 1.1. We assume that we are in the exceptional case $P=P_1$ and $l(w)=2$. If we have $\Lambda^{\mathrm{coh}}(\sigma,2)=0$ then

$$\mathrm{Im}_!(r)(\sigma_f)\otimes F=H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),F)(\sigma_f)$$

The situation is different if we are in the case $\Lambda(\sigma,2)\neq 0$.

The cohomology $H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Z})(\sigma_f)$ is a module under the Hecke algebra $\mathcal{H}=\otimes_{\ell\neq p}\mathcal{H}(G(\mathbb{Q}_\ell))//G(\mathbb{Z}_\ell)\otimes\mathcal{H}_{\mathcal{I}_2}$. More precisely we can say that $H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Z})(\sigma_f)$ is two dimensional F-vector space, the central part $\otimes_{\ell\neq p}\mathcal{H}(G(\mathbb{Q}_\ell)//G(\mathbb{Z}_\ell)$ acts by a homomorphism

$$h/\sigma_f):\otimes_{\ell\neq p}\mathcal{H}(G(\mathbb{Q}_\ell)//G(\mathbb{Z}_\ell))\to\mathcal{O}_F$$

and on this two dimensional space we have an action of $\mathcal{H}_{\mathcal{I}_2}$. This is the module $\mathrm{St}(1,p^{-2})^{\mathcal{I}_2}$.

At the same time we have the module $H^{1+l(w')}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2))(\sigma_f\otimes|\gamma_2|_f)$. These two modules are isomorphic as modules for $\otimes_{\ell\neq p}\mathcal{H}(G(\mathbb{Q}_\ell)//G(\mathbb{Z}_\ell))$, i.e. $h/\sigma_f)=h/\sigma'_f)$.

Theorem 1.2. We assume again that we are in the exceptional case $P = P_1, w = s_2 s_1, w' = s_2$. For a σ_f which occurs in $H^{1+l(w)}(\partial_{P_1})$ we consider the map from the inverse image

$$r(\sigma_f) : (H^2(\Gamma_0(p)\backslash\mathbb{H}_2, F) \oplus H^3(\Gamma_0(p)\backslash\mathbb{H}_2, F)[\sigma_f] \to \\ H^{1+l(w')}(\partial_{P_1})(\sigma_f \otimes |\gamma_2|_f)) \oplus H^{1+l(w)}(\partial_{P_1})(\sigma_f) \tag{77}$$

If $\Lambda(\sigma, 2) \neq 0$ then

$$\mathrm{Im}(r[\sigma_f]) = \mathrm{Im}(T^{\mathrm{loc}}_{P_1}(\mathrm{St}(1, p^{-2}))) \oplus \ker(T^{\mathrm{loc}}_{P_1}(\mathrm{St}(1, p^{-2}))) \tag{78}$$

Proof. The Lie -algebra cohomology $H^{1+l(w)}(\mathfrak{m}, K_\infty^{M_1}, \mathcal{D}_{k,\nu} \otimes \mathbb{C}(w \cdot 0)))$ is one dimensional and in [10] we explain how to choose a canonical generator $\omega_{\varepsilon(\nu)}$ in this vector space. Here $\varepsilon(\nu) = \pm 1$. This generator provides an identification

$$h \mapsto [\omega_+ \otimes h_f]; \ \ \mathrm{Ind}_{P_1(\mathbb{A}_f)}^{G(\mathbb{A}_f)} V_{\sigma_f} \xrightarrow{\sim} H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2), \mathbb{Z})(\sigma_f) \otimes \mathbb{C} \tag{79}$$

We have to understand the behavior $\mathrm{Eis}(z)(\omega_+ \otimes h_f \otimes |\gamma_2|^z)$ at $z = 0$ and hence we have to look at the constant term

$$\mathcal{F}^{P_i} \circ \mathrm{Eis}(z)(\omega_+ \otimes h_f) = \omega_+ \otimes h + \mathcal{L}(\sigma, z) T^{\mathrm{loc}}(z)(\omega_+ \otimes h_f) \tag{80}$$

and evaluate at $z = 0$. We are in the case where $\mathcal{L}(\sigma, z)$ has a first order pole at $z = 0$. The local intertwining operator is a product of local intertwining operators at all places and only the factor $T_p(z)$ depends on z and is holomorphic at $z = 0$.

Now $h_f = h_p \otimes \prod_{\ell \neq p} h_\ell$ where the h_ℓ are the spherical function and if $T_p(0)(h_p) \neq 0$ then the expression in (80) has a first order pole at $z = 0$ and we can take the residue

$$\mathrm{Res}_{z=0} \mathcal{F}^{P_i} \circ \mathrm{Eis}(z)(\omega \otimes h_f) = \mathrm{Res}_{z=0}(\mathcal{L}(\sigma, z)) T^{\mathrm{loc}}(0)(\omega \otimes h_f). \tag{81}$$

Before we continue we say a few words about the intertwining operator

$$T^{\mathrm{loc}}_\infty(z) : \mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})} V_{\sigma_\infty} \otimes |\gamma_2|_\infty^z \to \mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})} V_{\sigma_\infty} \otimes |\gamma_2|_\infty^{1-z}$$

This local operator does not depend on z, we evaluate at $z = 0$. We study the effect of the intertwining operator on the Lie-algebra complexes

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})} V_{\sigma_\infty}) \to \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})} V_{\sigma_\infty} \otimes |\gamma_2|_\infty)$$

We apply the formula of Delorme which computes the cohomology of the two above complexes and finds that

$$H^2(\mathfrak{g}, K_\infty, \mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})} V_{\sigma_\infty})) = 0 \text{ and } \dim H^3(\mathfrak{g}, K_\infty, \mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})} V_{\sigma_\infty})) = 1, \tag{82}$$

the generating cohomology class in degree 3 is represented by our form ω_+. There is an element $\omega_2 \in \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}, \mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})} V_{\sigma_\infty}))$ which is not closed, i. e. $d\omega_2 = \psi \neq 0$ but such that ψ is in the kernel of $T^{\mathrm{loc}}(0)$. Hence we get a closed form $T^{\mathrm{loc}}(0)(\omega_2) = \omega_- \in H^2(\mathfrak{g}, K_\infty, \mathrm{Ind}_{P_1(\mathbb{R})}^{G(\mathbb{R})} V_{\sigma_\infty})) \otimes |\gamma_2|)$

We claim

$$H^2(\mathfrak{g}, K_\infty, V_{\sigma_\infty} \otimes |\gamma_2|) \text{ is one dimensional and generated by } [\omega_-].$$

Hence we see that taking the residue (assumptions as above)

$$\mathrm{Res}_{z=0}\mathrm{Eis}(z)(\omega_2 \otimes h_f) \tag{83}$$

provides a non zero cohomology class in $H^{1+l(w')}(\sigma_f \otimes |\gamma_2|)$ where now $l(w') = 1$.

