Computing the homology of $\mathrm{SL}_3(\mathbb{Z})$ with infinite-dimensional coefficient modules

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Abstract

Let p be a prime and Γ a congruence subgroup of $\mathrm{SL}_3(\mathbb{Z})$ which is Iwahori at p. Suppose f is a noncritical Hecke eigenclass in the homology of Γ with trivial $\mathbb{Q}_p$-coefficients. In [4] we outlined a method to compute to any desired degree of accuracy a lift of f to a homology class F with coefficients in a module of p-adic distributions with trivial highest weight. Then we studied how to deform F to an analytic family of homology classes in distribution modules with varying highest p-adic weight. In this paper we explain how to realize these deformations in an actual computer program, and we report on our initial computations of examples. The calculation boils down to row reduction of an infinite matrix over a ring R of power series in three variables over $\mathbb{Z}_p$. To carry this out, we must approximate, using finite quotients of R.

2010 Mathematics Subject Classification. **11F55**. 11F85.
Keywords. eigencurve, p-adic deformation, homology, arithmetic group.

1 Introduction

Fix a prime p. Let Γ be a congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$ contained in the Iwahori subgroup at p. Let λ_0 be a dominant integral weight for GL_n, and V_{λ_0} the finite dimensional irreducible representation with highest weight λ_0 over $\mathbb{Q}_p$. For any p-adic weight λ, let $\mathbf{D}_\lambda$ be the infinite-dimensional module of distributions defined in Section 2.

The homology of Γ with coefficients in V_{λ_0} has great number-theoretic interest, in view of Langlands' program connecting automorphic forms, motives and Galois representations (see, for example, Venkatesh's Fields Medal Lecture [11]). It has become increasingly important to see how to vary the homology as the highest weight λ varies p-adically, in such a way that the residual homology class modulo p remains constant. One way to do this is to replace the V_λ-coefficients by a very large coefficient module $\mathbf{D}_\lambda$ in which the highest weight can vary p-adically over a weight space Ω (section 4). For example, see David Hansen's work [9].

From now on let $n = 3$. All of the cuspidal automorphic representations of cohomological type can already be detected in the homology of degrees and 2 and 3 of a congruence subgroup of $\mathrm{SL}_3(\mathbb{Z})$. It is easier to compute the degree 3 homology of $\mathrm{SL}_3(\mathbb{Z})$ than the degree 2 homology and so, as in [4], we will work in degree 3 homology. In [4] we addressed the computational problem of determining the p-adic weights λ over which a homology class for Γ can vary p-adic analytically. In degree 3, there are no boundaries[1], so the space of 3-cycles is the same as H_3. We gave an explicit method for lifting a given 3-cycle f_0 with coefficients in V_{λ_0} to a 3-cycle F_0 with coefficients in $\mathbf{D}_{\lambda_0}$. We

[1]A congruence subgroup of $\mathrm{SL}_3(\mathbb{Z})$ has virtual cohomological dimension 3. Therefore, we can and do use a chain complex in which H_3 is the same as the space of 3-cycles.

Advanced Studies: Euro-Tbilisi Mathematical Journal Special Issue (9 - 2021), pp. 47–65.

Tbilisi Centre for Mathematical Sciences.

Received by the editors: 18 December 2020.
Accepted for publication: 09 June 2021.

then showed how to create an infinite inhomogeneous matrix equation $M(\lambda)x = b$ with coefficients in a certain ring of p-adic power series whose solutions compute p-adic analytically varying 3-cycles with coefficients in $\mathbf{D}_\lambda$ for varying λ that specialize to F_0 at $\lambda = \lambda_0$.

We look for conditions on the variety Z of $\lambda \in \Omega$ such that $M(\lambda)x = b$ has a solution. For example, such conditions might yield a "forbidden" closed subset of λ's for which there are no solutions, or they might give information on what the germ of Z at 0 looks like, or (if Z is smooth) an equation of its tangent plane at 0. For brevity, in the rest of this paper, we will call such information "constraints on λ."

Our goal then is to look for constraints on $\lambda \in \Omega$ that must be satisfied for a solution to $M(\lambda)x = b$ to exist. (It is possible that in some cases there will be no constraints.)

In this paper we explain how we have implemented this in computer programs, and we report on the results of three sample computations. The size of the computation grows quickly as p, λ_0 and the tame level N of Γ get larger. For this reason, we will assume from now on that $\lambda_0 = 0 := (0, 0, 0)$, so that V_0 is the trivial representation, and in our actual computations, p is a very small prime and $N \leq 100$.

The language of eigenvarieties may help put our work into perspective. If $S \supseteq \Gamma$ is a semigroup such that Γ and $s^{-1}\Gamma s$ are commensurable for all $s \in S$ we call (Γ, S) a *Hecke pair*. Given such an S, the free group on the double cosets $\Gamma s \Gamma$ with $s \in S$ carries a natural multiplicative structure [6, Section 1.1]. The resulting *Hecke algebra* is denoted $\mathcal{H}(\Gamma, S)$ and acts naturally on the homology of Γ. We refer to the elements of $\mathcal{H}(\Gamma, S)$ as *Hecke operators*. We will be interested in simultaneous eigenclasses of the Hecke operators and the associated systems of eigenvalues. We call an assignment $\mathcal{H}(\Gamma, S) \to \mathbb{C}_p^\times$ sending a given Hecke operator to its eigenvalue on a particular simultaneous eigenclass a *Hecke eigenpacket*.

Let U_p denote the Hecke operator corresponding to the double coset $\Gamma\,\mathrm{diag}(1, p, \ldots, p^{n-1})\Gamma$, which is of special importance in the theory.

Let Φ_0 be a Hecke eigenpacket occurring on some numerically noncritical Hecke eigenclass in $\mathrm{H}_*(\Gamma, \mathbf{D}_0)$. The "eigenvariety" through Φ_0 is a rigid-analytic space that parameterizes all the p-adic deformations of Φ_0 which have nonzero eigenvalues for the U_p-operator. Each point x on the eigenvariety through Φ_0 corresponds to a Hecke eigenpacket occurring on an eigenclass in $\mathrm{H}_*(\Gamma, \mathbf{D}_{\lambda_x})$, and the Hecke eigenvalue for any particular Hecke operator T varies p-adically as a function of x. We refer to $\pi : x \mapsto \lambda_x$ as the "projection to weight space."

When Φ_0 is cuspidal, the eigenvariety is two-dimensional by [9, Thm 4.5.1]. If Φ_0 is not cuspidal, but is an Eisenstein eigenpacket ("Eisenstein" means coming from the Borel-Serre boundary) and if in addition it is ordinary ("ordinary" means that the eigenvalue of U_p is a p-adic unit), then unpublished computations of Hansen show that the projection of the eigenvariety to weight space is surjective. These theorems of Hansen give us a place to start in interpreting our computational results.

In [4] and in this paper, we do not deform a Hecke eigenpacket, rather we deform the 3-cycle that represents a given homology class. The reason is that to compute and diagonalize the Hecke operators on the homology with coefficients in the infinite-dimensional module $\mathbf{D}_\lambda$ is very time and space consuming. In this initial paper, we will ignore the Hecke operators on $\mathrm{H}_3(\Gamma, \mathbf{D}_\lambda)$. We still use the Hecke operators to find an initial Hecke eigenclass $f_0 \in \mathrm{H}_3(\Gamma, V_0)$, and we use U_p to lift f_0 to a class F_0 with $\mathbf{D}_0$-coefficients. In subsequent work we plan to introduce Hecke operators more fully into our computations.

The relationship between the two deformation problems (of the Hecke eigenpacket versus the

3-cycle) is not straightforward. Lemma 32 of [2] strongly suggests that at least in the ordinary case the dimension of the deformation space of the 3-cycle should be the same as the dimension of the eigenvariety, and we would expect this to hold in the non-ordinary case also.

Thus we expect to find that when f_0 is cuspidal, that it deforms in a 2-dimensional space, and we may hope by appropriate computation to find obstructions to the deformation problem that would give information about that 2-dimensional subspace. (In the Eisenstein case we don't expect to find any such obstructions.)

The projection π from the eigenvariety to weight space is locally finite on the source. We expect the same to be true of the deformation space of the 3-cycle. The weight space has dimension 3. But even if the dimension of the defomation space is 2, it is possible that its projection to weight space might be surjective because π is not globally finite in general. In subsequent work we plan to use the U_p-operator to look locally on the eigenvariety. If we fail to find obstructions on the projection of the whole deformation space, we will restrict attention to a space of 3-cycles on which we impose a bound on the slope of the U_p-operator. We then expect that the projection to weight space of the deformation space will not be surjective in the cuspidal case. We then hope to discover obstructions that will give nontrivial information about the germ of the deformation space at various points above 0.

Our calculations go through three basic steps:

1. Compute $\mathrm{H}_3(\Gamma, \mathbb{Q})$ and the actions of several Hecke operators, including U_p, on $\mathrm{H}_3(\Gamma, \mathbb{Q})$. Choose a Hecke eigenclass $f_0 \in \mathrm{H}_3(\Gamma, \overline{\mathbb{Q}_p})$ and let K be $\mathbb{Q}_p$ adjoined the Hecke eigenvalues of f_0, so that $f_0 \in \mathrm{H}_3(\Gamma, K)$.

