Notes on Chow rings of flag varieties G/B and classifying spaces BG

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Abstract

Let G be a connected compact Lie group and T its maximal torus. The composition of maps $H^*(BG) \to H^*(BT) \to H^*(G/T)$ is zero for positive degree, while it is far from exact. We change $H^*(G/T)$ by Chow ring $CH^*(X)$ for X some twisted form of G/T, and change $H^*(BG)$ by $CH^*(BG)$. Then we see that it becomes near to exact but still not exact, in general. We also see that the difference for exactness relates to the generalized Rost motive in X.

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1 Introduction

Let p be a prime number. Let G and T be a connected compact Lie group and its maximal torus. Given a field k with $ch(k) = 0$, let G_k and T_k be a split reductive group and a split maximal torus over the field k, corresponding to G and T. Let B_k be the Borel subgroup containing T_k. Let us write by BG_k its classifying space of G_k defined by Totaro [To1].

For a smooth algebraic variety X over k (resp. toplogical space), let $CH^*(X) = CH^*(X)_{(p)}$ (resp. $H^*(X) = H^*(X)_{(p)}$) mean p-localized Chow ring over k (resp. p-localized ordinary cohomology ring). In general, to compute $CH^*(BG_k)$ or $H^*(BG)$ are difficult problems. At first, we consider them modulo torsion elements. We consider the following diagram

$$(1.1) \qquad \begin{array}{ccc} CH^*(BG_k)/Tor & \xrightarrow[i^*]{(1)} & CH^*(BB_k)^W \\ {\scriptstyle (2)}\downarrow{\scriptstyle cl} & & \cong\downarrow \\ H^*(BG)/Tor & \xrightarrow[i^*]{(3)} & H^*(BT)^W \end{array}$$

where Tor is the ideal generated by torsion elements, cl is the cycle map, and $W = N_G(T)/T$ is the Weyl group.

When $H^*(G)$ is torsion free, we know that $Tor = 0$ and all maps $(1), (2), (3)$ are isomorphic, and $H^*(BT)^W$ is well known. So we assume that $H^*(G)$ have p-torsion, throughout this paper. By the existence of the Becker-Gottlieb transfer, the maps $(1), (3)$ are injections. Moreover when G is simply connected, (1) is always not surjective ([Ya3]), while for many cases (3) are surjective. (For cases that (3) are not surjective are founded by Feshbach [Fe], Benson-Wood [Be-Wo]). In any way, $CH^*(BG_k)/Tor$ is isomorphic to a proper subring of $CH^*(BB_k)^W$ for each simply connected G.

To study $CH^*(BG_k)/Tor$, we consider twisted flag varieties. Let $\mathbb{G}$ be a G_k-torsor. Then $\mathbb{F} = \mathbb{G}/B_k$ is a (twisted) form of the flag variety G_k/B_k. The fibering $G/T \xrightarrow{j} BT \xrightarrow{i} BG$ induces

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the maps

$$(1.2)\quad CH^*(BG_k) \stackrel{i^*}{\to} CH^*(BB_k) \stackrel{j^*}{\to} CH^*(\mathbb{F}),$$

whose composition $j^*i^*=0$ for $*>0$. But it is far from exact when $\mathbb{G}\cong G_k$ the split group. Here exact means $Ker(j^+)=Ideal(Im(i^+))\subset CH^+(BB_k)$ (where + means the positive degree parts). However, we observe that it becomes near exact when $\mathbb{G}$ is sufficient twisted, while it is still not exact for most cases. To see this fact, let us write the difference

$$(1.3)\quad D_{CH}(\mathbb{G})=Ker(j^+(\mathbb{G}))/(Ideal(Im(i^+)).$$

Note that this invariant $D_{CH}(\mathbb{G})$ becomes smaller, if $\mathbb{F}$ becomes strongly twisted. In particular. we will see that it is quite small for the versal flag variety $CH^*(\mathbb{F})$.

Here *versal* is defined as follows. Let us consider an embedding of G_k into the general linear group GL_N for some large N. This makes GL_N a G_k-torsor over the quotient variety $S=GL_N/G_k$. Define the *versal G_k-torsor* E to be the G_k-torsor over the function field $k(S)$ given by the generic fiber of $GL_N\to S$. (For details, see [Ga-Me-Se], [To2], [Me-Ne-Za], [Ka].) The corresponding flag variety $E/B_{k(S)}$ is called the *versal* flag variety, which is considered as the most complicated twisted flag variety (for given G_k). It is known that the Chow ring $CH^*(E/B_{k(S)})$ is not dependent to the choice of generic G_k-torsors E (Remark 2.3 in [Ka]).