But if we take as local component $h_p = h_p^{(0)}$ then (52) yields

$$T_p^{\mathrm{loc}}(h_p^{(0)}) = c(p)(p^{-z} - 1)h_p^{(0)} + O(z^2) \tag{84}$$

and hence we see that $\mathcal{L}(\sigma, z)T^{\mathrm{loc}}(z)(h_p^{(0)})$ is holomorphic at $z = 0$ and we see that

$$\mathrm{Eis}(z)(\omega_+ \otimes h_f)$$

is holomorphic at $z = 0$ and provides a non trivial cohomology class whose restriction to the boundary gives us back the class $[\omega_+ \otimes h_f]$. Q.E.D.

We summarize:

Theorem 1.3. *In the exact sequence*

$$0 \to H_!^\bullet(\Gamma\backslash X, F) \to H^\bullet(\Gamma\backslash X, F) \to \mathrm{Im}^\bullet(r) \to 0$$

we have complete understanding of $\mathrm{Im}^\bullet(r)$ *(as module under the Hecke algebra) in terms of spaces of cusp forms of level p and of weight 2, 4.*

a) We have have a splitting

$$\mathrm{Im}^\bullet(r) = \mathrm{Im}_!^\bullet(r) \oplus \mathrm{Im}_{\mathrm{Eis}}^\bullet(r) \text{ and } \mathrm{Im}_!^\bullet(r) = \mathrm{Im}_{!,P_1}^\bullet(r) \oplus \mathrm{Im}_{!,P_2}^\bullet(r)$$

b) For the parabolic P_2 we have only one connected component ξ for which $\Gamma_{M_{P\xi}} \neq \mathrm{Gl}_2(\mathbb{Z})$. *Hence we get*

$$\mathrm{Im}_{!,P_2}^\bullet(r) = \mathrm{Im}_{!,P_2}^4(r) = 2 \text{ copies of the space of cusp forms of weight two for } \Gamma_{M_{P\xi}}.$$

(See (22))

c) The case of the parabolic group P_1 is the most interesting case.

c1) Here we have in degree 4

$$\mathrm{Im}_{!,P_1}^4(r) = 2 \text{ copies of the space of cusp forms of weight two for } \Gamma_{M_{P\xi}}.$$

(For each of the two components we get one copy of the space of cusp forms)

c2) In degree $\mathrm{q}=2$ *and degree* $\mathrm{q}=3$ *we have to look at the decomposition into eigenspaces*

$$\mathrm{Im}^{\bullet}_{!,P_1}(r)=\bigoplus_{\sigma_f}\mathrm{Im}^{\bullet}_{!,P_1}(r)(\sigma_f)\subset H^{1+l(w')}(\partial_{P_1})(\sigma_f\otimes|\gamma_2|_f^2))\oplus H^{1+l(w)}(\partial_{P_1})(\sigma_f)$$

If $\Lambda(\sigma,2)\neq 0$ *then we get one copy of the space of modular cusp forms of weight 4 for* $\Gamma_{M_{P\xi}}$ *in degree 2 and one copy in degree 3.*

If $\Lambda(\sigma,2)=0$ *we get two copies of the space of modular cusp forms of weight 4 for* $\Gamma_{M_{P\xi}}$ *in degree 3 and nothing in degree 2.*

d) For the Borel subgroup B we refer the computation of the Eisenstein part of the cohomology of the boundary (see (28)). We get

$$\mathrm{Im}^0_{\mathrm{Eis}}(r)=H^0_{\mathrm{Eis}}(\partial(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Q})=\mathbb{Q},\mathrm{Im}^2_{\mathrm{Eis}}(r)=H^2_{\mathrm{Eis}}(\partial(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Q})=\mathbb{Q}$$

$$\mathrm{Im}^4_{\mathrm{Eis}}(r)=\{(x_1,x_2,x_3,x_4)\in H^4_{\mathrm{Eis}}(\partial(\Gamma_0(p)\backslash\mathbb{H}_2),\mathbb{Q})=\mathbb{Q}^4\mid\sum x_i=0\}=\mathbb{Q}^3$$

On all these spaces the Hecke operator T_ℓ *acts by the eigenvalue* $\ell^3+\ell^2+\ell+1$.

1.5 What can the computer do for us?

In principle we want to extend the computations in [10] Chapter 3 section 3.3.5 to this situation here. In the following our coefficient system will be the ring $\mathbb{Z}_S$, where S is a controlled finite set of primes. At the beginning this set is empty. We have the exact sequence

$$0\to H^{\mathrm{q}}_!(\Gamma_0(p)\backslash\mathbb{H}_2,\mathbb{Z})\xrightarrow{j}H^{\mathrm{q}}(\Gamma_0(p)\backslash\mathbb{H}_2,\mathbb{Z})\xrightarrow{r}H^{\mathrm{q}}(\partial(\Gamma\backslash\mathbb{H}_2),\mathbb{Z})\to H^{\mathrm{q}+1}_c(\Gamma\backslash\mathbb{H}_2,\mathbb{Z})\qquad(85)$$

The first challenge is to compute the modules in this exact sequence. We have control of the torsion and we include all the primes which occur in the torsion into our set S. We are interested in the degree $\mathrm{q}=3$ and get the exact sequence

$$0\to H^3_!(\Gamma_0(p)\backslash\mathbb{H}_2,\mathbb{Z}_S)\xrightarrow{j}H^3(\Gamma_0(p)\backslash\mathbb{H}_2,\mathbb{Z}_S)\xrightarrow{r}\mathrm{Im}^3(r)\to 0$$

and we get free modules

$$H^3_!(\Gamma_0(p)\backslash\mathbb{H}_2,\mathbb{Z}_S)=\mathbb{Z}_Sx_1\oplus\mathbb{Z}_Sx_2\oplus\cdots\oplus\mathbb{Z}_Sx_s$$

$$\mathrm{Im}^3(r)=\mathbb{Z}_Sy_1\oplus\mathbb{Z}_Sy_2\oplus\cdots\oplus\mathbb{Z}_Sy_t$$

$$H^3(\Gamma_0(p)\backslash\mathbb{H}_2,\mathbb{Z}_S)=H^3_!(\Gamma_0(p)\backslash\mathbb{H}_2,\mathbb{Z}_S)\oplus\mathbb{Z}_S\tilde{y}_1\oplus\mathbb{Z}_S\tilde{y}_2\oplus\cdots\oplus\mathbb{Z}_S\tilde{y}_t$$

such that $r(x_i)=0$ and $r(\tilde{y}_j)=y_j$.