2. Using U_p, lift f_0 to a Hecke eigenclass $F_0 \in \mathrm{H}_3(\Gamma, \mathbf{D}_0)$ which reduces to f_0.

3. Letting λ be a variable weight, look for constraints on λ imposed by assuming F_0 is part of an analytic family extending to weight λ.

Step (1) is standard, see for example [1]. Step (2) uses an iterative process (Proposition 3.10 in [4]) to approximate F_0 to any desired degree of accuracy. In step (3) we use a characteristic 0 version of [1, Theorem 2.1], already used in Step 1, to embed $\mathrm{H}_3(\Gamma, D_\lambda)$ in $E_\lambda = \mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})} D_\lambda$. We then analyze a system $M(\lambda)x = b$ of linear equations that express the conditions that a general $\Psi_\lambda \in E_\lambda$ actually lies in $\mathrm{H}_3(\Gamma, D_\lambda)$ and that $\Psi_0 = F_0$.

The entries of $M(\lambda)$ belong to a certain ring R_r^0 which is a large subring of the power series ring $\mathbb{Z}_p[[k_1, k_2, k_3]]$ in three variables, where the weight $\lambda = (k_1, k_2, k_3)$. We work in practice over finite quotients A of R_r^0. This means performing row reduction on a matrix with infinitely many rows and infinitely many columns with entries in R_r^0 or A. In the course of the row reduction, we may or may not obtain constraints on λ, whose satisfaction is necessary for the system of equations to have a solution.

Part of the computational interest in our paper resides in how we deal with row reduction of infinite matrices over the complicated ring R_r^0. The details are given in section 5.

Briefly, what we do is look at reductions of M modulo the ideal generated by some p^e. Even modulo p^e, M still has infinitely many rows and columns. However, we show that each row only has finitely many nonzero entries. So we choose a (large) finite number of rows and form the submatrix $\overline{M}$ of the reduction of M modulo p^e including only those rows and the columns in which they have nonzero entries. We now have a finite linear system $\overline{M}\overline{x} = \overline{z}$ where $\overline{z}$ is the reduction of the subvector of b corresponding to the rows we have chosen.

When we do this, $\overline{M}$ has many more rows than columns. A random system of linear equations with many more equations than variables would be expected to have no solutions. If we find an obstruction to solving $\overline{M}\overline{x} = \overline{z}$ we can interpret what it says about the deformation space through F_0. If we find no obstruction, we repeat the process, using more or different rows, or increasing e or both. If we never find an obstruction, this is evidence in the cuspidal case that the eigenvariety, as it goes to infinity in the direction of increasing slope of the U_p-eigenvalue, may be fractal-like with a large projection to weight space. In the Eisenstein case, we do not expect to find any obstruction, and this is a good check on our computations.

Section 2 is devoted to establishing some notation and recalling important definitions. Section 3 deals with details on the computation of $\mathrm{H}_3(\Gamma, -)$, of the U_p-operator, and of liftings from finite-dimensional coefficients V_0 to infinite-dimensional coefficients $\mathbf{D}_0$.

Section 4 explains how we work explicitly with 3-cycles representing elements in $\mathrm{H}_3(\Gamma, \mathbf{D}_\lambda)$ in order to form the matrix $M(\lambda)$. Section 5 has a detailed discussion of our study of $\overline{M}\overline{x} = \overline{z}$. Section 6 provides a review of our numerical results at the present time. We also describe our plans for the next phases of this research.

Some of the material in sections 2, 3, and 4 is unavoidably repeated from [4], to make this paper self-contained.

Many thanks to David Hansen for very helpful conversations about eigenvarieties.

2 Definitions and notations

Let p be a prime number. Fix a finite dimensional extension K of $\mathbb{Q}_p$, with ring of integers $\mathbb{O}_K$. Choose K large enough to contain all Hecke eigenvalues that will come under consideration. Let $\mathbb{C}_p$ denote the completion of a fixed algebraic closure of K, and let $\mathbb{O}_p$ denote its ring of integers.

Let G be the algebraic group scheme GL_3, B (resp. B^{opp}) the group of upper (resp. lower) triangular matrices in G, and N (resp. N^{opp}) the group of unipotent matrices in B (resp. B^{opp}). Let T be the group of diagonal matrices so that $B = TN$ and $B^{\mathrm{opp}} = N^{\mathrm{opp}}T$. Let I denote the Iwahori subgroup of $G(\mathbb{Z}_p)$, that is, the collection of elements in $G(\mathbb{Z}_p)$ whose reduction modulo $p\mathbb{Z}_p$ is upper triangular. Fix L relatively prime to p and let $\Gamma_0(L)$ be the subgroup of $\mathrm{SL}_3(\mathbb{Z})$ consisting of those matrices whose first row is congruent to $(*, 0, 0)$ modulo L. Let $\Gamma = \Gamma_0(L) \cap I$.

Set

$$\mathfrak{X} := \mathrm{im}\left(I \to N^{\mathrm{opp}}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)\right).$$

It is isomorphic to $B(\mathbb{Z}_p)$.

Define weight space to be $\mathcal{W} = \mathrm{Hom}_{cont}(T(\mathbb{Z}_p), \mathbb{C}_p)$. Let $\Omega \subset \mathcal{W}$ be an open affinoid consisting of analytic characters, and fix some $\lambda \in \Omega$. Consider the collection of $\mathbb{C}_p$-valued functions

$$M_\lambda := \{f : \mathfrak{X} \to \mathbb{C}_p \;\mid\; f(tg) = \lambda(t)f(g) \text{ for } t \in T(\mathbb{Z}_p) \text{ and } g \in \mathfrak{X}\}.$$

A function on $N(\mathbb{Z}_p)$ is $\mathbb{Q}_p$-rigid analytic if it is of the form

$$f\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \sum_{i,j,k} c_{ijk} x^i y^j z^k$$

with $c_{ijk} \in \mathbb{Q}_p$ where $c_{ijk} \to 0$ as $i + j + k \to \infty$.

Define

$$\mathbf{A}_\lambda := \left\{ f : \mathfrak{X} \to \mathbb{C}_p \;\middle|\; \begin{array}{c} f \text{ restricted to } N(\mathbb{Z}_p) \text{ is a } \mathbb{Q}_p\text{-rigid analytic function,} \\ f(tg) = \lambda(t) f(g) \text{ for } t \in T(\mathbb{Z}_p) \end{array} \right\}.$$

Note that any function in $\mathbf{A}_\lambda$ is uniquely determined by its restriction to $N(\mathbb{Z}_p)$, so $\mathbf{A}_\lambda$ is isomorphic to the Tate algebra $\mathbb{Q}_p\langle x, y, z\rangle$.

For each triple of nonnegative integers (a, b, c), let f^λ_{abc} denote the unique extension to $\mathbf{A}_\lambda$ of the function that sends $\left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{smallmatrix}\right)$ to $x^a y^b z^c$. When we are working with a fixed λ (as here) we will usually write f_{abc} for f^λ_{abc}. The $\mathbb{Q}_p$-span of the f_{abc} forms a dense subset of $\mathbf{A}_\lambda$. We then set $\mathbf{D}_\lambda = \mathrm{Hom}_{cont}(\mathbf{A}_\lambda, K)$, the space of continuous $\mathbb{Q}_p$-linear functionals from $\mathbf{A}_\lambda$ to K. For $\mu \in \mathbf{D}_\lambda$, the values $\mu(f_{abc})$ are called the *moments* of μ. An element $\mu \in \mathbf{D}_\lambda$ is uniquely determined by its moments. Continuity implies that for fixed μ, the set $\{\mu(f_{abc})\}$ is bounded. Let δ^λ_{ijk} be the element of $\mathbf{D}_\lambda$ which takes value 1 on f_{ijk} and vanishes on the other f_{abc}. Let D_λ be the unit ball in $\mathbf{D}_\lambda$,

$$D_\lambda = \{\mu \in \mathbf{D}_\lambda : \mu(f_{ijk}) \in \mathbb{O}_K \text{ for all } \; i, j, k\}.$$

Note that $\mathbf{D}_\lambda = D_\lambda \otimes_{\mathbb{O}_K} K$ and $\delta^\lambda_{ijk} \in D_\lambda$.

Let π be the diagonal matrix with diagonal entries 1, p and p^2, and let Σ_0 be the semigroup generated by I and π. Section 2.2 of [4] gives an action of Σ_0 on $\mathbf{D}_\lambda$, preserving D_λ. Let U denote the Hecke operator corresponding to the double coset $\Gamma\backslash\pi/\Gamma$.

