On nilpotent and solvable MR-groups

Mikheil Amaglobeli

Faculty of Exact and Natural Sciences, Ivane Javakhishvili Tbilisi State University, 13 University St., Tbilisi 0186, Georgia;
Andrea Razmadze Mathematical Institute of Ivane Javakishvili Tbilisi State University, 6 Tamarashvili Str., Tbilisi 0177, Georgia

E-mail: mikheil.amaglobeli@tsu.ge

Abstract

In the present paper, the central series and series of commutants in MR-groups are introduced. Moreover, various definitions of nilpotency in this category are compared.

2010 Mathematics Subject Classification. **20F10**. 20J15, 20E06, 20E10.
Keywords. Lyndon R-group, Hall R-group, MR-group, α-commutator, tenzor completion, R-commutant, nilpotent MR-group, solvable MR-group.

1 Preliminaries

The notion of a exponential R-group, where R is an arbitrary associative ring with unity, was introduced by R. Lyndon [13]. A. G. Myasnikov and V. N. Remeslennikov [15] gave a more precise definition of a R-group by introducing a complementary axiom. In particular, their modified notion of a exponential MR-group is the direct generalization of the notion of a R-module to the case of noncommutative groups. M. G. Amaglobeli and V. N. Remeslennikov [5] called R-groups with a complementary axiom MR-groups (R is the ring). A systematic study of MR-groups was undertaken in [15, 5, 1, 16, 10, 6, 7, 4, 8, 2, 3]. The results of these studies have turned out useful for the solution of well-known Tarski's problems.

Let us recall the basic definitions and facts from the works [13, 15].

Let $L_{gr} = \{\cdot\,,\ ^{-1}, e\}$ be the group language signature, where $\cdot$ is the binary operation of multiplication, $^{-1}$ is the unary operation of inversion, e is the constant symbol for the group unit. We enrich the group language L_{gr} up to the language $L^*_{gr} = L_{gr} \cup \{f_\alpha \mid \alpha \in R\}$, where f_α is the unary algebraic operation.

Definition 1.1 ([13])**.** A set G is called a Lyndon R-group if the operations $\cdot\,,\ ^{-1}$, e, $\{f_\alpha \mid \alpha \in R\}$ are defined on it and the following axioms are fulfilled (for brevity, the expression $f_\alpha(g)$ will be written below in the form g^α):

(a) group axioms;

(b) for all $g, h \in G$ and $\alpha, \beta \in R$, the following equalities are fulfilled:

(1) $g^1 = g$, $g^0 = e$, $e^\alpha = e$;
(2) $g^{\alpha+\beta} = g^\alpha g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$;
(3) $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$.

Tbilisi Mathematical Journal Special Issue (7 - 2021), pp. 99–106.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 20 December 2019.
Accepted for publication: 21 May 2020.

Denote by L_R the category of Lyndon R-groups. The above axioms are the identities of the language L^*_{gr} and therefore the class L_R is the variety of algebraic systems of the language L^*_{gr}, and from the general theorems of universal algebra it follows that we can speak of R-homomorphisms, free R-groups, and so on.

There exist abelian Lyndon R-groups which are not R-modules (see [9] that presents a detailed study of a free abelian R-group). To the Lyndon axioms the authors of [15] added the additional axiom (quasi identity)

(4) $\forall\, g,h \in G,\ \alpha \in R\ [g,h] = e \longrightarrow (gh)^\alpha = g^\alpha h^\alpha$ (*MR*-axiom),

where $[g,h] = g^{-1}h^{-1}gh$.

Definition 1.2 ([15])**.** A group G is called an *MR*-group if on G the operation g^α is defined for all $g \in G$, $\alpha \in R$ and the axioms (1)–(4) are fulfilled.

Denote by M_R the class of all R-groups satisfying the axioms (1)–(4). It is obvious that $L_R \supset M_R$. It is likewise clear that this class is a quasi variety in the language L^*_{gr} and for it there are the notions of a free *MR*-group, R-homomorphism, and so on. Moreover, each abelian *MR*-group is a R-module and vice versa.