Now comes the hardest part, we have to compute an explicit expression for a Hecke operator (See [10], section 6.3.2)

$$\begin{gathered} T^{\mathrm{coh},0}_{\ell,\chi}(x_i) = \sum a_{i,j} x_j \\ T^{\mathrm{coh},0}_{\ell,\chi}(\tilde{y}_\nu) = \sum_\mu b_{\nu,\mu} x_\mu + \sum_j b_{\nu,j} \tilde{y}_j \end{gathered} \tag{86}$$

here $\chi : \mathbb{G}_m \to \mathcal{T}$ is a cocharacter and ℓ a prime. For the following it may be sufficient to take $\ell = 2$ and for χ one of the two fundamental cocharacters. The highest weight is $\lambda = 0$.

We pass to a splitting field $F/\mathbb{Q}$ let $\mathcal{O}_{F,S}$ its ring of S integers. We pick a component σ_f, with $H^{1+l(w)}(\partial_{[P_1]}(\Gamma_0(p)\backslash\mathbb{H}_2), F)(\sigma_f) \neq 0$, we assume that $\Lambda(\sigma, 2) \neq 0$. We consider the exact sequence

$$0 \to H^3_!(\Gamma_0(p)\backslash\mathbb{H}_2, \mathcal{O}_{F,S}) \to H^3(\Gamma_0(p)\backslash\mathbb{H}_2, \mathcal{O}_{F,S})[\sigma_f] \to \mathrm{Im}(r)^3(\sigma_f) \to 0, \tag{87}$$

here $\mathrm{Im}(r)^3(\sigma_f) = \mathrm{Im}(r)^3 \cap H^3(\partial(\Gamma\backslash\mathbb{H}_2), \mathcal{O}_{F,S})$ then $\mathrm{Im}^3(r)(\sigma_f)$ is a free rank one $\mathcal{O}_{F,S}$ module.

Under these assumptions the Manin-Drinfeld principle holds, i.e. we have

$$\mathrm{Hom}_{\mathcal{H}}(\mathrm{Im}^3(\sigma_f), H^3_!(\Gamma_0(p)\backslash\mathbb{H}_2, F)) = 0.$$

Remark: In the letter to Goresky and McPherson in [8] I made some very speculative calculations which give some support to this assumption. These calculation were based on the topological trace formula and on the fundamental lemma. These calculations also implied that the Manin-Drinfeld principle can fail if $\Lambda(\sigma, 2) = 0$. I will come back to this later. These facts have also been confirmed in a personal conversation with Jim Arthur in Oberwolfach and follow from his general results.

End remark

We get a decomposition

$$H^3(\Gamma_0(p)\backslash\mathbb{H}_2, F)[\sigma_f] = H^3_!(\Gamma_0(p)\backslash\mathbb{H}_2, F) \oplus \mathrm{Im}^3(r)(\sigma_f) \otimes F \tag{88}$$

Now we consider the cohomology with coefficients in $\mathcal{O}_{F,S}$ The decomposition induces a decomposition *up to isogeny*

$$H^3(\Gamma_0(p)\backslash\mathbb{H}_2, \mathcal{O}_{F,S})[\sigma_f] \supset H^3_!(\Gamma_0(p)\backslash\mathbb{H}_2, \mathcal{O}_{F,S}) \oplus \widetilde{\mathrm{Im}^3(r)}(\sigma_f) \tag{89}$$

where $\widetilde{\mathrm{Im}^3(r)}(\sigma_f) = \mathrm{Im}^3(r)(\sigma_f) \cap H^3(\Gamma_0(p)\backslash\mathbb{H}_2, \mathcal{O}_{F,S})$.

1.5.1 The main question

What can we say about the structure of the finite Hecke-module

$$H^3(\Gamma_0(p)\backslash\mathbb{H}_2, \mathcal{O}_{F,S})[\sigma_f](H^3_!(\Gamma_0(p)\backslash\mathbb{H}_2, \mathcal{O}_{F,S})[\sigma_f] \oplus \widetilde{\mathrm{Im}^3(r)}(\sigma_f)) = \mathrm{Im}^3(r)_{\mathrm{int}}(\sigma_f)/\widetilde{\mathrm{Im}^3(r)}(\sigma_f)? \tag{90}$$

The $\mathcal{O}_F$ module $\mathrm{Im}^3(r)_{\mathrm{int}}(\sigma_f)$ is locally free of rank one and therefore

$$\mathrm{Im}^3(r)(\sigma_f)/\widetilde{\mathrm{Im}^3(r)}(\sigma_f) = \mathcal{O}_F/(\mathfrak{n}(\sigma_f)) \tag{91}$$

where $(\mathfrak{n}(\sigma_f))$ is a non zero integral ideal. This ideal is *the denominator of the Eisenstein class* and this denominator can be computed in a given case once we have an effective program for the computation of the cohomology and the Hecke operator.

We may for instance assume that we find a σ_f which is defined over $\mathbb{Q}$ (There are a few of them in the modular forms data bank). Then we go back to (86) and we can assume that $\mathrm{Im}^3(r)(\sigma_f) = \mathbb{Z}_S y_1$ and hence the second equation in (86) says

$$T^{\mathrm{coh},0}_{\ell,\chi}(\tilde{y}_1) = \sum_\mu a_{\nu,\mu} x_\mu + h(\sigma_f)(T^{\mathrm{coh},0}_{\ell,\chi})(\tilde{y}_1) = x_T + h(\sigma_f)(T^{\mathrm{coh},0}_{\ell,\chi})(\tilde{y}_1).$$

We want to modify $\tilde{y}_1$ to $\tilde{y}_1 + x_0$ such that

$$T^{\mathrm{coh},0}_{\ell,\chi}(\tilde{y}_1 + x_0) = h(\sigma_f)(T^{\mathrm{coh},0}_{\ell,\chi})(\tilde{y}_1 + x_0))$$

then $\mathrm{Im}^3(r)(\sigma_f) \otimes \mathbb{Q} = \mathbb{Q}(\tilde{y}_1 + x_0)$.