If λ is a dominant weight with respect to the Borel subgroup B^{opp}, we denote by V_λ the finite-dimensional irreducible right representation of GL_3 with highest weight λ. Lemma 2.6 of [10] gives a Σ_0-equivariant map $\rho_\lambda \colon \mathbf{D}_\lambda \to V_\lambda(K)$. In the special case $\lambda = 0$, ρ_0 is the map $\mathbf{D}_0 \to K$ sending μ to $\mu(f_{0,0,0})$.[2]

3 Computational models and liftings

Let Σ be the semigroup generated by $\mathrm{SL}_3(\mathbb{Z})$ and $\pi = \mathrm{diag}(1, p, p^2)$. Suppose that V is any right $\mathbb{Q}_p[\Sigma]$-module. Exactly as in the proof of Theorem 2.1 of [1], except working over a field of characteristic 0 instead of p, we obtain a vector space isomorphism

$$\Psi\colon\ \mathrm{H}_3(\mathrm{SL}_3(\mathbb{Z}), V) \to \mathcal{V}$$

where $\mathcal{V}$ is the subspace of V satisfying

1. $v \cdot m = \mathrm{sgn}(m) v$ for all monomial matrices $m \in \mathrm{SL}_3(\mathbb{Z})$, where $\mathrm{sgn}(m)$ is the sign of the permutation induced by m.

2. $v + v \cdot h + v \cdot (h^2) = 0$,

where

$$h = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[2] For a general dominant integral weight λ we have to modify the usual action of π on $V_\lambda(K)$ in order for ρ_λ to be Σ_0-equivariant. This modified action is called the "star" action. However, in this paper, because we assume $\lambda_0 = 0$, the star action is the same as the trivial action, and we can omit further mention of the star action.

Definition 3.1. A vector $w \in W$ *represents* a homology class $f \in \mathrm{H}_3(\mathrm{SL}_3(\mathbb{Z}), W)$ if $\Psi(f) = w$.

Let f be some homology class in $\mathrm{H}_3(\mathrm{SL}_3(\mathbb{Z}), V)$ and suppose f is represented by $v \in V$. To compute the action of the Hecke operator $T(\pi) = \mathrm{SL}_3(\mathbb{Z}_p)\pi\,\mathrm{SL}_3(\mathbb{Z}_p)$ on f we first write

$$\mathrm{SL}_3(\mathbb{Z})\pi\,\mathrm{SL}_3(\mathbb{Z}) = \coprod_j \mathrm{SL}_3(\mathbb{Z})\pi A_j$$

as a disjoint union of cosets of $\mathrm{SL}_3(\mathbb{Z})$. Then we find (using, for example the Ash-Rudolph algorithm [5, Section 4]) unimodular matrices $M_{j,k} \in M_3(\mathbb{Z})$ such that the modular symbol $[(\pi A_j)^{-1}]$ is homologous to $\sum_k [M_{jk}]$. Then we have [1, Lemma 9.1] that $f|T(\pi) \in \mathrm{H}_3(\mathrm{SL}_3(\mathbb{Z}), V)$ is represented by

$$\sum_{j,k} v \cdot M_{j,k}\pi A_j \in V.$$

Fix once and for all matrices A_j and $M_{i,j}$, and use the above formula to define an operator $T(\pi)$ on V by

$$v|T(\pi) = \sum_{j,k} v \cdot M_{j,k}\pi A_j \tag{1.1}$$

whether or not v represents a homology class. If $v = \Psi(f)$ does represent a homology class f then we have that $v|T(\pi) = \Psi(f|T(\pi))$ represents $f|T(\pi)$.

We consider the semigroup Ξ generated by π and the set of matrices in $\mathrm{GL}_3(\mathbb{Q})$ whose entries have denominators prime to pL, whose determinant is positive and has numerator prime to pL, and which are congruent to an upper-triangular matrix modulo p, and whose first row is congruent to $(*, 0, 0)$ modulo L. Let W be any right Σ_0-module.

Recall the definitions of Hecke pair and Hecke algebra from the introduction. Since the Hecke pairs (Γ, Ξ) and $(\mathrm{SL}_3(\mathbb{Z}), \Sigma)$ are compatible [6, Def. 1.1.2], we get an action of Σ on $\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}(W)$ given by

$$f|\sigma(x) = \begin{cases} f(y)\tau & \text{if} \quad \sigma x = y\tau \quad \text{with} \quad y \in \mathrm{SL}_3(\mathbb{Z}), \tau \in \Xi \\ 0 & \text{if} \quad \sigma x \notin \mathrm{SL}_3(\mathbb{Z})\Xi. \end{cases}$$

With this action, the Shapiro Isomorphism

$$\mathrm{H}_3(\mathrm{SL}_3(\mathbb{Z}), \mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}(W)) \to \mathrm{H}_3(\Gamma, W)$$

is Hecke-equivariant with respect to the map of Hecke algebras

$$\Theta\colon\ \mathcal{H}(\mathrm{SL}_3(\mathbb{Z}), \Sigma) \to \mathcal{H}(\Gamma, \Xi)$$

given by restriction of functions. Note that $\Theta(T(\pi)) = U$, where $U = U_p$ is the Hecke operator discussed in the introduction. Denote $\mathcal{H}(\Gamma, \Xi)$ by $\mathcal{H}$.

Remark 3.2. The map ρ_0 induces a map

$$\mathrm{Ind}\,\rho_0\colon\ \mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})} D_0 \to \mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})} \mathbb{Z}_p.$$

This maps sends the subspace of $\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})} D_0$ representing the homology $\mathrm{H}_3(\Gamma, D_0)$ to the subspace of $\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})} \mathbb{Z}_p$ representing $\mathrm{H}_3(\Gamma, \mathbb{Z}_p)$ and agrees with the usual induced map on homology.

In [4] we had a critical number called m_λ. In particular $m_0 = 1$, so now when we quote theorems from [4] we replace m_λ by 1. Also, $V_0 = K$.

Definition 3.3. A U-eigenvector $f \in \mathrm{H}_3(\Gamma, K)$ with U-eigenvalue α having slope $\mathrm{ord}_p(\alpha) < 1$ is called *numerically noncritical.*

Given a numerically noncritial U-eigenvector $f \in \mathrm{H}_3(\Gamma, K)$, Theorem 6.4.1 of [7] asserts the existence of a unique U-eigenvector $F \in \mathrm{H}_3(\Gamma, \mathbf{D}_0)$ such that $\rho_\lambda(F) = f$. We say F *lifts* f.

Scaling f by a power of p, we may (and will) assume without loss of generality that $f \in \mathrm{H}_3(\Gamma, \mathbb{O}_K)$. Even so, Theorem 6.4.1 of [7] only guarantees that F is in $\mathrm{H}_3(\Gamma, \mathbf{D}_0)$ not $\mathrm{H}_3(\Gamma, D_0)$, where D_0 is the unit ball in $\mathbf{D}_0$. But Theorem 3.5 of [4] says that $F \in p^{-\mathrm{ord}_p(\alpha)}\,\mathrm{H}_3(\Gamma, D_0)$.

We now quote and discuss the key theorem from [4] that gives us a way to lift f to F.

Theorem 3.4. *Let $f \in \mathrm{H}_3(\Gamma, \mathbb{O}_K)$ be an $\mathcal{H}$-eigenvector with U-eigenvalue α such that $\mathrm{ord}_p(\alpha) < 1$. Let $F \in \alpha^{-1}\,\mathrm{H}_3(\Gamma, D_0)$ be the unique $\mathcal{H}$-eigenvector lifting f. Let $z \in \mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,\mathbb{O}_K$ represent f and let $Z \in \alpha^{-1}\,\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0$ represent F. Suppose further that $W \in \mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0$ is any element which maps to z under $\mathrm{Ind}\,\rho_0\colon\ \mathrm{Ind}\,D_0 \to \mathrm{Ind}\,\mathbb{O}_K$.*

Then for each n, α^{n-1} divides $W|U^n$ in $\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0$ and $\frac{W|U^n}{\alpha^{n-1}}$ converges to αZ in the sense that for any k we can force $\left(\frac{W|U^n}{\alpha^{n-1}} - \alpha Z\right)$ to be in $p^k\,\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0$ by taking n sufficiently large. In particular

$$\left(\frac{W|U^n}{\alpha^{n-1}} - \alpha Z\right) \in p^{n(1-\mathrm{ord}_p(\alpha))}\,\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0.$$

We can't hope to make exact calculations in $\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0$ since representing an arbitrary element of D_0 requires knowing the p-adic coefficients of infinitely many of the $\delta_{i,j,k}$. We can't even compute in $\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0/p^n D_0$ as this still requires knowing infinitely many mod-p^n coefficients. Instead we work modulo a filtration on $\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0$ whose filtered pieces have finite index in $\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0$. In [4] we discussed a filtration introduced by Bonita Graham in her thesis, but for our calculations we actually used the following filtration, which is a slight reindexing of the one introduced in [10]. We use this filtration because it allows us to work with fewer moments when we compute lifts. This is not a significant change, and we recall below the relevant results from [4], mutatis mutandis.

We also note that when doing computations we represent elements of $\mathbb{Z}_p$ by elements of $\mathbb{Z}/p^N\mathbb{Z}$ for an N sufficiently large to give good enough approximations for our purposes.