In this paper, a versal G_k-torsor $\mathbb{G}$ means this $G_{k(S)}$-torsor E, and Chow ring $CH^*(\mathbb{G}/B_k)$ means this $CH^*(E/B_{k(S)})$, which is defined over $k(S)$ but not k. Exchanging k to $k(S)$ in (1,2), we also define $D_{CH}(\mathbb{G})$ (note $CH^*(BB_k)\cong CH^*(BB_{k(S)})$). Moreover, when G is of type (I) (see §2 below), it is known $CH^*(\mathbb{G}/B_k)\cong CH^*(\mathbb{G}'/B_k)$ for the versal $\mathbb{G}$ (over $k(S)$) and each non-trivial G_k-torsor $\mathbb{G}'$.

By Petrov-Semenov-Zainoulline ([Pe-Se-Za], [Se-Zh]), it is known that the p-localized motive $M(\mathbb{F})_{(p)}$ of $\mathbb{F}$ is decomposed as

$$(1.4)\quad M(\mathbb{F})_{(p)}=M(\mathbb{G}/B_k)_{(p)}\cong R(\mathbb{G})\otimes(\oplus_i\mathbb{T}^{\otimes s_i})$$

where $\mathbb{T}$ is the Tate motive and $R(\mathbb{G})$ is some motive called generalized Rost motive. (It is the original Rost motive ([Ro], [Vo1,2], [Pe-Se-Za], [Ya4]) when G is of type (I)). Hence we have maps

$$(1.5)\quad CH^*(BB_k)\stackrel{j^*}{\to} CH^*(\mathbb{F})\stackrel{pr.}{\to} CH^*(R(\mathbb{G})).$$

From Merkurjev and Karpenko [Me-Ne-Za], [Kar], we know that the first map j^* is also surjective when $\mathbb{G}$ is a versal G_k-torsor.

For ease of computations, we mainly consider the $mod(p)$ theories for (1.2)

$$(1.6)\quad CH^*(BG_k)/p\stackrel{i_p^*}{\to} CH^*(BB_k)/p\stackrel{j_p^*}{\to} CH^*(\mathbb{F})/p.$$

Let us define $D(\mathbb{G})=Ker(j_p^+)/(Ideal(Im(i_p^+))$. Then we see

Lemma 1.1. Let $\mathbb{G}$ be versal. Then we have the surjection

$$pr: D(G_k)/D(\mathbb{G})\to CH^+(R(\mathbb{G}))/p.$$

We will see that $D(\mathbb{G})$ are quite small in some cases. For example we have

Theorem 1.1. Let $(G,p)=(SO(2\ell+1),2)$ and $\mathbb{G}$ be versal. Then $D(\mathbb{G})\cong 0$, that is the above sequence (1.6) is exact.

Theorem 1.2. Let $(G(N),p)=(Spin(N),2)$ and $\mathbb{G}(N)$ be versal. Then we have $lim_{\infty\leftarrow N}D(\mathbb{G}(N))=0$.

Recall that $CH^*(BB_k)\cong S(t)=\mathbb{Z}_{(p)}[t_1,...t_\ell]$ with $|t_i|=2$. Let us write c_i the i-th elementary symmetric function in $S(t)$ and let $e=c_1^4$. The notation $\Lambda(a,...,b)$ means the $\mathbb{Z}/2$-exterior algebra generated by $a,...,b$

Proposition 1.3. Let $(G,p)=(Spin(7),2)$ and $\mathbb{G}$ be versal. Then we have additively

$$D(\mathbb{G})\cong \Lambda(c_2c_3,e_4)^+\otimes S(t,c)\quad for\ S(t,c)=S(t)/(c_2,c_3,e_4).$$

The plan of this paper is the following. In §2, we recall the Chow ring $CH^*(\mathbb{F})$ for a nontrivial G_k-torsor $\mathbb{G}$. In §3 we note some elementary relations between $CH^*(\mathbb{F})$ and $CH^*(BG_k)$. In §4 we note some facts for $CH^*(BB_k)^W/Tor$. In §5, §6, we try to compute $D(\mathbb{G})$ for $G=PU(p),SO(n)$. In §7, §8, we try to study $D(\mathbb{G})$ for $G=Spin(n)$ for general n. In §9, §10, we study $Spin(7),Spin(9)$. In §11, §12 we study the case $(G,p)=(F_4,3)$. In §13, we study the case $G=E_6,E_7$ and $p=3$.