For this we have to solve

$$(-T^{\mathrm{coh},0}_{\ell,\chi} + h(\sigma_f)(T^{\mathrm{coh},0}_{\ell,\chi})\mathrm{Id})x_0 = x_T. \tag{92}$$

Now the Manin-Drinfeld principle says that we can find a ℓ and a χ such that $-T^{\mathrm{coh},0}_{\ell,\chi}+h(\sigma_f(T^{\mathrm{coh},0}_{\ell,\chi})\mathrm{Id}$ induces an injection on $H^3_!(\Gamma_0(p)\backslash\mathbb{H}_2, \mathbb{Z}_S)$. This implies that we can solve (92) with $x_0 \in H^3_!(\Gamma_0(p)\backslash\mathbb{H}_2, \mathbb{Q})$, the denominator of this element is the denominator of the Eisenstein class.

Now it is shown in [13],[14] that there is an array of complex numbers

$$\{\dots, \Omega(\tau\sigma_f), \dots\}_{\tau\in\mathrm{Gal}(\mathbb{Q}([\sigma_f])/\mathbb{Q}(\sigma_f)))}$$

which is well defined up to a unit in $\mathcal{O}^\times_{F,S}$ such that

$$\frac{1}{\Omega(\tau\sigma_f)}\frac{\Lambda(\tau\sigma_f,2)}{\Lambda(\tau\sigma_f,3)} \in F$$

and this expression transforms under Galois in the right way, i.e,

$$\tau(\frac{1}{\Omega(\sigma_f)}\frac{\Lambda(\sigma_f,2)}{\Lambda(\sigma_f,3)}) = \frac{1}{\Omega(\tau\sigma_f)}\frac{\Lambda(\tau\sigma_f,2)}{\Lambda(\tau\sigma_f,3)}$$

This number has a denominator ideal $\mathfrak{Den}(\frac{1}{\Omega(\iota,\sigma_f)}\frac{\Lambda(\tau\sigma_f,2)}{\Lambda(\tau\sigma_f,3)})$ and my conjectural answer says that

$$\mathfrak{n}(\sigma_f) \simeq \mathfrak{Den}(\frac{c(p)}{\Omega(\sigma_f)}\frac{\Lambda(\sigma_f,2)}{\Lambda(\sigma_f,3)}) \tag{93}$$

where $\simeq$ means that we have to invert a small number of small primes.

We think that it of great interest to provide experimental data, which confirm (or falsify) this conjecture. We get these data for a given prime p and σ_f once we can tackle task A) and B) in the sequence (85).

1.6 The motivation

Finally I want to say a few words why I expect that this should be true. The following considerations are not very precise, we speculate about certain objects namely the mixed motives, for which we give some kind of a definition in [11]. Motives and mixed motives are also the topic in the two volumes [16]. In these two volumes many of the results and even definitions are conditional and depending on the truth of (accepted) conjectures.

We will see that certain versions of these well accepted conjectures imply half of the conjecture above: The right hand side divides the left hand side, we get an estimate of the denominator from below.

We pass to the smaller subgroup $\Gamma_{00}(p)$ (see (1.3.3) and consider the Borel-Serre compactification $\Gamma_{00}(p)\backslash\mathbb{H}_2 \xrightarrow{i} \overline{\Gamma_{00}(p)\backslash\mathbb{H}_2}$. The number of connected components in the different boundary strata becomes larger, we have 4 connected components for each of the two maximal maximal parabolic subgroups P_1, P_2 and 8 connected components for the Borel stratum. We are mainly interested in the connected components of $\partial_{P_1}(\Gamma_{00}(p)\backslash\mathbb{H}_2)$. These connected components correspond to the right action orbits of $\{e, s_1\}$ on the Weyl group W, we make a list

$$\{\xi_1,\xi_2,\xi_3,\xi_4\} := \{\{e,s_1\},\{s_2,s_2s_1\},\{s_1s_2,s_1s_2s_1\},\{s_2s_1s_2,s_2s_1s_2s_1\}\}$$

of these orbits. Let Y_1, Y_2, Y_3, Y_4 be the corresponding boundary strata. We add Y_3 and Y_4 to $\Gamma_{00}(p)\backslash\mathbb{H}_2$ and get an inclusion

$$i_0 : \Gamma_{00}(p)\backslash\mathbb{H}_2 \hookrightarrow \Gamma_{00}(p)\backslash\mathbb{H}_2 \cup Y_3 \cup Y_4,$$

the set on the right hand side is open in $\overline{\Gamma_{00}(p)\backslash\mathbb{H}_2}$. We extend the sheaf $\mathbb{Z}$ on $\Gamma_{00}(p)\backslash\mathbb{H}_2$ by zero to $\Gamma_{00}(p)\backslash\mathbb{H}_2 \cup Y_3 \cup Y_4$, i.e. we consider the sheaf $i_{0,!}(\mathbb{Z})$. Now we have the inclusion i_∞ : $\Gamma_{00}(p)\backslash\mathbb{H}_2 \cup Y_3 \cup Y_4 \hookrightarrow \overline{\Gamma_{00}(p)\backslash\mathbb{H}_2}$ and define the sheaf $\mathbb{Z}^\# := i_{\infty,*} \circ i_{0,!}(\mathbb{Z})$. We recollect that for the Borel-Serre compactification the direct image functor $i_{\infty,*}$ functor is exact.

We get an exact sequence of sheaves $0 \to \mathbb{Z}^\# \to i_*(\mathbb{Z}) \to i_*(\mathbb{Z})/\mathbb{Z}^\# \to 0$ and hence we get a diagram of cohomology groups

$$
\begin{array}{ccccccc}
 & H^3(\Gamma_{00}(p)\backslash\mathbb{H}_2, i_!(\mathbb{Z})) & & & & & \\
 & \searrow & & & & & \\
H^2(Y_3\cup Y_4,\mathbb{Z}) \to & H^3(\Gamma_{00}(p)\backslash\mathbb{H}_2,\mathbb{Z}^{\#}) & \to & H^3(\Gamma_{00}(p)\backslash\bar{\mathbb{H}}_2,\mathbb{Z}) & \to H^3(Y_3\cup Y_4,\mathbb{Z}) \to & & \\
 & & \searrow r_{12}^{\#} & \downarrow r_{12} & & & \\
 & & & H^3(Y_1\cup Y_2,\mathbb{Z}) & & & \\
 & & & \searrow & & & \\
 & & & & H^4(\Gamma_{00}(p)\backslash\mathbb{H}_2, i_!(\mathbb{Z})) & &
\end{array} \tag{94}
$$