Definition 3.5. The filtration Fil on D_0 is defined by

$$\mathrm{Fil}^n D_0 := \left\{\ \mu \in \ker(\rho_o)\ :\ \mathrm{ord}_p\left(\mu(f_{rst})\right) \geq n - \left\lceil \frac{(r+s+t)}{2} \right\rceil\ \text{ for } r,s,t \geq 0 \right\}.$$

We extend this filtration on D_0 to a filtration on $\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0$ by defining

$$\mathrm{Fil}^n\,\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0 = \{f \in \mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0 : f(\mathrm{SL}_3(\mathbb{Z})) \subseteq \mathrm{Fil}^n D_0\}. \tag{1.2}$$

The following two lemmas are the key results we need about Fil.

Lemma 3.6. *The filtration Fil on $\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0$ is stable under Σ, and for each $n \geq 0$,*

$$\mathrm{Fil}^n\,\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0|\pi \subset \mathrm{Fil}^{n+1}\,\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})}\,D_0.$$

Lemma 3.7. If $W, Z \in \operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} D_0$ such that

$$W \equiv Z \pmod{\operatorname{Fil}^n \operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} D_0},$$

then

$$W|U \equiv Z|U \pmod{\operatorname{Fil}^{n+1} \operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} D_0}.$$

With this filtration in hand, we are now prepared to compute an approximation to the lift of a given U-eigenclass $f \in \mathrm{H}_3(\Gamma, \mathbb{O}_K)$. To make everything concrete, we introduce some more notation. Fix once and for all a set A of coset representatives for $\mathrm{SL}_3(\mathbb{Z})/\Gamma$. For $\alpha \in A$ let $\omega_\alpha \in \operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} \mathbb{O}_K$ be the characteristic function of the coset $\alpha\Gamma$ and for each triple $I = (i, j, k)$ of nonnegative integers, let $\delta^0_{\alpha,I}$ be the function in $\operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} D_0$ supported on $\alpha\Gamma$ and sending α to δ^0_I.

Suppose f is represented by an element $v = \sum_{\alpha \in A} a_\alpha \omega_\alpha \in \operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} \mathbb{O}_K$. We choose $W \in \operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} D_0$ such that $(\operatorname{Ind} \rho_0)(W) = v$. The simplest choice, which we make, is to take $W = \sum_{\alpha \in A} a_\alpha \delta^0_{\alpha,(0,0,0)}$.

We would like to compute $W|U$ but in general $W|U$ will not be a *finite* linear combination of $\delta_{\alpha,I}$, even though W itself is. If we write $W|U = \sum_{\alpha,I} c_{\alpha,I} \delta^0_{\alpha,I}$ then we can use equation (1.1) and the formulas in Section 2.3 of [10] to compute any finite number of the $c_{\alpha,I}$ to any degree of p-adic accuracy we desire. In particular, we can choose an explicit element $W_1 \in \operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} D_0$ such that

$$W_1 \equiv W|U \pmod{\operatorname{Fil}^1 \operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} D_0}.$$

and W_1 is a finite linear combination of the $\delta_{\alpha,I}$. Likewise, we can find $W_i \in \operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} D_0$ such that

$$W_i \equiv W_{i-1}|U \pmod{\operatorname{Fil}^i \operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} D_0}$$

and W_i is again a finite linear combination of the $\delta_{\alpha,I}$.

The following lemma follows from Lemma 3.7 and is proved in [4].

Lemma 3.8. With the W_i chosen as above,

$$W_i \equiv W|U^i \pmod{\operatorname{Fil}^i \operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} D_0}$$

for each $i \geq 1$.

In the following proposition, the coefficients $b_{\alpha,r,s,t}$ are "unknown" numbers we are trying to approximate and the $c^n_{\alpha,r,s,t}$ are the numbers we can compute to any desired degree of accuracy.

Proposition 3.9. Let $f|U = \beta U$. Let $F \in \beta^{-1} \mathrm{H}_3(\Gamma, D_0)$ be the unique U-eigenvector lifting f. Let $Z \in \beta^{-1} \operatorname{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} D_0$ represent F and write $Z = \sum b_{\alpha,r,s,t} \delta^0_{\alpha,r,s,t}$. Let W and the W_n be as in the discussion before Lemma 3.8 and write $W_n = \sum c^n_{\alpha,r,s,t} \delta^0_{\alpha,r,s,t}$. If

$$n \geq \frac{\left\lceil \frac{r+s+t}{2} \right\rceil - \operatorname{ord}_p(\beta)}{1 - \operatorname{ord}_p(\beta)}$$

then β^{n-1} divides $c^n_{\alpha,r,s,t}(x)$.

Moreover, in this case

$$\operatorname{ord}_p\left(\beta b_{\alpha,r,s,t}(x) - \frac{c^n_{\alpha,r,s,t}(x)}{\beta^{n-1}}\right) \geq n\left(1 - \operatorname{ord}_p(\beta)\right) + \min\left(0, \operatorname{ord}_p(\beta) - \left\lceil \frac{r+s+t}{2} \right\rceil\right).$$

So, choosing n sufficiently large, we can compute any given finite collection of the $\beta b_{\alpha,r,s,t}(x)$ to any desired degree of accuracy by computing the corresponding $c^n_{\alpha,r,s,t}(x)$. For example, if we wish to compute $\beta b_{\alpha,r,s,t}(x)$ modulo p for all $r+s+t<d$, it suffices to take

$$n \geq \frac{1+\max(0,\lceil \frac{d}{2} \rceil - \mathrm{ord}_p(\beta))}{1-\mathrm{ord}_p(\beta)}.$$

4 Explicit p-deformations

The weight space is 3-dimensional, with variable $\lambda=(k_1,k_2,k_3)$. If $p>2$ let $\Omega=\mathbb{O}_p^3$, and if $p=2$ let $\Omega=2\mathbb{O}_2^3$. We view $\lambda=(k_1,k_2,k_3)\in\Omega$ as the character of $T(\mathbb{Z}_p)$ which sends $t=\mathrm{diag}(t_1,t_2,t_3)$ to ${t_1}^{k_1}{t_2}^{k_2}{t_3}^{k_3}$ for t in the open unit polydisc $\mathcal{B}^3$ of $\mathbb{Z}_p^3$ centered at 1 and which is trivial when restricted to the torsion of $T(\mathbb{Z}_p)$.

Remark 4.1. With these domains for λ and t, D_λ/p is isomorphic to D_0/p as $\mathrm{GL}_3(\mathbb{Z}_p)$-modules. Therefore modulo p, there is no difference between computing $\mathrm{H}_3(\Gamma,D_0)$ and $\mathrm{H}_3(\Gamma,D_\lambda)$.

Let F_0 be a U-eigenclass in $\mathrm{H}_3(\Gamma,D_0)$ represented as before by an element $v_0\in\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})} D_0$. Then we may write

$$v_0=\sum_{\alpha,I} a^0_{\alpha,I}\delta^0_{\alpha,I}$$

where α runs over our fixed set A of coset representatives for $\mathrm{SL}_3(\mathbb{Z})/\Gamma$ and I runs over the set B of triples (r,s,t) of nonnegative integers. We let $Y=A\times B$ and for $g\in\mathrm{SL}_3(\mathbb{Z})$ we denote the unique element of A in the coset $g\Gamma$ by $\{g\}$.

If F_0 is non-ordinary, there may not be deformations of F_0 far from the origin of Ω, so we introduce a parameter $r\geq 0$. Even in the ordinary case, if $p=2$ we always take $r\geq 1$. We let $\mathcal{W}_r$ be the closed subset of Ω consisting of characters of the torus $\{(k_1,k_2,k_3)||k_i|_p\leq 1/p^r\}$ and let R_r be the algebra

$$R_r=\left\{\sum_{i,j,k} c_{ijk}k_1^i k_2^j k_3^k : c_{ijk}\in K, |p^{r(i+j+k)}c_{ijk}|\to 0\right\}.$$

We let R_r^0 be the unit ball in R_r,

$$R_r^0=R_r\cap\mathbb{O}_K[[p^{-r}k_1,p^{-r}k_2,p^{-r}k_3]].$$

Note that elements of R_r converge on $\mathcal{W}_r$, and R_r^0 consists of those elements of R_r which take values in $\mathbb{O}_p$.

Recall the map Ψ from the beginning of Section 3.

Suppose, for each α and I, we are given an element $a_{\alpha,I}(\lambda)\in R_r^0$ such that $a_{\alpha,I}(0)=a^0_{\alpha,I}$. Let $v=(a_{\alpha,I})\in(R_r^0)^Y$. Then for each $\lambda\in\mathcal{W}_r$ we may consider

$$v(\lambda)=\sum_{\alpha,I} a_{\alpha,I}(\lambda)\delta^\lambda_{\alpha,I}\in\mathrm{Ind}_\Gamma^{\mathrm{SL}_3(\mathbb{Z})} D_\lambda.$$

For a given λ, $v(\lambda)$ may or may not lie in the image of Ψ and hence may or may not represent a class in $\mathrm{H}_3(\Gamma,D_\lambda)$.