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2 $CH^*(\mathbb{G}/B_k)$

We recall arguments for $H^*(G/T)$ in the algebraic topology. By Borel, the mod(p) cohomology of the Lie group G is (for p odd)

$$H^*(G;\mathbb{Z}/p)\cong P(y)/p\otimes\Lambda(x_1,...,x_\ell),\quad |x_i|=odd$$

where $P(y)$ is a truncated polynomial ring over $\mathbb{Z}_{(p)}$ generated by *even* dimensional elements y_i, and $\Lambda(x_1,...,x_\ell)$ is the $\mathbb{Z}/p$-exterior algebra generated by $x_1,...,x_\ell$. When $p=2$, we consider the graded ring $grH^*(G;\mathbb{Z}/2)$ which is isomorphic to the right hand side ring above.

When G is simply connected and $P(y)$ is generated by just one generator, we say that G is of type (I). Except for $(E_7,p=2)$ and $(E_8,p=2,3)$, all exceptional (simple) Lie groups are of type (I). The spin groups $G=Spin(n)$ are of type (I) for $7\le n\le 10$. Note that in these cases, it is known $rank(G)=\ell\ge 2p-2$.

We consider the fibering ([Tod2], [Mi-Ni]) $G\xrightarrow{\pi} G/T\xrightarrow{i} BT$ and the induced spectral sequence

$$(2.1)\quad E_2^{*,*}=H^*(BT;H^*(G;\mathbb{Z}/p))\Longrightarrow H^*(G/T;\mathbb{Z}/p).$$

Here we can write $H^*(BT)\cong S(t)=\mathbb{Z}[t_1,...,t_\ell]$ with $|t_i|=2$.

It is well known that $y_i\in P(y)$ are permanent cycles and that there is a regular sequence $(\bar{b}_1,...,\bar{b}_\ell)$ in $H^*(BT)/(p)$ such that $d_{|x_i|+1}(x_i)=\bar{b}_i$ ([Tod2], [Mi-Ni]).

We know that G/T is a manifold such that $H^*(G/T)$ is torsion free and is generated by even degree elements. We also see that there is a filtration in $H^*(G/T)_{(p)}$ such that

$$grH^*(G/T)_{(p)}\cong P(y)\otimes S(t)/(b_1,...,b_\ell)$$

where $b_i\in S(t)$ with $b_i=\bar{b}_i\ mod(p)$.

Recall $BP^*(-)$ is the Brown-Peterson theory with the coefficient $BP^* = \mathbb{Z}_{(p)}[v_1,...]$, $|v_i| = -2(p^i-1)$. Then we have

$$grBP^*(G/T) \cong BP^* \otimes grH^*(G/T).$$

Let $Q_i : H^*(X;\mathbb{Z}/p) \to H^{*+2p^n-1}(X;\mathbb{Z}/p)$ be the Milnor operation. There is a relation between Q_i-action on $H^*(X;\mathbb{Z}/p)$ and v_i-action on $BP^*(X)$.

Lemma 2.1. Let $d(x) = b \neq 0 \in H^*(BT;\mathbb{Z}/p)$ in the above spectral sequence (2.1). Then we can take a lift $b \in BP^*(BT)$ such that

$$b = \sum_{i=0} v_i y(i) \in \ BP^*(G/T)/I_\infty^2 \quad \textit{with } I_\infty = (p, v_1, ...)$$

where $y(i) \in H^*(G/T;\mathbb{Z}/p)$ with $\pi^* y(i) = Q_i x$.

For the algebraic closure $\bar{k}$ of k, let us write $\bar{X} = X|_{\bar{k}}$. Then considering (2.1) over $\bar{k}$, we see

$$(2.2) \quad CH^*(\bar{R}(\mathbb{G}))/p \subset P(y)/p, \quad CH^*(\oplus_i \mathbb{T}^{\otimes s_i}) \cong S(t)/(b_1,...,b_\ell).$$

Moreover when $\mathbb{G}$ is versal, we can see ([Ya4]) that $CH^*(R(\mathbb{G}))$ is additively generated by products of $b_1,...,b_\ell$ in (2.2) i.e., $CH^*(\bar{R}(\mathbb{G})/p \cong P(y)$. Hence we have surjections $CH^*(BB_k) \to CH^*(\mathbb{F}) \stackrel{pr}{\to} CH^*(R(\mathbb{G}))$.

For ease of notations, let us write

$$A(b) = \mathbb{Z}/p[b_1,...,b_\ell], \quad (b) = Ideal(b_1,...,b_\ell) \subset S(t)/p.$$

By giving the filtration on $S(t)$ by b_i, we can write (additively)

$$grS(t)/p \cong A(b) \otimes S(t)/(b).$$

Namely, $x \in S(t)/p$ is written as

$$x = \sum_I b(I)t(I) \quad \textit{for } b(I) \in A(b), \quad \textit{and} \ \ 0 \neq t(I) \in S(t)/(b).$$

In particular, we have maps $A(b) \stackrel{i_A}{\to} CH^*(\mathbb{F})/p \to CH^*(R(\mathbb{G}))/p$. We also see that this composition map is surjective.