We consider the tensor product of this diagram by $\mathbb{Q}$. On this diagram we have an action of the Hecke algebra $\mathcal{H}^{(p)}$. We recall that an irreducible isomorphism type σ_f, which occurs in the cohomology is defined by its restriction to $\mathcal{H}^{(p)}$. (Strong multiplicity one). We pick a σ_f, for simplicity we assume that σ_f is defined over $\mathbb{Q}$. Then we know that $H^3(Y_1\cup Y_2,\mathbb{Q}) = H^3(Y_1\cup Y_2,\mathbb{Q})(\sigma_f)\oplus H^{\perp}$, where the first summand is the σ_f isotypical component. Then we have seen that $H^3(Y_1\cup Y_2,\mathbb{Q})(\sigma_f) = \mathbb{Q}^2$ and we also know that the image

$$\mathrm{Im}(r_{12}^{\#}\otimes\mathbb{Q})(\sigma_f) = \mathrm{Im}(r_{12}\otimes\mathbb{Q})(\sigma_f) = \mathbb{Q}\subset\mathbb{Q}^2.$$

We may replace $\mathbb{Q}$ by a ring $\mathbb{Z}_S$. The set S contains those primes which yield inner congruences between σ_f and other isotypical subspaces. Then $H^3(Y_1\cup Y_2,\mathbb{Z}_S)(\sigma_f)$ will be a direct summand in the Hecke module $H^3(Y_1\cup Y_2,\mathbb{Z}_S)$. The set S should also contain those primes which divide the order of the torsion of $H^4_c(\Gamma_{00}(p)\backslash\mathbb{H}_2, i_!(\mathbb{Z}))$.

Then it is clear that $\mathrm{Im}(r_{12}^{\#}\otimes\mathbb{Z}_S) = \mathrm{Im}(r_{12})\otimes\mathbb{Z}_S = \mathbb{Z}_S$ is a direct summand in $H^3(Y_1\cup Y_2,\mathbb{Z}_S)(\sigma_f) = \mathbb{Z}_S^2$. Under our assumptions (91) $\mathfrak{n}(\sigma_f)\in H^3(Y_1\cup Y_2,\mathbb{Z}_S)$ can be lifted to an eigenclass $\mathrm{Eis}(\mathfrak{n}(\sigma_f))\in H^3(\Gamma_{00}(p)\backslash\mathbb{H}_2,\mathbb{Z}_S)$ and the inverse image of this eigenclass in $H^3(\Gamma_{00}(p)\backslash\mathbb{H}_2,\mathbb{Z}_S^{\#})$ provides a short exact sequence of Hecke modules

$$0\to\mathbb{Z}_S\to\mathcal{K}(\sigma_f)\to\mathbb{Z}_S\mathrm{Eis}(\mathfrak{n}(\sigma_f))\to 0 \tag{95}$$

where the $\mathbb{Z}_S$ on the left is $\mathrm{Im}(H^2(Y_3\cup Y_4,\mathbb{Z}_S)\to H^3(\Gamma_{00}(p)\backslash\mathbb{H}_2,\mathbb{Z}_S^{\#}))$.

The point is that we can interpret $\mathcal{K}(\sigma_f)$ as a mixed Tate motive, more precisely as a mixed Kummer-Tate motive.

Of course we believe that $\Gamma_{00}(p)\backslash\mathbb{H}_2$ is the set of complex points of a scheme $\mathcal{Y}(p)/\mathbb{Z}[\frac{1}{p}]$ and we consider the embedding into its toroidal compactification

$$i:\mathcal{Y}(p)/\mathbb{Z}[\frac{1}{p}]\hookrightarrow\mathcal{X}(p)/\mathbb{Z}[\frac{1}{p}]\hookleftarrow\mathcal{Y}_\infty(p)/\mathbb{Z}[\frac{1}{p}]:j_\infty \tag{96}$$

where $\mathcal{Y}_\infty(p)/\mathbb{Z}[\frac{1}{p}]$ is the complement of the open part $\mathcal{Y}(p)/\mathbb{Z}[\frac{1}{p}]$. It is a divisor with normal crossings. The scheme $\mathcal{Y}(p)/\mathbb{Z}[\frac{1}{p}]$ has a stratification, the strata are labelled by the connected components of the boundary strata $\partial_{P_1},\partial_{P_2},\partial_B$. We are mainly interested in the boundary stratum attached to P_1. It has four connected components $\mathcal{Y}_1,\ldots,\mathcal{Y}_4$. Again we define the sheaf $\mathbb{Z}^{\#}=$

$i_{*,\infty} \circ i_{0,!}(\mathbb{Z})$, but now we have to be careful, because $i_{*,\infty}$ is not exact, we have to take the derived direct image.

We view $(\mathcal{X}(p), \mathbb{Z}_S^{\#})$ as a mixed motive (whatever that means), in any case we can consider its Betti-realisation $H_B(\mathcal{X}(p), \mathbb{Z}_S^{\#}) = H^\bullet(\mathcal{X}(p)(\mathbb{C}), \mathbb{Z}_S^{\#})$, its de-Rham realisation $H_{de-Rh}(\mathcal{X}(p), \mathbb{Q}^{\#})$ and for all primes ℓ its etale realisations $H^\bullet(\mathcal{X}(p) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_\ell^{\#})$. Furthermore we have the comparison isomorphism

$$I_{B-dRh} : H_B(\mathcal{X}(p)(\mathbb{C}), \mathbb{C}^{\#}) \xrightarrow{\sim} H_{de-Rh}(\mathcal{X}(p), \mathbb{Q}^{\#}) \otimes \mathbb{C}),$$

(See [11]). The triple

$$(\text{Betti realisation, de-Rham realisation}, I_{B-dRh})) \tag{97}$$

will be called the Betti-de-Rham realisation.

We assumed that the set S contains all primes which divide the order the torsion of some $H_c^4(...)$, then we know that for $\ell \notin S$ and $m >> 0$

$$H^3(\mathcal{X}(p)(\mathbb{C}), \mathbb{Z}_S^{\#}) \otimes \mathbb{Z}/\ell^m\mathbb{Z}_S = H^3_{et}(\mathcal{X}(p) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}/\ell^m\mathbb{Z}), \tag{98}$$

the right hand side is a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-module. We can form the projective limit and get

$$H^3(\mathcal{X}(p)(\mathbb{C}), \mathbb{Z}_S^{\#}) \otimes \mathbb{Z}_\ell = H^3_{et}(\mathcal{X}(p) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_\ell) \tag{99}$$

and we get an action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the left hand side. This provides a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-module structure on $\mathcal{K}(\sigma_f) \otimes \mathbb{Z}_\ell$. The results of Pink (See[19]) say that we have an exact sequence of Galois-modules

$$0 \to \mathbb{Z}_\ell(-1) \to \mathcal{K}(\sigma_f) \otimes \mathbb{Z}_\ell \to \mathbb{Z}_\ell(-2) \to 0. \tag{100}$$

This Galois-module is unramified outside ℓ, p.