Definition 4.2. Given a collection $v = (a_{\alpha,I}) \in (R_r^0)^Y$, the *validity locus* of v is the set

$$Z(v) = \left\{ \lambda \in \mathcal{W}_r : v(\lambda) = \sum_{\alpha,I} a_{\alpha,I}(\lambda)\delta^\lambda_{\alpha,I} \in \mathrm{Im}(\Psi) \right\}.$$

For $\lambda \in Z(v)$, let $F(\lambda)$ be the homology class represented by $v(\lambda)$. If $F(0) = F_0$, the map $\lambda \mapsto F(\lambda)$ is a deformation of F_0 over $Z(v)$. Corollary 4.3 of [4] asserts that $Z(v)$ is a rigid analytic subset of $\mathcal{W}_r$.

We examine the constraints we obtain on the $a_{\alpha,I}(\lambda)$ by insisting that $v(\lambda)\cdot(1+h+h^2) = 0$ and that each $v(\lambda)(m - \mathrm{sgn}(m)) = 0$ for m a monomial matrix in $\mathrm{SL}_3(\mathbb{Z})$, as is necessary and sufficient for $v(\lambda)$ to lie in the image of Ψ, i.e. to represent a homology class. These constraints can be interpreted by a matrix with an infinite number of rows and columns whose entries depends on λ. This matrix is the augmented matrix M of an inhomogeneous infinite system of linear equations in an infinite-dimensional vector space. If this system has solutions for a given λ, then every finite augmented submatrix of M is consistent. The subvariety of weight space over which F deforms will consist of exactly those λ for which this system has solutions.

To compute M, we need to be able to compute $\delta^\lambda_{\alpha,I}\cdot\tau$ for $\tau \in \mathrm{SL}_3(\mathbb{Z})$. Now $(\delta^\lambda_{\alpha,I}\cdot\tau)$ is supported on the coset of $\tau^{-1}\alpha$. Write $\tau^{-1}\alpha = \beta\gamma$ where $\beta = \{\tau^{-1}\alpha\} \in A$ is the representative for the coset of $\tau^{-1}\alpha$ chosen above and $\gamma \in \Gamma$. Then $(\delta^\lambda_{\alpha,I}\cdot\tau)(\beta) = \delta^\lambda_I\gamma^{-1}$.

We need to compute $\delta^\lambda_I\gamma^{-1}$. To do so, consider

$$\delta^\lambda_I\gamma^{-1}(f_{q,s,t}) = \delta^\lambda_I(\gamma^{-1}f_{q,s,t}).$$

Now, $\gamma^{-1} \in \Gamma$ is a three by three matrix of integers which we can compute explicitly, depending only on α and τ. Say $\gamma^{-1} = (a_{ij})$. Then, writing $\lambda = (k_1, k_2, k_3)$, Lemma 2.1 of [10] gives a formula for $\gamma^{-1}f_{q,s,t}$. We take this opportunity to note and correct an error in the formula as given in [10] and repeated in [4]. In particular, what were referred to there as the minors of γ should have been the *cofactors* of γ. We have made the necessary sign adjustments below so that our m_{ij} really are the minors of γ. Further, in [10] the character λ was not chosen to act trivially on torsion. For $k \in \mathbb{Z}_p$ ($k \in 2\mathbb{Z}_2$ if $p = 2$) let μ_k be the character of $\mathbb{Z}_p^\times$ which is trivial on roots of unity and sends $t \in 1 + p\mathbb{Z}_p$ to t^k. Then we have

$$\begin{aligned}(\gamma^{-1}f_{q,s,t})(x,y,z) = {}& \\ &\mu_{k_1-k_2}(a_{11}+a_{21}x+a_{31}y)\mu_{k_2-k_3}(m_{33}-m_{13}y+m_{23}z+m_{13}xz)\\ &(a_{11}+a_{21}x+a_{31}y)^{-q-s}(m_{33}-m_{13}y+m_{23}z+m_{13}xz)^{-t}\\ &(a_{12}+a_{22}x+a_{32}y)^q\,(a_{13}+a_{23}x+a_{33}y)^s\,(m_{32}-m_{12}y+m_{22}z+m_{12}xz)^t\end{aligned}$$

where m_{ij} is the ij-minor of γ^{-1}. Since $\gamma^{-1} \in I$, we know that p divides each of $a_{21}, a_{31}, a_{32}, m_{12}, m_{13}$, and m_{23} while a_{11} and m_{33} are p-adic units.

We can expand each of these factors as power series in x, y, z, k_1, k_2, k_3. Multiplying all these expression together we obtain an expression for $(\gamma^{-1}f_{q,s,t})(x,y,z)$ as a power series

$$(\gamma^{-1}f_{q,s,t})(x,y,z) = \sum_{I=(i,j,k)} c^{\tau,\alpha}_{I,q,s,t}(\lambda)x^iy^jz^k$$

with each $c^{\tau,\alpha}_{I,q,s,t}(\lambda)$ a power series in λ.

Proposition 4.3. With the $c^{\tau,\alpha}_{(i,j,k),(q,s,t)}$ as above we have

1. If $p \geq 3$, each $c^{\tau,\alpha}_{(i,j,k),(q,s,t)} \in p^{\lceil \frac{i+j+k-(q+s+t)}{2} \rceil \frac{p-2}{p-1}} R^0_0$.

2. If $p = 2$, each $c^{\tau,\alpha}_{(i,j,k),(q,s,t)} \in p^{\lceil \frac{i+j+k-(q+s+t)}{4} \rceil} R^0_1$.

Proof. The $c^{\tau,\alpha}_{(i,j,k),(q,s,t)}$ are defined as the coefficients in the expresion

$$(\gamma^{-1} f_{q,s,t})(x,y,z) = \sum_{I=(i,j,k)} c^{\tau,\alpha}_{I,q,s,t}(\lambda) x^i y^j z^k.$$

We fix (q,s,t) and examine the formula for $(\gamma^{-1} f_{q,s,t})(x,y,z)$ as a product of 7 factors given above. First consider the product of the last 5 factors, none of which depend on $\lambda = (k_1,k_2,k_3)$. Call this product g_0. It is a power series in x,y,z with coefficients in $\mathbb{Z}_p$. As in the proof of [10, Prop. 4.1] the coefficient of a monomial in x,y,z of total degree d appearing in g_0 has p-adic valuation at least $\max(0, \left\lceil \frac{d-(q+s+t)}{2} \right\rceil)$.

Let $\zeta, \eta \in \mathbb{Z}_p$ be the Teichmuller representatives for the classes of a_{11} and m_{33} modulo p. Then we can write the first two factors in our product as

$$\begin{aligned} g_1 &= \mu_{k_1-k_2}(a_{11} + a_{21}x + a_{31}y) \\ &= \mu_{k_1-k_2}\left(\zeta^{-1}(a_{11} + a_{21}x + a_{31}y)\right) \\ &= \left(\zeta^{-1}(a_{11} + a_{21}x + a_{31}y)\right)^{k_1-k_2} \end{aligned}$$

and

$$\begin{aligned} g_2 &= \mu_{k_2-k_3}(m_{33} - m_{13}y + m_{23}z + m_{13}xz) \\ &= \mu_{k_2-k_3}\left(\eta^{-1}(m_{33} - m_{13}y + m_{23}z + m_{13}xz)\right) \\ &= \left(\eta^{-1}(m_{33} - m_{13}y + m_{23}z + m_{13}xz)\right)^{k_2-k_3}. \end{aligned}$$

Suppose now that $p \geq 3$. Recall that for $k, t \in \mathbb{Z}_p$ the exponential $(1+pt)^k$ is defined and can be expressed (via the binomial theorem) as a power series in t as

$$(1+pt)^k = \sum_{n=0}^{\infty} \binom{k}{n} (pt)^n$$

where

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-(n-1))}{n!}.$$

Since p divides each of a_{21}, a_{31} as well as $\zeta^{-1}a_{11} - 1$ we can use the binomial theorem to write g_1 as

$$
\begin{aligned}
(\zeta^{-1}(a_{11}+a_{21}x+a_{31}y))^{k_1-k_2} &= (1+(\zeta^{-1}a_{11}-1)+\zeta^{-1}(a_{21}x+a_{31}y))^{k_1-k_2} \\
&= \sum_{n=0}^{\infty}\binom{k_1-k_2}{n}\left((\zeta^{-1}a_{11}-1)+\zeta^{-1}(a_{21}x+a_{31}y)\right)^n .
\end{aligned}
$$

For a fixed n, $\binom{k_1-k_2}{n}\left((\zeta^{-1}a_{11}-1)+\zeta^{-1}(a_{21}x+a_{31}y)\right)^n$ is a sum of terms $\binom{k_1-k_2}{n}p^n a_{u,v,n}x^u y^v$ with $a_{u,v,n}\in\mathbb{Z}_p$ and $u+v\le n$. We thus have a sum over n, u, and v. We will see that these terms tend to 0 p-adically, and so we can reorder the terms to obtain

$$
\sum_{n=0}^{\infty}\binom{k_1-k_2}{n}\left((\zeta^{-1}a_{11}-1)+\zeta^{-1}(a_{21}x+a_{31}y)\right)^n = \sum_{u,v}\left(\sum_{n\ge u+v}\frac{p^n}{n!}a_{u,v,n}f_n(k_1-k_2)\right)x^u y^v
$$

where $f_n(k)=n!\binom{k}{n}\in\mathbb{Z}[k]$ and $a_{u,v,n}\in\mathbb{Z}_p$.