Most of natural examples of exponential R-groups are provided by the class M_R. For example, a free Lyndon exponential R-group is an *MR*-group, a unipotent group over the field K of zero characteristic is an *MR*-group, an arbitrary pro-p-group is an $M\mathbb{Z}_p$-group over the ring of integer p-adic numbers $\mathbb{Z}_p$, and so on (for other examples see [15]).

Definition 1.3 ([15])**.** A homomorphism of R-groups $\varphi : G \to G^*$ is called a ***R-homomorphism*** if $\varphi(g^\alpha) = \varphi(g)^\alpha$ for any $g \in G$, $\alpha \in R$.

Definition 1.4 ([15])**.** For $g,h \in G$, $\alpha \in R$, we call the element $(g,h)_\alpha = h^{-\alpha}g^{-\alpha}(gh)^\alpha$ the α-commutator of elements g and h.

It is obvious that for $\alpha = -1$ the α-commutator $(g,h)_\alpha$ coincides with an ordinary commutator $[h^{-1}, g^{-1}]$. We clearly have $(gh)^\alpha = g^\alpha h^\alpha (g,h)_\alpha$ and $G \in M_R \iff ([g,h] = e \implies (g,h)_\alpha = e)$. The latter equivalence leads to the definition of an M_R-ideal.

Definition 1.5 ([15])**.** A normal *MR*-subgroup $H \trianglelefteq G$, $G \in L_R$ is called an M_R-ideal if $(g,h)_\alpha \in H$ for any $g,h \in G$, $\alpha \in R$.

Proposition 1.6 ([15])**.** Let $G \in L_R$.

(a) If $\varphi : G \to G^*$ is a R-homomorphism of groups from M_R, then $\operatorname{Ker}\varphi$ is an M_R-ideal in G.

(b) If H is an M_R-ideal in G, then $G/H \in M_R$.

In [15] it is shown that n the study of exponential *MR*-groups the key role is played by the operation of tensor completion. This operation naturally generalizes the notion of scalar rings extension for modules to the noncommutative case. The idea of such generalization for the class of nilpotent groups is expounded in [14].

Definition 1.7 ([15])**.** Let G be an *MR*-group, $\mu : R \to S$ be a ring homomorphism. Then an MS-group $G^{S,\mu}$ is called the tensor S-completion of an *MR*-group G if $G^{S,\mu}$ satisfies the following universal properties:

(a) there exists an R-homomorphism $\lambda : G \to G^{s,\mu}$ such that $\lambda(G)$ S-generates $G^{s,\mu}$, i.e. $\langle\lambda(G)\rangle_S = G^{s,\mu}$;

(b) for any MS-group H and any R-homomorphism $\varphi : G \to H$ consistent with μ (i.e. $\varphi(g^\alpha) = (\varphi(g))^{\mu(\alpha)}$) there exists an S-homomorphism $\psi : G^{s,\mu} \to H$ that makes the diagram

$$\begin{tikzcd} G \arrow[r, "\lambda"] \arrow[d, "\varphi"'] & G^{S,\mu} \arrow[dl, dashed, "\exists\,\psi"] \\ H & \end{tikzcd} \qquad (\varphi = \lambda\psi)$$

commutative.

Note that if G is an abelian *MR*-group, them $G^{S,\mu} \cong G \underset{R}{\otimes} S$ is the tensor product of a R-module G by the ring S. In [15] it is proved that for any *MR*-group G and any homomorphism $\mu : R \to S$ the tensor completion $G^{S,\mu}$ exists and is unique with an accuracy of an R-homomorphism. The realization of tensor completion of an *MR*-group in the form of a concrete construction by the technique of combinatorial theory of groups is exponded in [2].