Lemma 2.2. ([Ya4]) Suppose that there are $f_1(b),...,f_s(b) \in A(b)$ such that

$$CH^*(R(\mathbb{G}))/p \cong A(b)/(f_1(b),...,f_s(b)).$$

Moreover if $f_i(b) = 0$ for $1 \le i \le s$ also in $CH^*(\mathbb{F})/p$, we have the isomorphism

$$CH^*(\mathbb{F})/p \cong S(t)/(p, f_1(b),...,f_s(b)).$$

For a ring B, let $B\{a,...,b\}$ mean the B-free module generated by $a,...,b$.

Lemma 2.3. Let $pr : CH^*(\mathbb{F})/p \to CH^*(R(\mathbb{G}))/p$, and $0 \neq b \in Ker(pr)$. Then $b = \sum b'u'$ with $b' \in A(b)$, $u' \in S(t)^+/(p, b_1,...,b_\ell)$ i.e., $|u'| > 0$.

Using these, we can prove

Theorem 2.1. ([Ya4]) Let G be of type (I) and $rank(G)=\ell$. Let $\mathbb{G}$ be a non-trivial G_k-torsor. Then $2p-2\le \ell$, and we can take $b_i\in S(t)=CH^*(BB_k)$ for $1\le i\le \ell$ such that there are isomorphisms

$$CH^*(R(\mathbb{G}))/p\cong \mathbb{Z}/p\{1,b_1,...,b_{2p-2}\},$$
$$CH^*(\mathbb{G}/B_k)/p\cong S(t)/(p,b_ib_j,b_k|1\le i,j\le 2p-2<k\le \ell).$$

We note that the above theorem also holds when $\mathbb{G}$ is versal.

3 Relation between $\mathbb{G}/B_k$ and BG

In this section, we consider $CH^*(X)/I$ for some ideal I (e.g., $CH^*(X)/p$). Let us write it simply $h^*(X)$ and $I=I(h)$.

We note here the following lemma for each G_k-torsor $\mathbb{G}$ (not assumed twisted).

Lemma 3.1. For the above $h^*(X)$, the composition of the following maps is zero for $*>0$

$$h^*(BG_k)\to h^*(BB_k)\to h^*(\mathbb{G}/B_k).$$

Proof. Take U (e.g., GL_N for a large N) such that U/G_k approximates the classifying space BG_k [To3]. Namely, we can take $\mathbb{G}=f^*U$ for the classifying map $f:\mathbb{G}/G_k\to U/G_k$. Hence we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{F}=\mathbb{G}/B_k & \longrightarrow & U/B_k \\ \downarrow & & \downarrow \\ Spec(k)\cong \mathbb{G}/G_k & \longrightarrow & U/G_k \end{array}$$

where U/B_k (resp. U/G_k) approximates BB_k (resp. BG_k). Since $h^*(Spec(k))=CH^*(Spec(k))/I(h)=0$ for $*>0$, we have the lemma. Q.E.D.

The above sequences of maps in the lemma is not exact, in general. However we get some informations from $h^*(\mathbb{F})$ to $h^*(BG_k)$. In particular, we get much informations of $h^*(BG_k)$ from $h^*(\mathbb{F})$ than that from $h^*(G_k/B_k)$ when $\mathbb{F}$ is twisted.

Let us write the induced maps

$$h^+(BG_k)\stackrel{i^+}{\to} h^+(BB_k)\stackrel{j(\mathbb{G})^+}{\to} h^+(\mathbb{G}/B_k)$$

where $h^+(-)$ is the ideal of the positive degree parts. Let us define

$$D_h(\mathbb{G})=Ker(j^+)/(Ideal(Im(i^+))).$$

Let $\mathbb{G}$ be versal and k' is some extension of k. Then

$$D_h(\mathbb{G})\subset D_h(\mathbb{G}|_{k'})\subset D_h(G|_{\bar{k}})\cong D_h(G_k).$$

For ease of arguments we mainly consider the case $h^*(X)=CH^*(G)/p$, and write this $D_h(\mathbb{G})$ simply by $D(\mathbb{G})$.