Since $\mathrm{ord}_p(n!)\le\frac{n-1}{p-1}$ we have $\mathrm{ord}_p(\frac{p^n}{n!})\ge n-\frac{n-1}{p-1}$ which tends to infinity monotonically as n does. We thus see that the coefficient in g_1 of a term $x^u y^v$ with $u+v=d$ is in $p^{d-\frac{d-1}{p-1}}\mathbb{Z}_p[[\lambda]]$. A similar analysis applies to g_2, though since $(m_{33}-m_{13}y-m_{23}z+m_{13}xz)^n$ can have a term of degree as high as $2n$, we get that the coefficient of a term $x^u y^v z^w$ in this expansion with $u+v+w=d$ is in $p^{\lceil d/2\rceil-\frac{\lceil d/2\rceil-1}{p-1}}\mathbb{Z}_p[[\lambda]]$.

Note that both $d-\frac{d-1}{p-1}$ and $\lceil d/2\rceil-\frac{\lceil d/2\rceil-1}{p-1}$ are greater than

$$
\lceil d/2\rceil(1-\frac{1}{p-1})=\lceil d/2\rceil\frac{p-2}{p-1}.
$$

Suppose finally that when we are multiplying together the power series we have obtained for g_0, g_1, and g_2 we multiply a monomial term of degree d from g_0 with monomial terms of degrees e and f from g_1 and g_2. Then the resulting monomial has degree $D=d+e+f$ and its coefficient has p-adic valuation at least

$$
\left\lceil\frac{d-(q+s+t)}{2}\right\rceil+\left(\left\lceil\frac{e}{2}\right\rceil+\left\lceil\frac{f}{2}\right\rceil\right)\frac{p-2}{p-1}\ge\left\lceil\frac{D-(q+s+t)}{2}\right\rceil\frac{p-2}{p-1}.
$$

Thus $c^{\tau,\alpha}_{(i,j,k),(q,s,t)}\in p^{\lceil\frac{i+j+k-(q+s+t)}{2}\rceil\frac{p-2}{p-1}}\mathbb{Z}_p[[\lambda]]$.

Returning to the power series expansion of g_1, note that the coefficient of a monomial in x, y, z is a sum of p-adic integral multiples of $\frac{p^n}{n!}f_n(k_1-k_2)$. Since $\mathrm{ord}_p(\frac{p^n}{n!})>n-\frac{n}{p-1}=n\frac{p-2}{p-1}$ is positive and tends to infinity as n increases, it follows that each of these coefficients are in R_0^0. The same analysis applies to g_2, and the coefficients of the monomial terms in g_0 are integral and don't involve λ. Thus $c^{\tau,\alpha}_{(i,j,k),(q,s,t)}\in R_0^0$. So $c_{(i,j,k),(q,s,t)}\in p^{\lceil\frac{i+j+k-(q+s+t)}{2}\rceil\frac{p-2}{p-1}}\mathbb{Z}_p[[\lambda]\cap R_0^0\subset p^{\lceil\frac{i+j+k-(q+s+t)}{2}\rceil\frac{p-2}{p-1}}R_0^0$, completing the proof of part (1).

If $p=2$ then the power series expansion for $(1+pt)^k$ only converges if $k\in 2\mathbb{Z}_2$, which we are assuming is the case. Writing $k_i=2(k_i/2)$ and carrying out an analysis similar to the case $p>2$ gives part (2) of the proposition. Q.E.D.

Having $v(\lambda)\cdot(1+h+h^2)=0$ is equivalent to having

$$a_{\alpha,(q,s,t)}(\lambda)+\sum_I\left(c^{h,\{h\alpha\}}_{I,q,s,t}(\lambda)a_{\{h\alpha\},I}(\lambda)+c^{h^2,\{h^2\alpha\}}_{I,q,s,t}(\lambda)a_{\{h^2\alpha\},I}(\lambda)\right)=0 \tag{1.3}$$

for each fixed α,q,s,t. Rather than viewing this sum as a power series in λ we will view it as a *linear* expression in the $a_{\alpha,I}(\lambda)$ whose coefficients lie in R^0_0 (or R^0_1 if $p=2$).

Likewise, for each monomial matrix m in $\mathrm{SL}_3(\mathbb{Z})$, the constraint $v(\lambda)\cdot(m-\mathrm{sgn}(m))=0$ is equivalent to the system

$$a_{\alpha,(q,s,t)}(\lambda)-\mathrm{sgn}(m)\sum_I c^{m,\{m\alpha\}}_{I,q,s,t}(\lambda)a_{\{m\alpha\},I}(\lambda)=0. \tag{1.4}$$

It suffices to consider just the equations corresponding to any three elements m_1,m_2,m_3 that generate the group of signed permutations. Call these equations $(1.4m_1)$, $(1.4m_2)$, $(1.4m_3)$.

Thus we have a collection of linear constraints on the $a_{\alpha,I}(\lambda)$, with 4 such constraints for each α,q,s,t. We are only interested in solutions to this system that satisfy $a_{\alpha,I}(0)=a^0_{\alpha,I}$, so we add dummy variables $b^j_{\alpha,I}$ for each α,I and each $j=1,2,3$ and the constraints

$$a_{\alpha,I}(\lambda)-p^{-r}k_1b^1_{\alpha,I}(\lambda)-p^{-r}k_2b^2_{\alpha,I}(\lambda)-p^{-r}k_3b^3_{\alpha,I}(\lambda)=a^0_{\alpha,I}. \tag{1.5}$$

The constraint (1.5), for any $a_{\alpha,I}(\lambda)\in R^0_r$, has solutions in R^0_r precisely when $a_{\alpha,I}(0)=a^0_{\alpha,I}$.

Definition 4.4. (1) Let M be the infinite matrix with coefficients in R^0_r whose columns are indexed by $Y\times\{0,1,2,3\}$ and whose rows are indexed by $Y\times\{h,m_1,m_2,m_3,con\}$, constructed as follows. The columns indexed by $\alpha,I,0$ correspond to $a_{\alpha,I}(\lambda)$ and the columns indexed by α,I,k correspond to $b^k_{\alpha,I}(\lambda)$ for $k=1,2,3$. Each row in M expresses the left hand side one of the equations (1.3), $(1.4m_1)$, $(1.4m_2)$, $(1.4m_3)$, or (1.5), so that all the equations are included.

(2) Let z be the column vector with whose rows are indexed by $Y\times\{h,m_1,m_2,m_3,con\}$ and having an entry of 0 in each row corresponding to an equation of type (1.3) or (1.4), and having the entry $a^0_{\alpha,I}$ in the row corresponding to the equation of type (1.5) for each (α,I).

The following result is Theorem 4.4 of [4].

Theorem 4.5. For every $r\ge 0$, the following two conditions on an element λ_0 of $\mathcal{W}_r$ are equivalent.

(i) There exists $(a_{\alpha,I})\in(R^0_r)^Y$ with $\sum_{\alpha,I}a_{\alpha,I}(0)\delta^0_{\alpha,I}$ representing F_0 and $\sum_{\alpha,I}a_{\alpha,I}(\lambda_0)\delta^{\lambda_0}_{\alpha,I}$ representing a class in $\mathrm{H}_3(\Gamma,D_{\lambda_0})$.

(ii) There exists a vector $x\in(R^0_r)^{Y\times\{0,1,2,3\}}$ such that $(Mx)(0)=z(0)$ and $(Mx)(\lambda_0)=z(\lambda_0)$.

5 Explicit computation and row reduction

Starting with a U-eigenclass $f_0\in\mathrm{H}_3(\Gamma,V_0)$ with eigenvalue β, construct a lift $F_0\in\mathrm{H}_3(\Gamma,D_0)$ of βf_0 as described in section 3. Fix an $r\ge 0$ with $r>0$ if $p=2$. Let M and z be as in Definition 4.4. The system $Mx=z$ over the ring R^0_r has infinitely many rows and columns. We want to investigate whether there is a solution x over R^0_r, as such a solution would represent a deformation of F_0 across all of $\mathcal{W}_r$. If not, then F_0 will deform only over a proper analytic subset Z of $\mathcal{W}_r$, cut out by some (as yet unknown) nonzero ideal J of R^0_r. We would then see solutions to $Mx=z$ over R^0_r/J. Our goal is to determine if there are any constraints to solving $Mx = Z$ and, if so, to get information about J and hence about the subset of weight space over which F_0 deforms.

In the discussion below, we let J stand for an ideal such that $Mx = z$ has solutions over R_r^0/J. In particular, if there are no obstructions at all on the weights of solutions then we may take $J = (0)$.