2 Nilpotent R-groups

Let $c > 1$ be a natural number. Denote by $\mathcal{N}_{c,R}$ the category of nilpotent R-groups of nilpotence c from the class L_R, i.e. of the R-groups where the identity

$$\forall\, x_1, \ldots, x_{c+1} \;\; [x_1, \ldots, x_{c+1}] = e$$

is fulfilled, and by $\mathcal{N}^0_{c,R}$ the category of nilpotent *MR*-groups of step c. The structure of R-groups without the axiom of choice (MR) is very complicated and that's why only the *MR*-group is studied in most of the works. In the rest of this paper only the *MR*-groups will be considered.

Hall *MR*-groups. In order to introduce this notion, following [11] we need to restrict the class of considered rings.

Definition 2.1. A ring R is called ***a binomial ring*** if R is the integer domain containing $\mathbb{Z}$ as a subring and, together with every element $\alpha \in R$, contains all binomial coefficients

$$C^n_\alpha = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\,, \;\; n \in \mathbb{N}.$$

Any field of zero characteristic, a ring of polynomials over the field and a ring of integer numbers are examples of binomial rings.

Definition 2.2. A nilpotent group G of nilpotence c is called ***a Hall R-group*** (R is a binomial ring) if an element $x^\alpha \in G$ is uniquely defined for any $\alpha \in R$ and $x \in G$, and the following axioms ($x, y, x_1, \ldots, x_n \in G,\; \alpha, \beta \in R$) are fulfilled for all elements of the group G and the ring R:

(1) $x^1 = x$, $x^{\alpha+\beta} = x^\alpha x^\beta$, $x^{\alpha\beta} = (x^\alpha)^\beta$;

(2) $(y^{-1}xy)^\alpha = y^{-1}x^\alpha y$;

(3) $x_1^\alpha \cdots x_n^\alpha = (x_1, \ldots, x_n)^\alpha \tau_2^{C_\alpha^2}(X) \cdots \tau_c^{C_\alpha^c}(X)$, where $X = \{x_1, \ldots, x_n\}$, $\tau_k(X)$ is the k-th Petrescu word.

Recall that for any natural k the k-th Petrescu word is recursively defined by the formula $\tau_k(X) = \tau_k(x_1, \ldots, x_n) = \tau_1^{C_k^1}(X)\tau_2^{C_k^2}(X) \cdots \tau_{k-1}^{C_k^{k-1}}(X)\tau_k^{C_k^k}(X)$ in a free group F with the generators $x_1, \ldots, x_n$. In particular $\tau_1(X) = x_1x_2 \cdots x_n$, $\tau_2(X) = \prod\limits_{i>j,\ i,j=1}^{n} [x_i, x_j] \mod \gamma_3(F)$, where $\gamma_3(F)$ is the third member of the lower third series of the group F. It is well known (see e.g. [11]) that for any $n \in \mathbb{N}$, $\tau_k(X)$ belongs to a subgroup $\gamma_k(F)$, i.e. to the k-th member of the third central series of the group F, $\gamma_1(F) = F$.

Denote by $\mathcal{HN}_{c,R}$ the variety of nilpotent elements of the class $\leq c$ of Hall R-groups.

If G happens to be abelian, then the axioms (1)–(3) reduce to ordinary axioms of R-modules. Indeed, $\tau_2, \tau_3, \ldots$ lie in $G' = e$ (see [11, Theorem 6.3]). Thus an arbitrary exponential nilpotent Hall R-group over the binomial ring R is an *MR*-group. Any torsion-free finitely generated nilpotent group may serve as an example of a Hall exponential *MR*-group. A particular case is the group $UT_n(R)$ of all unitriangular matrices with elements from R (for other examples see [11]).