Recall $grS(t)/p\cong A(b)\otimes S(t)/(b)$. For $f_1,...,f_s\in A(b)$, let us write by

$$A(b)(f_1,...,f_s)\quad (resp.\ S(t)(f_1,...,f_s))$$

the ideal in $A(b)$ (resp. $S(t)/p$) generated by $f_1,...,f_s$. Then it is almost immediately

Lemma 3.2. We can write additively

$$S(t)(f_1,...,f_s)\cong A(b)(f_1,...,f_s)\otimes S(t)/(b).$$

Proof. Each element $x\in S(t)(f_1,...,f_s)$ can be written as

$$x=\sum_J(\sum_i b(J)_if_i)t(J),\quad for\ b(J)_i\in A(b),\ \ 0\neq t(J)\in S(t)/(b).$$

Q.E.D.

Lemma 3.3. Let $\mathbb{G}$ be versal. Then there are maps

$$D(G_k)/D(\mathbb{G})\subset CH^*(\mathbb{F})/p\stackrel{pr}{\to}CH^*(R(\mathbb{G}))/p,$$

such that $pr(D(G_k)/D(\mathbb{G}))=CH^+(R(\mathbb{G}))/p$.

Proof. We consider the map $S(t)/p\cong CH^*(BB_k)/p\stackrel{j^*(\mathbb{G})}{\to}CH^*(\mathbb{G}/B_k)/p$. By the definition, we have

$$D(G_k)/(D(\mathbb{G}))\cong(Ker(j^*(G_k)/Im(i^+))/(Ker(j^*(\mathbb{G}))/Im(i^+))$$
$$\cong Ker(j^*(G_k))/Ker(j^*(\mathbb{G}))\subset S(t)/(Kerj^*(\mathbb{G}))\cong CH^*(\mathbb{F})/p.$$

Recall that $CH^*(G_k/B_k)/p\cong P(y)\otimes S(t)/(b)$. So $Ker(j(G_k))=(b)$. From lemma 3.2,

$$(b)=S(t)(b_1,...,b_\ell)\cong(A(b)(b_1,...,b_\ell)\otimes S(t)/(b)\cong A(b)^+\otimes S(t)/(b).$$

Since $prS(t)/(b)=\mathbb{Z}/p\{1\}$ and from Lemma 2.3, we have the lemma. Q.E.D.

Corollary 3.1. Let $\mathbb{G}$ be versal. Suppose there are $f_1(b),...,f_s(b)$ in $A(b)$ such that

$$CH^*(\mathbb{F})/p\cong S(t)/(p,f_1(b),...,f_s(b)).$$

Then $D(G_k)/D(\mathbb{G})\cong CH^+(R(\mathbb{G}))/p\otimes S(t)/(b)$.

Proof. The ideal $Ker(j^+(\mathbb{G}))$ is shown from

$$Kerj^*(\mathbb{G})\cong S(t)(f_1(b),...,f_s(b))\cong(A(b)(f_1(b),...,f_s(b))\otimes S(t)/(b).$$

Hence we have $D(G_k)/D(\mathbb{G})\cong A(b)^+/(f_1(b),...,f_s(b))\otimes S(t)/(b)$. Q.E.D.

Corollary 3.2. Let $\mathbb{G}$ be versal, and assume the supposition in Lemma 2.2. Moreover assume $Im(i^*)\subset A(b)$. Then there is $\tilde{D}(\mathbb{G})\subset D(\mathbb{G})$ such that

$$D(\mathbb{G})\cong\tilde{D}(\mathbb{G})\otimes S(t)/(b).$$

From above corollaries, we have a very weak version of the decomposition theorem by Petrov-Semenov-Zainoulline [Pe-Se-Za], without using deep theories of motives.

Corollary 3.3. Let $\mathbb{G}$ be versal, and assume the supposition in Lemma 2.2. Then we have an additive decomposition of the $mod(p)$ Chow ring

$$CH^*(\mathbb{G}/B_k)/p \cong S(t)/(b) \oplus D(G_k)/D(\mathbb{G})$$

$$\cong (\mathbb{Z}/p\{1\} \oplus CH^+(R(\mathbb{G}))/p) \otimes S(t)/(b) \cong CH^*(R(\mathbb{G})) \otimes S(t)/(b).$$

Example. Let G be of type (I). Then

$$Kerj^+(G_k) \cong Ideal(b_1, ..., b_\ell) \subset S(t)/p = CH^*(BB_k)/p,$$

$$Kerj^+(\mathbb{G}) \cong Ideal(b_ib_j, b_k | 1 \le i, j \le 2p-2 < k \le \ell) \subset S(t)/p.$$

Hence $D(G_k)/D(\mathbb{G}) \cong \mathbb{Z}/p\{b_1, ..., b_{2p-2}\} \otimes S(t)/(b)$.