If we put all these realisations and the comparison isomorphisms into one object we get a mixed Kummer-Tate motive

$$\{0 \to \mathbb{Z}(-1) \to \mathcal{K}(\sigma_f) \to \mathbb{Z}(-2) \to 0\}, \tag{101}$$

which we we simply call $\mathcal{K}(\sigma_f)$. We get the extension class

$$[\mathcal{K}(\sigma_f)] \in \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Z}[1/p]}}(\mathbb{Z}(-2), \mathbb{Z}(-1)). \tag{102}$$

We consider the different realisations of this extension class (See [11], 1.7.1-1.7.2)

$$\begin{gathered} [\mathcal{K}(\sigma_f)]_{B-dRh} \in \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-2), \mathbb{Z}(-1)) = \mathbb{R} \\ [\mathcal{K}(\sigma_f)]_{et,\ell} \in \mathrm{Ext}^1_{et,\ell}(\mathbb{Z}_\ell(-2), \mathbb{Z}_\ell(-1)) = \mathbb{Q}_\ell^\times \end{gathered} \tag{103}$$

Now we are confronted with a fundamental problem:

Since $[\mathcal{K}(\sigma_f)]$ is obtained from a global object (a mixed motive over $\mathbb{Q}$) we should have some relationship between the extension classes in the different realisations.

The only conceivable solution to this problem I can think of is:

All the numbers $[\mathcal{K}(\sigma_f)]_{et,\ell}$ are equal to the same rational number

$$\mathcal{K}(\sigma_f)^\dagger \in \mathbb{Q}^\times$$

and moreover

$$e^{[\mathcal{K}(\sigma_f)]_{B-dRh}} = \mathcal{K}(\sigma_f)^\dagger$$

If this is not the case then we call $\mathcal{K}(\sigma_f)$ an *exotic* mixed Kummer-Tate motive.

I think it would be a disaster if such an object exist.

Now let us assume that $\mathcal{K}(\sigma_f)$ is not exotic. In [8], in [11] and [12] we developed a strategy to compute the Betti-de-Rham extension classes of mixed Anderson-Tate motives. Our $\mathcal{K}(\sigma_f)$ is an Anderson-Kummer -Tate motive. We apply this strategy to this case and get

$$[\mathcal{K}(\sigma_f)]_{B-dRh} = \mathfrak{n}(\sigma_f)\frac{c(p)}{\Omega(\sigma_f)}\frac{\Lambda(\sigma_f,2)}{\Lambda(\sigma_f,3)}\log(p). \tag{104}$$

The number $\Omega(\sigma_f)$ is a *relative period* it is well defined up to an element in $\mathbb{Z}_S^\times$. (See further down section 1.8). The $\log(p)$ comes from taking the limit $\lim_{z\to 0}(p^{-z}-1)\zeta(z+1)$ for $c(p)$ see (56). Since we assumed that σ_f is defined over $\mathbb{Q}$ the factor $\frac{1}{\Omega(\sigma_f)}\frac{\Lambda(\sigma_f,2)}{\Lambda(\sigma_f,3)}$ is a rational number. Then we get the formula

$$\mathcal{K}(\sigma_f)^\dagger = p^{\mathfrak{n}(\sigma_f)\frac{c(p)}{\Omega(\sigma_f)}\frac{\Lambda(\sigma_f,2)}{\Lambda(\sigma_f,3)}} \tag{105}$$

Since we know that the exponent must be an integer, we conclude that for a prime $\ell \notin S$ for which $\ell^m | \mathfrak{Den}\left(\frac{c(p)}{\Omega(\sigma_f)}\frac{\Lambda(\sigma_f,2)}{\Lambda(\sigma_f,3)}\right)$ we must have $\ell^m | \mathfrak{n}(\sigma_f)$, and we get half of (93).

Therefore we see:

If for a σ_f the assertion (93) turns out to be wrong then $\mathcal{K}(\sigma_f)$ must be exotic.

If σ_f is not defined over $\mathbb{Q}$ then basically the same reasoning works, but we have to argue with motives with coefficients.

We get a very precise estimate for the denominator, we have good control over the primes in S. We just remark that also the denominator of $c(p)$ should contribute to the denominator of the Eisenstein class. This would provide *denominators of local origin* whereas the denominators induced by the denominators in ratio of L-values are *denominators of global origin.* Of course it might be interesting to see what happens if we have a cancellation, i.e. if a prime ℓ occurs as well in the numerator of the ratio of L-values and in the denominator of $c(p)$.

Finally we remark that we have discussed a similar construction of Anderson-Kummer motives for certain congruence subgroups $\Gamma \subset \mathrm{Gl}_2(\mathbb{Z})$ in [11] Section 2. In this case we can show that the mixed Anderson-Kummer motives are not exotic and we get estimates for the denominators of the Eisenstein classes.

1.7 $\Lambda(\sigma, 2) = 0$

This section is even more speculative. Our speculation does not provide a result that can be verified or falsified by a computer. One might argue that this section does not belong into this article. On the other hand the Saito-Kurokawa lift - or the more general results of Arthur- predict some subtle assertions concerning the structure of the cohomology, which then may serve as a test for correctness of the algorithm.

Of course we can also ask what happens if $\Lambda(\sigma_f, 2) = 0$, We have seen that this must be the case if the root number $\varepsilon(\sigma_f) = -1$, but it may also happen in the other case. For the following we also refer to [8] Kapitel III, 3.1. and we hope to explain it in [11] in greater generality.