Let us show that the structure of groups from $\mathcal{N}_{c,R}$ much differs from the structure of Hall R-groups from $\mathcal{HN}_{c,R}$. For this, we reduce the structure of a free *MR*-group in the variety $\mathcal{HN}_{c,R}$ as this is done in the work [5] of M. G. Amaglobeli and V. N. Remeslennikov. Our consideration will be limited to two binomial rings $R = \mathbb{Q}[t]$, $R = \mathbb{Q}(t)$. Denote by G_0 a free 2-step nilpotent R-group in the variety $\mathcal{HN}_{c,R}$ with generators x, y. It is well known that the Maltsev base of this group consists of three elements x, y, $[y,x]$. A general form of an element $g \in G_0$ is $g = x^\gamma y^\delta [y,x]^\varepsilon$, $\gamma, \delta, \varepsilon \in R$. In particular, in this group the commutant G_0' is a free R-module of rank 1 with a generator $[y,x]$. If now G is a free *MR*-group in the variety $\mathcal{HN}^0_{c,R}$, then, as shown in [5], G' is a free R-module of infinite rank and the base of this module is found.

Series of commutants in *MR*-groups. Let G be an arbitrary *MR*-group. Assume

$$(G,G)_R = \big\langle (g,h)_\alpha \mid \ \ g,h \in G, \ \ \alpha \in R \big\rangle_R.$$

We will call a subgroup $(G,G)_R$ a ***R*-commutant** of the group G.

Using general theorems of the theory of group varieties (see e.g. [17]) it is not difficult to prove

Proposition 2.3. For any *MR*-group G the following statements are true:

(1) a R-commutant of G is a verbal *MR*-subgroup defined by the word $[x,y] = x^{-1}y^{-1}xy$;

(2) a R-commutant is the smallest M_R-ideal by which the factor group is abelian.

For $G \in M_R$, we call a R-commutant $(G,G)_R$ ***the first R-commutant*** and denote it by $G^{(1,R)}$. A R-commutant of $G^{(1,R)}$ is called ***the second R-commutant*** and denoted by $G^{(2,R)}$, and so on. There arises a decreasing series of R-commutants

$$G = G^{(0,R)} \geq G^{(1,R)} \geq \cdots \geq G^{(n,R)} \geq \cdots . \tag{2.1}$$

Definition 2.4. A exponential *MR*-group G is called ***solvable*** of there exists a natural number n such that $G^{(n,R)} = e$.

By induction with respect to n it is easy to show that the ordinary n-th commutant $G^{(n)}$ is contained in $G^{(n,R)}$. Hence an n-step solvable group in the category M_R is n-step solvable in the category of groups.

Let us proceed to the definition of the lower central series in the category of exponential *MR*-groups. The first member of this series is the R-commutant of the group G which we denote by $G_{(1,R)}$. Assume that the n-th member of the lower central series $G_{(n,R)}$ has already been defined. Then $G_{(n+1,R)} = id([G, G_{(n,R)}])$, i.e. $G_{(n+1,R)}$ is the M_R-ideal generated by the reciprocal commutant of G and $G_{(n,R)}$. There arises the lower central series

$$G = G_{(0,R)} \geq G_{(1,R)} \geq \cdots \geq G_{(n,R)} \geq \cdots . \tag{2.2}$$

Definition 2.5. A exponential *MR*-group will be called l***ower R-nilpotent*** if there exists a natural number n such that $G_{(n,R)} = e$. The smallest number n with such a property is called ***the step of R-nilpotence***.

Since the ordinary member of the lower central series $G_{(R)}$ is contained in $G_{(n,R)}$, the n-step lower nilpotent group in the category M_R is a nilpotent group of step $\leq n$ in the category of groups. From the definition of series (2.1), (2.2) and the definition of a verbal *MR*-subgroup it directly follows that for any natural number n and ring R the groups $G^{(n,R)}$ and $G_{(n,R)}$ are verbal *MR*-subgroups. Hence there arise the following questions.

Question 1. Is it true that $G^{(n,R)} = idG^{(n)}$, $G_{(n,R)} = idG_{(n)}$, where $G^{(n)}$ is the n-th member of an ordinary series of commutants, and $G_{(n)}$ is the n-th member of the lower central series?