Of course we can't actually compute with the infinite system $Mx = z$. But, fixing a small e, Propositon 4.3 tells us that any given row only has finitely many entries which are nonzero modulo p^e. So we select a finite number of rows out of the matrix M to form a submatrix, then reduce this modulo p^e, and then drop all the zero columns. We thus obtain a finite matrix over $R_r^0/p^e R_r^0$. Call this finite matrix $\overline{M}$ and let $\overline{z}$ be the column vector we obtain from z by the same reduction and truncation. Then the existence of an x solving $Mx = z$ over some R_r^0/J implies the existence of a solution $\overline{x}$ to $\overline{M}\overline{x} = \overline{z}$ over $R_r^0/(J, p^e)$. It is this latter system that we actually study on the computer.

In practice we first choose a bound d and include in $\overline{M}$ all rows of M encoding equations of type (1.3) and (1.4) with $q + s + t \le d$ (and any α). We then choose a small e and look at the finitely many columns where those rows have a nonzero entry mod p^e. We also include the rows of M corresponding to equations of type (1.5) for the variables $a_{\alpha,I}$ corresponding to these columns. Each such additional row will have nonzero entries in the columns corresponding to the variables $a_{\alpha,I}$ and $b^1_{\alpha,I}, b^2_{\alpha,I}, b^3_{\alpha,I}$ and we include precisely these columns in $\overline{M}$. We arrange the rows of $\overline{M}$ by placing all rows of type (1.5) below any rows of type (1.3) or (1.4). We order the columns by fixing an ordering on the pairs α, I that occur and group first the columns corresponding to the variables $a_{\alpha,I}$ and then those corresponding to $b^1_{\alpha,I}$, then to $b^2_{\alpha,I}$, and finally to $b^3_{\alpha,I}$. The matrix $\overline{M}$ we obtain then has the block form

$$\overline{M} = \left(\begin{array}{c|c|c|c} B & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline I & -k_1 I & -k_2 I & -k_3 I \end{array}\right). \tag{1.6}$$

Let ξ be the column vector obtained from $\overline{z}$ by including only the rows in the lower section of $\overline{M}$. Then the equation $\overline{M}\overline{x} = \overline{z}$ can be represented by the augmented matrix

$$\left(\begin{array}{c|c|c|c|c} B & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ \hline I & -k_1 I & -k_2 I & -k_3 I & \xi \end{array}\right).$$

Performing elementary row operations on this augmented matrix doesn't change the solvability of the associated system of equations, so we begin by subtracting the appropriate multiples of the lower rows from the upper rows to obtain

$$\left(\begin{array}{c|c|c|c|c} \mathbf{0} & k_1 B & k_2 B & k_3 B & -B\xi \\ \hline I & -k_1 I & -k_2 I & -k_3 I & \xi \end{array}\right).$$

The bottom rows of this augmented matrix do not put any constraints on the solvability of the associated linear system, and so we see that our equation $\overline{M}\overline{x} = \overline{z}$ has a solution if and only if

$$\left(\begin{array}{c|c|c|c|c} \mathbf{0} & k_1 B & k_2 B & k_3 B & -B\xi \end{array}\right)$$

is the augmented matrix of a system of linear equations that has a solution.

Let $\overline{M}_0$ be the matrix $\left(\mathbf{0} \mid k_1 B \mid k_2 B \mid k_3 B\right)$, so the equation $\overline{M}\overline{x} = \overline{z}$ is equivalent to $\overline{M}_0\overline{x} = -B\xi$.

Let $\overline{J} = (J + p^e R_r^0)/p^e R_r^0$ be the image of J in $R_r^0/p^e R_r^0$. Note that if $\overline{M}_0 \overline{x} \equiv -B\xi \pmod{\overline{J}}$ then for any row vector w over $R_r^0/p^e R_r^0$ of the appropriate length, $w\overline{M}_0 \overline{x} \equiv -wB\xi \pmod{\overline{J}}$ and so $wB\xi$ must be in the ideal of $R_r^0/p^e R_r^0$ generated by $\overline{J}$ and the entries of $w\overline{M}_0$.

Definition 5.1. Let L be the ideal of $R_r^0/p^e R_r^0$ generated by k_1, k_2, k_3.

Theorem 5.2. *Let e, $\overline{M}_0$, and ξ be as above. Let J be the ideal of R_r^0 cutting out the validity locus Z and let $\overline{J}$ be its projection to $R_r^0/p^e R_r^0$. Let w be a row vector over $R_r^0/p^e R_r^0$ of length the number of rows of $\overline{M}_0$ and let I be an ideal of $R_r^0/p^e R_r^0$ which contains all of the entries of wB. Then $\overline{J}$ contains an element of $wB\xi + IL$.*

Proof. Since J cuts out Z, the equation $Mx = z$ has a solution over R_r^0/J and hence there exists a column vector $\overline{x}$ over $R_r^0/p^e R_r^0$ such that $\overline{M}_0 \overline{x} \equiv -B\xi \pmod{\overline{J}}$. Thus $w\overline{M}_0 \overline{x} \equiv -wB\xi \pmod{\overline{J}}$ and $w\overline{M}_0\overline{x} + wB\xi \in \overline{J}$. Since I contains the entries of wB, and every nonzero entry in $\overline{M}_0$ equals an entry of B times either k_1, k_2 or k_3, IL contains the entries of $w\overline{M}_0$. Thus $w\overline{M}_0\overline{x} + wB\xi \in wB\xi + IL$, proving the theorem.

Q.E.D.

Definition 5.3. Let u be a row vector over $R_r^0/p^e R_r^0$. Define $\mathcal{I}(u)$ to be the ideal of $R_r^0/p^e R_r^0$ generated by the entries of u.

Note that if $wB\xi \in IL$ then the theorem gives us no information as then $wB\xi + IL = IL$ and $\overline{J}$ certainly contains $0 \in IL$. This will always be the case if $wB\xi$ is contained in $\mathcal{I}(wB)L$. So to obtain constraints on J, we seek a row vector w over $R_r^0/p^e R_r^0$ such that $\mathcal{I}(wB)L$ does not contain $wB\xi$.

Definition 5.4. Let I be an ideal of $R_r^0/p^e R_r^0$ and let w be a row vector over $R_r^0/p^e R_r^0$ such that $\mathcal{I}(wB) \subseteq I$ and $wB\xi \notin IL$. We call (I, w) a *poison pair* and say that w is a *poison row of type I*.

We will discuss three strategies for choosing I and then searching for a poison row of type I.

5.1 Looking for the tangent plane

If we let $I = L = (k_1, k_2, k_3)$, then Theorem 4.4 of [4] shows how to determine the mod p^e tangent space of Z at 0 from a poison row of type I, assuming that Z is smooth at 0. Note that if $I = L$ then $IL = L^2$ is the ideal of $R_r^0/p^e R_r^0$ generated by monomials of degree 2 in the weight variables k_i. We argued in [4] that $wB\xi$ must vanish at 0 and so $wB\xi = A_1 k_1 + A_2 k_2 + A_3 k_3 +$ higher order terms. We then proved that the tangent plane has an equation $c_1 k_1 + c_2 k_2 + c_3 k_3 = 0$ such that $c_i \equiv p^r A_i \pmod{p^e}$ for $i = 1, 2, 3$.

For this choice of $I = L$, we will have $\mathcal{I}(wB) \subseteq I$ if and only if the constant terms of all entries of wB are 0 in $\mathbb{Z}/p^e\mathbb{Z}$. So to search for poison rows of type I, let B_0 be matrix whose entries are the constant terms of the entries of B. We can then, in theory, carry out row reduction on the augmented matrix $[B_0| - B\xi]$ until the left hand block is upper triangular with the multiplicity of p dividing the diagonal entries increasing. This will allow us either to find a poison row of type I or show no such row exists for our chosen submatrix. In the latter case we can try increasing e, or including more rows in $\overline{M}$, or both.

The matrix B has many more rows than columns. We can reduce the complexity of the computation by removing some of those rows so that we are left with only a few more rows than columns. We may hope to find poison rows for these smaller linear systems, and working with them requires less computation. On the other hand, a failure to find a poison row for a matrix with fewer rows than B won't rule out the existence of a poison row for the full matrix B.

5.2 Determine I as we proceed

We don't need to specify I ahead of time, but rather can set $I = \mathcal{I}(wB)$ for each row w we want to test for poison. Given a particular w we can test $wB\xi$ for membership in the ideal $\mathcal{I}(wB)L$. This can be done, for example, using Gröbner bases [8]. If w is a poison row, we can look at the exact nature of $\mathcal{I}(wB)$ and hope that the conclusion of Theorem 5.2 gives us useful information about J.

One approach we considered for finding a suitable w is to carry out row operations directly on the augmented matrix $[B|B\xi]$. Unfortunately, since $R_r^0/p^eR_r^0$, which is the ring of polynomials in three variables over $\mathbb{Z}/p^e\mathbb{Z}$, is not a PID, we are not aware of a suitable normal form for B which would allow us to determine if a poison w exists. Nonetheless we attempted ad hoc calculations along these lines in some small cases. The calculation became large and unwieldy very quickly, and we abandoned the approach.