The Manin-Drinfeld fails if the root number $\varepsilon(\sigma_f) = -1$, In this case it follows from Arthur's work that $H^3_!((\Gamma_{00}(p)\backslash\mathbb{H}_2, \mathbb{Z})(\sigma_f) \neq 0$ and this is a module of rank two over the Hecke algebra $\mathcal{H}_{\mathcal{I}} \otimes \mathcal{H}^{(p)}$. (It is the so called Saito-Kurokawa lift). This tells us that the argument which lead us to (95) does not apply, the restriction $r_{12}^{\#}$ in the diagram((94) has a kernel $\mathcal{A}(\sigma_f) \subset H^3(\Gamma_{00}(p)\backslash\mathbb{H}_2, \mathbb{Z}^{\#})$ and we get a diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \to H^2(Y_3 \cup Y_4, \mathbb{Z}_S) & \to \mathcal{A}(\sigma_f) & & \to H^3_!((\Gamma_{00}(p)\backslash\mathbb{H}_2, \mathbb{Z}_S)(\sigma_f) & \to 0 & \\ & & \downarrow & & \downarrow & & \\ & & H^3(\Gamma_{00}(p)\backslash\mathbb{H}_2, \mathbb{Z}_S^{\#}) & & \to \mathcal{B}(\sigma_f) & & . \\ & & \downarrow & & \downarrow & & \\ & & H^3(Y_1 \cup Y_2, \mathbb{Z}_S) & & H^3(Y_1 \cup Y_2, \mathbb{Z}_S) & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \tag{106}$$

As before this is a diagram of Betti-cohomology groups of a mixed motive, The horizontal exact sequence and the vertical exact sequence on the right yield extension classes

$$\begin{aligned} [\mathcal{A}(\sigma_f)] &\in \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Z}[1/p]}}(H^3_!(\mathcal{X}(\sigma_f), \mathbb{Z}), \mathbb{Z}(-1)^2) \\ [\mathcal{B}(\sigma_f)] &\in \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Z}[1/p]}}(\mathbb{Z}(-2)^2, H^3_!(\mathcal{X}(\sigma_f), \mathbb{Z})) \end{aligned} \tag{107}$$

and we consider the entire diagram as an *enhanced biextension* of $H^3_!(\mathcal{X}, \mathbb{Z})(\sigma_f)$ by $\mathbb{Z}(-1)^2, \mathbb{Z}(-2)^2$, the enhancement is the introduction of $H^3(\Gamma_{00}(p)\backslash\mathbb{H}_2, \mathbb{Z}_S^{\#})$ in the middle.

If I understand correctly T. Scholl explains in [20] how to attach a bilinear pairing

$$< \, , \, >_{\mathcal{B}} \colon \mathbb{Z}(-1)^2 \times \mathbb{Z}(-2)^2 \to \mathbb{R} \tag{108}$$

to this enhanced biextension. (The subscript $_{\mathcal{B}}$ stands for Beilinson.)

We recall that

$$\begin{aligned} \mathbb{Z}_S(-2)^2 &= \bigoplus\nolimits_{\xi \in \{\xi_1, \xi_2\}} H^1_!(\Gamma_{M_{P\xi}} \backslash X^{M_{P\xi}}, \widetilde{\mathbb{Z}_S(w \cdot 0)})(\sigma_f) \\ \mathbb{Z}_S(-1)^2 &= \bigoplus\nolimits_{\xi \in \{\xi_3, \xi_4\}} H^1_!(\Gamma_{M_{P\xi}} \backslash X^{M_{P\xi}}, \widetilde{\mathbb{Z}_S(w' \cdot 0)})(\sigma_f \otimes |\gamma_2|^3) \end{aligned} \tag{109}$$

At first we define a local pairing $< , >_p$: $\mathbb{Z}_S(-2)^2 \times \mathbb{Z}_S(-1)^2 \to \mathbb{Q}$. To do this we refer to section 1.3.3. We have the identification

$$\begin{aligned} \bigoplus_{\xi\in\{\xi_1,\xi_2\}} H^1_!(\Gamma_{M_{P\xi}}\backslash X^{M_{P\xi}}, \widetilde{\mathbb{Z}_S(w\cdot 0)}) &= \mathbb{Z}_S(\delta_1 - \frac{1}{p}\delta_5) \oplus \mathbb{Z}_S(\delta_2 - \frac{1}{p}\delta_6) \\ \bigoplus_{\xi\in\{\xi_3,\xi_4\}} H^1_!(\Gamma_{M_{P\xi}}\backslash X^{M_{P\xi}}, \widetilde{\mathbb{Z}_S(w'\cdot 0)})(\sigma_f \otimes |\gamma_2|^3) &= \mathbb{Z}_S(\delta_3 - \frac{1}{p}\delta_7) \oplus \mathbb{Z}_S(\delta_4 - \frac{1}{p}\delta_8). \end{aligned} \tag{110}$$

On the second summand we define the obvious scalar product

$$< \delta_i - \frac{1}{p}\delta_{i+4}, \delta_j - \frac{1}{p}\delta_{j+4} >= \delta_{ij}.$$

Now we can define the above local pairing: If $h_p \in \mathbb{Z}_S(\delta_1 - \frac{1}{p}\delta_5) \oplus \mathbb{Z}_S(\delta_2 - \frac{1}{p}\delta_6) \subset (\mathrm{St}(1,p^{-2}))^{\mathcal{I}}$ then

$$T^{\mathrm{st}}_{P_1}(\mathrm{St}(1,\mathrm{p}^{-2})(h_p) \in \bigoplus_{i=1}^{4} \mathbb{Z}_S(\delta_i - \frac{1}{p}\delta_{i+4}).$$

We project to the second summand then and get an element

$$T^{\mathrm{st},34}_{P_1}(\mathrm{St}(1,p^{-2}))(h_p) \in \mathbb{Z}_S(\delta_3 - \frac{1}{p}\delta_7) \oplus \mathbb{Z}_S(\delta_4 - \frac{1}{p}\delta_8).$$

Now we define

$$< h_p, h'_p >_p := < T^{\mathrm{st},34}_{P_1}(\mathrm{St}(1,p^{-2})(h_p), h'_p > . \tag{111}$$

For the element $h_p^{(0)} = -p(\delta_1 - \frac{1}{p}\delta_5) + (\delta_2 - \frac{1}{p}\delta_6)$ our Mathematica computation yields $T^{\mathrm{st}}_{P_1}(\mathrm{St}(1,p^{-2})(h_p^{(0)}) = 0$ and therefore $h_p^{(0)}$ lies in the kernel of the local pairing. We consider the orthogonal complement V of $T^{\mathrm{st}}_{P_1}(\mathrm{St}(1,p^{-2})(\mathbb{Z})$, in $H^2(Y_3 \cup Y_4, \mathbb{Z})$, then our pairing becomes a pairing between the two rank one $\mathbb{Z}_S$ modules $< , >_p$: $H^2(Y_3 \cup Y_4, \mathbb{Z}_S)/V \times H^3(Y_3 \cup Y_4, \mathbb{Z}_S)/\mathbb{Z}_S h_p^{(0)} \to \mathbb{Q}$. The Mathematica code also gives that the module of values of the pairing is the fractional ideal

$$(\frac{1}{p^2(p+1)(p^2+p+1)}) \subset \mathbb{Z}.$$

Now I claim that we get a "Gross-Zagier" formula for the value of this pairing:

The value of the height pairing is given by

$$< h_p, h'_p >_{\mathcal{B}} = < h_p, h'_p >_p \frac{\Lambda'(\sigma_f, 2)}{\Omega(\sigma_f)\Lambda(\sigma_f, 3)} \tag{112}$$

I do not state this as a theorem because some computations still have to be checked.