This question can be reformulated as follows:

(a) is the verbal subgroup $G^{(n,R)}$ generated by the word $v_n = [v_{n-1}(\overline{x}), v_{n-1}(\overline{y})]$, where $v_1 = [x, y]$?

(b) is the verbal subgroup generated by the commutator $[x_1, \ldots, x_n]$?

Question 2.

(a) Will an n-step nilpotent *MR*-group be an n-step lower R-nilpotent group?

(b) Will an n-step solvable *MR*-group be n-step R-solvable?

Series (2.1), (2.2) can be continued up to any ordinal α. If α is not a limit ordinal, then $G^{(\alpha,R)}$ is obtained from $G^{(\alpha-1,R)}$, while $G_{(\alpha,R)}$ is obtained from $G_{(\alpha-1,R)}$ in the manner described above. If α is a limit ordinal, then

$$G^{(\alpha,R)} = \bigcap_{\beta<\alpha} G^{(\beta,R)}, \quad G_{(\alpha,R)} = \bigcap_{\beta<\alpha} G_{(\alpha,R)}.$$

Question 3. Let $F = F_R(X)$ be a free *MR*-group with base X. Do for any ring R there exist the ordinals α and β depending on R and such that $F^{(\alpha,R)} = e$ and $F_{(\alpha,R)} = e$?

We denote by $\underline{N}_{n,R}$ the class of lower R-nilpotent groups of step n. We also introduce other definitions of nilpotence in the category of step *MR*-groups. For this, by induction with respect to n we define the notion of a simple $\overline{\alpha}$-commutator of weight n, where $\overline{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})$. If $n = 2$, then $\overline{\alpha} = (\alpha)$ is the above-defined α-commutator $(g_1, g_2)_\alpha$ of elements g_1, g_2 from G. Assume that for $n \geq 2$ the simple $\overline{\alpha}$-commutators of weight n have already been defined. Then a simple $(\overline{\alpha}, \alpha_n)$-commutator is an element $(x, g_n)_{\alpha_n}$, where x is a simple $\overline{\alpha}$-commutator. Further, let $X = \{x_1, x_2, \dots\}$ be the set of letters. Denote by W_n the set $W_n = \{(\cdots((x_1, x_2)_{\alpha_1}, x_3)_{\alpha_2}, \dots, x_{n+1})_{\alpha_n} \mid \alpha_1, \dots, \alpha_n \in R\}$ of all simple $\overline{\alpha}$-commutators of weight $n+1$ of the letters $x_1, \dots, x_{n+1}$. Denote by $N_{n,R}$ the group variety defined by the set of R-words W_n. The groups of this variety are called ***R-nilpotent*** *MR*-groups of nilpotence step $\leq n$.

We denote by $\overline{N}_{n,R}$ the variety of R-groups defined by the word $v_n = [\cdots[[x_1, x_2], x_3], \dots, x_{n+1}]$. The groups of this variety are called ***upper nilpotent*** groups of step $\leq n$. The corresponding verbal *MR*-subgroup is denoted by $\overline{\gamma}_{n,R}$. We obviously have the inclusions $\underline{N}_{n,R} \subseteq N_{n,R} \subseteq \overline{N}_{n,R}$. Let us clarify the nature of these inclusions for small values of n.

Theorem 2.6. For $n = 1, 2$, all the three definitions of nilpotence coincide.

Proof. Let $n = 1$. Since, by assumption, the R-commutant $G_{(1,R)}$ of the group G is a verbal *MR*-subgroup generated by the commutator $[x, y] = x^{-1}y^{-1}xy$, we have $G_{(1,R)} = \gamma_{1,R}(G) = \overline{\gamma}_{1,R}(G)$. Hence it follows that all the three definitions of exponential *MR*-groups being abelian coincide.