4 $CH^*(BG)/Tor$

By Totaro, we have the Becker-Gottlieb transfer also in $CH^*(X)$. Hence we get the injection

$$(4.1) \quad CH^*(BG_k)/Tor \subset CH^*(BT)^W$$

for the Weyl group $W = N_G(T)/T$. From [Ya3], the above injection is always not surjective when $H^*(G)$ has p-torsion. In general, to get $CH^*(BG_k)$ is a difficult problem, but $CH^*(BG_k)/Tor$ seems more accessible.

Recall that $gr^*_{geo}(X)$ (resp. $gr^*_{top}(X)$) is the graded ring associated with the geometric (resp. topological) filtration of the algebraic K-theory $K^0_{alg}(X)$ (resp. the topological K-theory $K^0_{top}(X)$). Namely, it is isomorphic to the infinite term $E_\infty^{2*,*,0}$ (resp. $E_\infty^{2*,0}$) of the motivic (resp. usual) Atiyah-Hirzebruch spectral sequence.

Lemma 4.1. There is an isomorphism

$$CH^*(BG_k)/Tor \cong gr^*_{geo}(BG_k)/Tor.$$

Moreover if $CH^*(BG_k) \to gr^*_{top}(BG)/Tor$ (resp. $(BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)})/Tor$) is surjective, then

$$CH^*(BG_k)/Tor \cong gr^*_{top}(BG)/Tor \quad (resp.\ (BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)})/Tor).$$

Proof. We consider the commutative diagram

$$\begin{array}{ccc} CH^*(BG_k)/Tor & \xrightarrow{(1)} & CH^*(BB_k) \\ {\scriptstyle (2)}\downarrow & & \downarrow{\scriptstyle \cong} \\ gr^*_{geo}(BG_k)/Tor & \xrightarrow{(3)} & gr^*_{geo}(BB_k) \end{array}$$

There is the Becker-Gottlieb transfer, the map (1) is injective. Moreover the map (2) is surjective, and we have the first isomorphism. The second isomorphism follows from exchanging $gr^*_{geo}(-)$ by $gr^*_{top}(-)$ (or by $BP^*(-) \otimes_{BP^*} Z_{(p)}$). Q.E.D.

On the other hand Totaro defines the modified cycle map $\bar{cl}$ such that the composition $\rho\cdot\bar{cl}$

$$(4.2)\quad CH^*(X)\stackrel{\bar{cl}}{\to} BP^*(X)\otimes_{BP^*}\mathbb{Z}_{(p)}\stackrel{\rho}{\to} H^*(X;\mathbb{Z}_{(p)})$$

is the usual clycle map cl. Moreover Totaro conjectures that $\bar{cl}$ is isomorphic when $X=BG$ and $k=\bar{k}$. More weakly, if the modified cycle map $\bar{cl}$ $mod(Tor)$ is surjective, then we have $CH^*(BG_k)/Tor\cong (BP^*(BG)\otimes_{BP^*}\mathbb{Z}_{(p)})/Tor$.

By arguments similar to the proof of Lemma 4.1, (using $CH^*(B\bar{B}_k)\cong CH^*(BB_k)$) we have

Lemma 4.2. If $res: CH^*(BG_k)/Tor\to CH^*(B\bar{G}_k)/Tor$ is surjective, then it is isomorphic.

Corollary 4.1. Let G be simply connected. If $CH^*(B\bar{G}_k)/Tor$ is generated by Chern classes, then $res: CH^*(BG_k)/Tor\cong CH^*(B\bar{G}_k)/Tor$.

Proof. When G is simply conned, by Chevalley, we know $res: K^0(BG_k)\cong K^0(B\bar{G}_k)$. Hence a map $B\bar{G}_k\to BU(N)$ can be lift to a map $BG_k\to BU(N)$. This implies that any Chern class in $CH^*(B\bar{G}_k)$ can be lift to an element in $CH^*(BG_k)$. Q.E.D.

5 $PGL(3)$ for $p=3$

Now we consider in the case $(G,p)=(PU(p),p)$, which has p-torsion in cohomology, but it is not simply connected. Its mod p cohomology is

$$H^*(G;\mathbb{Z}/p)\cong \mathbb{Z}/p[y]/(y^p)\otimes\Lambda(x_1,...,x_{p-1})\quad |y|=2,\ |x_i|=2i-1.$$

So $P(y)/p\cong\mathbb{Z}/p[y]/(y^p)$ with $|y|=2$.