Extensions of the type above have been constructed in [8], 3.1, 3.2 and there I do not say anything about triviality or non triviality of these extensions. I hope to discuss this issue more thoroughly in [11].

1.8 Computational aspects

If we want to verify (93) we need to compute the relative period $\Omega(\sigma_f)$ and the values $\Lambda(\sigma,\nu)$ for $\nu = 2,3$ up to a very high precision. Of course we are we are happy if we find an algebraic number $\alpha \in F$ which is not to big and which approximates the numerical value $\frac{1}{\Omega(\sigma_f)}\frac{\Lambda(\sigma_f,2)}{\Lambda(\sigma_f,3)})$ up a very high precision -very high compared to the "size" (height) of the number α, then we accept that $\alpha = \frac{1}{\Omega(\sigma_f)}\frac{\Lambda(\sigma_f,2)}{\Lambda(\sigma_f,3)}$.

The computation of these relative periods is a little bit delicate. When we discussed the cohomology of the Siegel stratum we introduced the group $\Gamma^{(1)}_{M_{P\xi}} \subset \mathrm{Sl}_2(\mathbb{Z})$ and claimed that each σ_f occurs twice in the cohomology (see (26))

$$H^1_!(\Gamma^{(1)}_{M_{P\xi}}\backslash X^M, \mathcal{O}_F w\cdot 0)(\sigma_f) = \\ H^1_{!,+}(\Gamma^{(1)}_{M_{P\xi}}\backslash X^M, \mathcal{O}_F w\cdot 0)(\sigma_f) \oplus H^1_{!,-}(\Gamma^{(1)}_{M_{P\xi}}\backslash X^M, \mathcal{O}_F w\cdot 0)(\sigma_f).$$

Both modules on the right hand side are locally free $\mathcal{O}_F-$ modules of rank one let us assume that they are actually free. Then we can find an isomorphism $T^{\text{arith}}(\sigma_f)$ between these two $\mathcal{O}_F$ modules which of course unique up to an element in $\mathcal{O}_F^\times$. On the other hand we can use the Eichler-Shimura isomorphism and construct a canonical isomorphism

$$T^{\text{trans}}(\sigma_f): H^1_{!,+}(\Gamma^{(1)}_{M_{P\xi}}\backslash X^M, \mathcal{O}_F w\cdot 0)(\sigma_f)\otimes_{F,\iota}\mathbb{C} \to \\ H^1_{!,-}(\Gamma^{(1)}_{M_{P\xi}}\backslash X^M, \mathcal{O}_F w\cdot 0)(\sigma_f)\otimes_{F,\iota}\mathbb{C}, \tag{113}$$

these two isomorphisms differ by the relative period

$$\frac{1}{\Omega(\sigma_f)}T^{\text{trans}}(\sigma_f) = T^{\text{arith}}(\sigma_f)\otimes_F \mathbb{C}. \tag{114}$$

Of course we have to do this for σ_f and its conjugates $\tau\sigma_f$ und the action of the Galois group. This is of course a rather theoretical definition, it is not clear how we can these numbers in practice.

We have good control over the set S, of course we should keep it as small as possible. The primes $\ell \in S$ are essentially determined by the torsion in the cohomology, we do not employ the Hecke operators to define S, The primes dividing $p-1$ should be in S.

We will explain in [10] Chapter 8 that for a given σ_f can define two periods $\Omega_+(\sigma_f), \Omega_-(\sigma_f) \in \mathbb{R}_{>0}$, which are well defined up to a unit in $\mathcal{O}_F^\times$ such that

$$\frac{\Lambda(\sigma_f,2)}{\Omega_+(\sigma_f)}, \frac{\Lambda(\sigma_f,3)}{\Omega_-(\sigma_f)} \in \mathcal{O}_F \tag{115}$$

then our relative period s given by

$$\Omega(\sigma_f) = \frac{\Omega_+(\sigma_f)}{\Omega_-(\sigma_f)}. \tag{116}$$

Of course we have to explain how these periods are defined. The following will be explained in in [10] Chapter 8. If we want to pin them down up to an element in $\mathcal{O}_F^\times$ we have to compute the values $\Lambda(\sigma_f\otimes\chi,\nu)$ for $\nu = 2,3$ and all characters $\chi : \mathbb{Z}/p\mathbb{Z}^\times \to \mathbb{C}^\times$. Here we use the fact that we have rather effective algorithms, which compute the values $\Lambda(\sigma_f \otimes \chi, 2), \Lambda(\sigma_f \otimes, 3)$ with a very high precision. It follows from the theorems of Manin and Shimura that viewed as points in projective space

$$\begin{gathered} \{\dots, \Lambda(\sigma_f \otimes \chi, 2), \dots\}_\chi \in \mathbb{P}(F[\zeta_{p-1}]) \\ \{\dots, \Lambda(\sigma_f \otimes \chi, 3), \dots\}_\chi \in \mathbb{P}(F[\zeta_{p-1}]) \end{gathered} \tag{117}$$

Then we choose the periods $\Omega_\pm(\sigma_f)$ such that the arrays

$$\{\dots, \frac{\Lambda(\sigma_f \otimes \chi, 2)}{\Omega_+(\sigma)}, \dots\}_\chi;\ \{\frac{\Lambda(\dots, \sigma_f \otimes \chi, 3)}{\Omega_-(\sigma)}, \dots\}_\chi \tag{118}$$

form arrays of coprime integers in $\mathcal{O}_F[\zeta_{p-1}]$. (We ignore some classnumber problems.)

Then we expect that for the primes $\ell \notin S$ which divide $\frac{\Lambda(\sigma_f,3)}{\Omega_-(\sigma_f)}$ it is a rare event that at the same time $\ell | \frac{\Lambda(\sigma_f,2)}{\Omega_+(\sigma_f)}$. Therefore we have to look at the primes ℓ which divide the numerator $\frac{\Lambda(\sigma_f,3)}{\Omega_-(\sigma_f)}$, if we want to find primes dividing the denominator of the Eisenstein class.

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