Let $n = 2$. Let us prove that $\gamma_{2,R}(G) = \overline{\gamma}_{2,R}(G)$. It suffices to prove this equality only for groups from the variety $\overline{N}_{2,R}$, i.e. for groups that are 2-step nilpotent in the group category. Let $G \in \overline{N}_{2,R}$. From the commutator relations, for any group from this class and any $\alpha \in R$ we have $[x^\alpha, y] = [x, y]^\alpha$. To prove the equality $\gamma_{2,R}(G) = \overline{\gamma}_{2,R}(G)$ it suffices to show that $\gamma_{2,R}(G) = e$. In other words, any simple $\overline{\alpha}$-commutator of weight 3 is equal to e. To prove the latter statement, it suffices to check that any $\overline{\alpha}$-commutator of the form $(x_1, x_2)_\alpha$ belongs to the center $Z(G)$ of the group G. Indeed,

$$\begin{aligned} [(x_1, x_2)_\alpha, y] &= \left[x_2^{-\alpha} x_1^{-\alpha} (x_1 x_2)^\alpha, y\right] = [x_2^{-\alpha}, y]\,[x_1^{-\alpha}, y]\,[(x_1 x_2)^\alpha, y] \\ &= [x_2, y]^{-\alpha}[x_1, y]^{-\alpha}[x_1 x_2, y]^\alpha = [x_2, y]^{-\alpha}[x_1, y]^{-\alpha}[x_1, y]^\alpha[x_2, y]^\alpha = e. \end{aligned}$$

The check that $G_{(2,R)} = \gamma_{2,R}(G)$ is quite simple since $G_{(2,R)} = id([G_{(1,R)}, G])$ and $G_{(1,R)}$ as an *MR*-subgroup is generated by α-commutators which belong to the center of G. Q.E.D.

In [15] it is stated that tensor completions of abelian groups are abelian groups. In the general case the tensor completion in the category of all power *MR*-groups is constructed by using free structures and therefore in the noncommutative case it contains free subgroups. Nevertheless the following statement is true.

Theorem 2.7. If $G \in N_{2,R}$, then its tensor completion $G^S \in N_{2,S}$.

Proof. Preliminarily, we prove that in any exponential R-group G, for any $g, f \in G$ and $\alpha, \beta \in R$ the following identity is fulfilled for α-commutators

$$[g^\alpha, f] = [g, f]^\alpha \big(g, [g, f]\big)_\alpha. \tag{2.3}$$

Indeed, the axiom (3) of the definition of an *MR*-group states that $(f^{-1}gf)^\alpha = f^{-1}g^\alpha f$ for all $f, g \in G$ and $\alpha \in R$. We rewrite this equality raking into account that

$$\begin{gathered} f^{-1}gf = g^\alpha g^{-\alpha} f^{-1} g^\alpha f = g^\alpha [g^\alpha, f], \\ (f^{-1}gf)^\alpha = (gg^{-1}f^{-1}gf)^\alpha = \big[g[g, f]\big)^\alpha = g^\alpha [g, f]^\alpha \big(g, [g, f]\big)_\alpha. \end{gathered}$$

Cancelling g^α, we obtain the required result.

Let us prove the theorem. Denote by Z the R-center of the group G. Since the nilpotence step of G is equal to 2, the R-commutant $\gamma_1(G) \subseteq Z$. A straightforward check shows that Z^S is the central subgroup of the group G^S. Let us show that $\gamma_1(G^S) \subseteq Z^S$. Since $\gamma_1(G^S)$ is generated by α-commutators, for this it suffices to prove that any α-commutator is contained in Z^S. Since G^S is generated by $\langle\lambda(G)\rangle_S$ and since the ordinary commutant lies at the center, it is easy to check by using the commutator relations [12, p. 171] that the ordinary commutant is at the center. Furthermore, the identity (3) shows that in this case $[x^\alpha, y] = [x, y]^\alpha$. We check that the α-commutator $(x, y)_\alpha$ lies at the center of G^S:

$$\begin{aligned}[(x,y)_\alpha, z] = \left[y^{-\alpha}x^{-\alpha}(xy)^\alpha, z\right] &= [y^{-\alpha}, z]\,[x^{-\alpha}, z]\,[(xy)^\alpha, z] \\ &= [y,z]^{-\alpha}[x,z]^{-\alpha}[xy,z]^\alpha = [y,z]^{-\alpha}[x,z]^{-\alpha}[x,z]^\alpha[y,z]^\alpha = e.\end{aligned}$$