Note that if (I, w) is a poison pair, then w is poison of type I' for any I' with $\mathcal{I}(wB) \subseteq I' \subset I$. In particular, w is poison of type $\mathcal{I}(wB)$, and applying Theorem 5.2 to the pair $(\mathcal{I}(wB), w)$ gives the strongest constraint on the existence of a solution that we can assert using w. In the next subsection we describe a choice of I that is easier to work with than the choices above.

5.3 Specializing the weight variables

Choose values $a_1, a_2, a_3 \in \mathbb{Z}/p^e\mathbb{Z}$ not all zero and let $I = (k_1 - a_1, k_2 - a_2, k_3 - a_3)$. This amounts to specializing the weight variable k_i to equal a_i. Let f be the maximum value for which $a_1, a_2, a_3 \in p^f\mathbb{Z}/p^e\mathbb{Z}$. Then $\mathcal{I}(wB) \subset I$ if and only if all of the entries of $p^f wB$ vanish at (a_1, a_2, a_3). The factor of p^f comes from plugging a_i into the k_i-term that multiplies the corresponding block B after pivoting with the lower rows.

Such a w will be poison of type I if $wB\xi$ doesn't vanish at (a_1, a_2, a_3). We can again search for poison rows of type I by a row reduction. Let B_s be the specialization of B at $k_i = a_i$.

Then, as in the previous subsection, we can row reduce $[B_s| - B_s\xi]$ to find poison rows or determine that none exist. Any poison rows w found may then be used by applying Theorem 5.2 to $(\mathcal{I}(wB), w)$ which should give much more information than just applying it to (I, w).

Remark 5.5. Note that the calculation we have described depends on r, which is a priori unknown if f_0 isn't ordinary. In practice, we would attempt to carry out this calculation for $r = 0, 1, 2, \ldots$. By row reducing larger and larger submatrices of M, we might find many obstructions to the solution of $Mx = z$.

If we have chosen r too small for F_0 to deform all the way to the "edge" of $\mathcal{W}_r$, then we wouldn't expect these obstructions to be consistent with each other for any $\lambda \in \mathcal{W}_r$. On the other hand, if we do find consistent obstructions, we would view it as evidence that F_0 does in fact deform all the way to the "edge" of $\mathcal{W}_r$. Thus our calculations may give some idea as to the size of the neighborhood around 0 to which the eigenvariety extends. For this paper, however, we only computed with ordinary f_0.

6 Discussion of computational results

We have run our computer programs so far with three initial homology classes f_0, one with $p = 3$ and $N = 5$, one with $p = 2$ and $N = 53$ and one with $p = 3$ and $N = 79$.

The programs were written in C++. The $N = 5$ calculation was run on a Macbook Pro, the larger computations were run on computers in Wesleyan's High Performance Compute Cluster on nodes equipped with Intel Xeon E5-2660 chips. Our row reduction program became overwhelmed by fill-in in the matrix, after running for approximately two weeks (for either the N=53 or N=79 calculations) without discovering any poison rows.

6.1 $N = 5$, $p = 3$

There is a unique newform f of level 15 and weight 2. Let $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_3)$ be the associated Galois representation, and let $\omega : G_{\mathbb{Q}} \to \mathbb{Z}_3^{\times}$ be the cyclotomic character. Then $\mathrm{H}_3(\Gamma_0(5) \cap I_3, \mathbb{Z}_p)$ contains a class c_1 which is ordinary at 3 and whose associated Galois representation is $1 \oplus \omega\rho_f$ and a class c_2 which is ordinary at 3 and whose associated Galois representation is $\rho_f \oplus \omega^2$.

For both c_1 and c_2, as mentioned in section 1, unpublished calculations of David Hansen suggest there should be no obstructions in the weight to the deformation problem. Our computations are consistent with this.

We carried out similar calculations with c_1 and c_2, so let f be either of these classes. Let $\Gamma = \Gamma_0(5) \cap I_3$. Using the techniques of [1] we computed an element in $\mathrm{Ind}_{\Gamma}^{\mathrm{SL}_3(\mathbb{Z})} \mathbb{Z}_3$ representing the class f.

We first fixed $e = 2$ and calculated the mod 3^2 reduction of the rows of M corresponding to $\delta_{\alpha,(q,s,t)}$ with $q + s + t \leq 1$. There are 6448 such $\delta_{\alpha,(q,s,t)}$ and 25792 corresponding rows. It follows from the proof of Proposition 4.3 that if $(i + j + k) - (q + s + t) > 2$ then each $c^{\tau,\alpha}_{(i,j,k),(q,s,t)}$ is a multiple of 3^2. Thus we only need to consider the columns of M corresponding to $\delta_{\alpha,(i,j,k)}$ with $i+j+k \leq 3$. There are 32240 such $\delta_{\alpha,(i,j,k)}$ with $(i+j+k) \leq 3$, but only 12771 of the corresponding columns have nonzero entries in our chosen rows. Thus the matrix B appearing in equation (1.6) has 25792 rows and 12771 columns. We carried out row reduction modulo 3^2 on B_0 as discussed in section 5.1 and found that there were no poison rows.

We then worked modulo 3^3 instead of 3^2. We continued to work with the same rows of M, but now we needed to include columns corresponding to $\delta_{\alpha,(i,j,k)}$ with $i + j + k \leq 7$. (The jump from 3 to 7 here is due to fact that the denominator 3! appearing in the binomial coefficients contains a factor of 3.) There are 193440 such $\delta_{\alpha,(i,j,k)}$ with $i + j + k \leq 7$ to consider; however only 22691 of the corresponding columns have nonzero entries. The large number of 0 columns is due to the fact that only columns corresponding to a few triples (i, j, k) with $3 < i + j + j < 7$ can have nonzero entries. To search for poison rows for these matrices, we specialized the weight variables to particular values $a_1, a_2, a_3 \in \mathbb{Z}/3^3\mathbb{Z}$ as discussed in section 5.3. Again there were no poison rows.

As a check on our calculations we randomly changed approximately 5% of the entries of B, multiplying them by a random unit mod 3^3. When we row reduced these altered matrices, we always found many poison rows, as one would expect from an overdetermined system. This suggests that our calculations are correct, and the actual systems do not have obstructions.

6.2 $N = 53$, $p = 2$ and $N = 79$, $p = 3$

The $N = 53$ and $N = 79$ examples are cuspidal. The homology classes are taken from [3]. In neither of these cases have we found a poison row as of the time of writing.

Here are some sample sizes of the matrices involved: For $N = 53, p = 2$, when working mod 2^4 and choosing rows with $q + s + t \leq 1$, we have 961968 rows and 795913 nonzero columns in B after specializing (as in subsection 5.3) at $(1, 0, 0)$. For $N = 79$ working mod p^2 and again with $q + s + t \leq 1$, we have $5, 259, 072$ rows. After specializing B at $(0, 0, 0)$ (so we can look for the tangent plane as in subsection 5.1) there are $2,604,251$ nonzero columns.

The class of level 53 appears empirically to be congruent to an Eisenstein class mod $p = 2$. We still hope to find a poison row, but the congruence with the Eisenstein series may force us to look modulo a high power of p to find it. As of the time of writing, we have looked modulo p^4 but we have not yet located any poison rows.

The $N = 79$ example is also cuspidal, and does not appear to be congruent to an Eisenstein class mod 3. This should make the poison relatively easier to find, but the larger values of the parameters N and p in this case make the computational problem take up more space and time. The algorithm described in section 3 took over three and a half days just to compute the lift of the relevant homology vector from $\mathrm{Ind}_{\Gamma_0(73)\cap I_3}^{\mathrm{SL}_3(\mathbb{Z})} \mathbb{O}_3$ to $\mathrm{Ind}_{\Gamma_0(73)\cap I_3}^{\mathrm{SL}_3(\mathbb{Z})} D_0$ modulo $\mathrm{Fil}^4 \mathrm{Ind}_{\Gamma_0(73)\cap I_3}^{\mathrm{SL}_3(\mathbb{Z})} D_0$. We have not yet located any poison rows in this case either.

6.3 Future plans

Among levels less than 100, there are also cuspidal classes for $N = 61, 89$. We plan to investigate them, along with continuing computation for the preceding classes.

If we do not find any obstructions (poison rows) in any of these examples, we will alter the matrix M so that it will encode a bound on the p-adic valuation of the eigenvalue for U_p. This will increase the size of the computations, but it is more likely that we can show computationally that the projection to weight space of the deformation of a given 3-cycle is not surjective if we restrict to the slope $\leq h$ part for some $h \geq 0$. This means we will be working locally on the spectral variety, and we will not have to worry about fractal or other strange behavior as the slope tends to ∞.

Acknowledgments

We thank Wesleyan University for computer time supported by the NSF under grant number CNS-0619508 and CNS-0959856. We also thank the referee for suggestions that improved the exposition of the paper.

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