Since G is not simply connected, G is not of type (I) while $P(y)$ is generated by only one y. (Indeed, $CH^*(X)/p$ resembles that of type (I). Compare Theorem 2.4 and Theorem 5.2 below.)

By using the map $U(p-1)\to PU(p)$, we know $d_{2i}(x_i)=c_i$ for the elementary symmetric function in $H^*(BT_{U(p)})$. Then we have

$$grH^*(G/T;\mathbb{Z}/p)\cong\mathbb{Z}/p[y]/(y^p)\otimes S(t)/(c_1,...,c_{p-1}).$$

Lemma 5.1. We have $py^i=c_i\in H^*(G/T)_{(p)}$.

Theorem 5.1. Let $G=PU(p)$ and $\mathbb{F}=\mathbb{G}_k/B_k$. Then there are isomorphisms

$$CH^*(R(\mathbb{G}_k))/p\cong CH^*(R_1)/p\cong \mathbb{Z}/p\{1,c_1,...,c_{p-1}\},$$

$$CH^*(\mathbb{F})/p\cong S(t)/(p,c_ic_j|1\le i,j\le p-1).$$

By Vistoli [Vi], it is known that $CH^*(BG)/Tor\cong CH^*(BB_k)^W$. However its ring structure is not mentioned except for $p=3,5$. (As additive groups it isomorphic to $\mathbb{Z}_{(p)}[c_2,...,c_p]$, but they are not isomorphic as rings.)

We compute here $D(\mathbb{G})$ only for $PU(3)$

$$(*)\quad CH^*(\mathbb{F})/3\cong S(t)/(3,c_1^2,c_1c_2,c_2^2).$$

By Vistoli and Vezzosi (Theorem 14.2 in [Vi]), we have

$$CH^*(BG_k)/Tor \cong \mathbb{Z}_{(3)}[c_2', c_3', c_6']/(27c_6' - 4(c_2')^3 - (c_3')^2).$$

Each element c_i' is written using c_i in $(S(t) = CH^*(BB_k)$ (see page 48 in [Vi]) as

$$c_2' = 3c_2 - c_1^2, \quad c_3' = 27c_3 - 9c_1c_2 + 2c_1^3, \quad c_6' = 4c_2^3 + 27c_3^2 \ mod(c_1).$$

Hence the map i^* $mod(3)$ is given as

$$(**) \quad c_2' \mapsto -c_1^2, \quad c_3' \mapsto -c_1^3, \quad c_6' \mapsto c_2^3 \ mod(c_1).$$

Proposition 5.2. Let $(G,p) = PU(3), 3)$ and $\mathbb{G}$ be versal. Then

$$D(\mathbb{G}) \cong \mathbb{Z}/3\{c_1c_2, c_2^2, c_1c_2^2\} \otimes S(t)/(c_1, c_2).$$

Proof. The result follows form $(*).(**)$ and the quotient

$$(c_1^2, c_1c_2, c_2^2)/(c_1^2, c_1^3, c_2^3)$$

of ideals in $CH^*(B_k)/3 \cong S(t)/3$. Q.E.D.

6 $SO(2\ell+1)$

We consider the orthogonal groups $G = SO(m)$ and $p = 2$. The $mod(2)$-cohomology is written as (see for example [Tod-Wa], [Ni])

$$grH^*(SO(m); \mathbb{Z}/2) \cong \Lambda(x_1, x_2, ..., x_{m-1})$$

where $|x_i| = i$, and the multiplications are given by $x_s^2 = x_{2s}$.

For ease of argument, we only consider the case $m = 2\ell + 1$ so that

$$H^*(G; \mathbb{Z}/2) \cong P(y) \otimes \Lambda(x_1, x_3, ..., x_{2\ell-1})$$

$$grP(y)/2 \cong \Lambda(y_2, ..., y_{2\ell}), \quad letting\ y_{2i} = x_{2i} \ \ (hence\ y_{4i} = y_{2i}^2).$$

The Steenrod operation is given as $Sq^k(x_i) = \binom{i}{k}(x_{i+k})$. The Q_i-operations are given by Nishimoto [Ni]

$$Q_n x_{2i-1} = y_{2i+2^{n+1}-2}, \qquad Q_n y_{2i} = 0.$$

In particular, $Q_0(x_{2i-1}) = y_{2i}$ in $H^*(G; \mathbb{Z}/2)$. It is well known that the transgression $b_i = d_{2i}(x_{2i-1}) = c_i$ is the i-th elementary symmetric function on $S(t)$. (this element c_i is also represented by the i-th Chern class.) Hence we have