Q.E.D.

References

[1] M. G. Amaglobeli, *On the permutability of a functor of tensor completion with principal group operations*, Appl. Math. Inform. Mech. **15** (2010), no. 1, 3–10, 54.

[2] M. G. Amaglobeli, *The tensor completion functor in categories of exponential MR-groups*, Algebra Logika **57** (2018) no. 2, 137–148 (in Russian).

[3] M. Amaglobeli, *Exponential MR-groups: faithful R-completion*, Doklady Mathematics **93** (2019), no. 2, 33–35.

[4] M. G. Amaglobeli, Varieties of power *MR*-groups. *Doklady Mathematics* **101** (2020), no. 1, 1–4.

[5] M. G. Amaglobeli and V. N. Remeslennikov, *Free nilpotent R-groups of class 2*, Dokl. Akad. Nauk **443** (2012), no. 4, 410–413 (in Russian); translation in Dokl. Math. **85** (2012) no. 2, 236–239.

[6] M. G. Amaglobeli and V. N. Remeslennikov, *Extension of a centralizer in nilpotent groups*, Sibirsk. Mat. Zh. **54** (2013) no. 1, 8–19 (in Russian).

[7] M. Amaglobeli and V. Remeslennikov, *Algorithmic problems for class-2 nilpotents MR-groups*, Georgian Math. J. **22** (2015) no. 4, 441–449.

[8] M. G. Amaglobeli and V. N. Remeslennikov, *Foundations of the theory of varieties of nilpotent MR-groups*, Sibirsk. Mat. Zh. **57** (2016) no. 6, 1197–1207 (in Russian); translation in Sib. Math. J. **57** (2016) no. 6, 935–942.

[9] G. Baumslag, *Free abelian X-groups*, Illinois J. Math. **30** (1986) no. 2, 235–245.

[10] G. Baumslag, A. Myasnikov and V. Remeslennikov, *Discriminating completions of hyperbolic groups*, Geom. Dedicata **92** (2002) 115–143.

[11] Ph. Hall, *The Edmonton Notes on Nilpotent Groups*. Queen Mary College Mathematics Notes. Mathematics Department, Queen Mary College, London (1969).

[12] M. Hall, Jr. *The Theory of Groups*. Reprinting of the 1968 edition. Chelsea Publishing Co., New York (1976).

[13] R. C. Lyndon, Groups with parametric exponents. Trans. Amer. Math. Soc. **96** (1960), 518–533.

[14] A. G. Myasnikov and V. N. Remeslennikov, *Definability of the set of Maltsev bases and elementary theories of finite-dimensional algebras. II*, Sibirsk. Mat. Zh. **24** (1983) no. 2, 97–113 (in Russian).

[15] A. G. Myasnikov and V. N. Remeslennikov, *Degree groups. I. Foundations of the theory and tensor completions*, Sibirsk. Mat. Zh. **35** (1994) no. 5, 1106–1118 (in Russian); translation in Siberian Math. J. **35** (1994) no. 5, 986–996.

[16] A. G. Myasnikov and V. N. Remeslennikov, *Exponential groups. II. Extensions of centralizers and tensor completion of CSA-groups*, Internat. J. Algebra Comput. **6** (1996) no. 6, 687–711.

[17] H. Neumann, *Varieties of Groups*, Springer-Verlag New York, Inc., New York (1967).