Lemma 6.1. We have an isomorphism

$$grH^*(G/T) \cong P(y) \otimes S(t)/(c_1, ..., c_\ell).$$

Moreover, the cohomology $H^*(G/T)$ is computed completely by Toda-Watanabe [Tod-Wa] (e.g. $2y_{2i} = c_i \ mod(4)$). In $BP^*(G/T)/I_\infty^2$, we have a relation from Lemma 2.1 and the result by Nishimoto

$$(6.1) \quad c_i = 2y_{2i} + v_1y_{2i+2} + ... + v_jy_{2i+2(2^j-1)} + ...$$

Let T be a maximal torus of $SO(m)$ and $W = W_{SO(m)}(T)$ its Weyl group. Then $W \cong S_\ell^\pm$ is generated by permutations and change of signs so that $|S_k^\pm| = 2^k k!$. Hence we have

$$H^*(BT)^W \cong \mathbb{Z}_{(2)}[p_1, ..., p_\ell] \subset H^*(BT) \cong \mathbb{Z}_{(2)}[t_1, ..., t_\ell], \ |t_i| = 2$$

where the Pontriyagin class p_i is defined by $\Pi_i(1 + t_i^2) = \sum_i p_i$.

Here we recall for the Stiefel-Whitney classes w_i,

$$H^*(BG; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, ..., w_{2\ell+1}], \quad Q_0(w_{2i}) = w_{2i+1} \ mod(w_sw_t).$$

It is known $H^*(BG)$ has no higher 2-torsion and

$$H(H^*(BG; \mathbb{Z}/2); Q_0) \cong (H^*(BG)/Tor) \otimes \mathbb{Z}/2$$

where $H(A; Q_0)$ is the homology of A with the differential Q_0. This homology is isomorphic to $\mathbb{Z}/2[w_2^2, ..., w_{2\ell}^2]$. Hence we have

$$H^*(BG)/Tor \cong D \quad where \quad D = \mathbb{Z}_{(2)}[c_2, c_4, ..., c_{2\ell}],$$

for the Chern classes c_i. The isomorphism $j^* : H^*(BG)/Tor \to H^*(BT)^W$ is given by $c_{2i} \mapsto p_i$.

Now we consider the $mod(2)$ Chow ring when $\mathbb{G}$ is the split group G_k.

Lemma 6.2. We have the additive isomorphism

$$D(G_k) \cong \Lambda(c_1, .., c_\ell)^+ \otimes S(t, c) \quad with \ S(t, c) \cong S(t)/(c_1, ..., c_\ell).$$

Proof. Recall that

$$CH^*(G_k/B_k)/2 \cong H^*(G/T)/2 \cong P(y)/2 \otimes S(t)/(c_1, ..., c_\ell).$$

Hence we see

$$Ker(j) \cong (c_1, ..., c_\ell) \subset CH^*(BB_k)/2 \cong H^*(BT)/2.$$

Here $j : p_i \mapsto c_i^2 \ mod(2)$ by definition of the Pontryagin class p_i.

On the other hand, we know by Totaro [To1]

$$CH^*(B\bar{G}_k) \cong BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(2)} \cong \mathbb{Z}_{(2)}[c_2, ..., c_{2\ell+1}]/(2c_{odd}).$$

In fact, $CH^*(B\bar{G}_k)/Tor \cong CH^*(BG_k)/Tor$ from Lemma 4.3. Hence

$$CH^*(BG_k)/Tor \cong D \cong H^*(BT)^W$$

by $i : c_{2i} \mapsto p_i$. Thus the ideal generated by the image is $(Im(i)) \cong (c_2, c_4, ..., c_{2\ell}) \subset S(t)$. Since $j : p_i \mapsto c_i^2$, we have

$$Ker(j)/(Im(i)) \cong (c_1, ..., c_\ell)/(c_1^2, ..., c_\ell^2) \subset S(t)/(c_1^2, ..., c_\ell^2).$$

It is additively isomorphic to $\Lambda(c_1, ..., c_\ell)^+ \otimes S(t)/(c_1, ..., c_\ell)$, namely, each element $x \in D(G_k)$ is written as $x = \sum_I c(I)t(I)$ with $c(I) \in \Lambda(c_1, ..., c_\ell)^+$ and $t(I) \neq 0 \in S(t)/(2, c_1, ..., c_\ell)$. Q.E.D.

Recall that there is a surjection $D(G_k)\to CH^+(R(\mathbb{G}))/p$ from Lemma 3.3. We can see $c_1...c_\ell\neq 0$ in $CH^*(R(\mathbb{G}))/2$ (for example, using the torsion index $t(G)=2^\ell$ [To2]).

Theorem 6.1. (Petrov [Pe], [Ya4]) Let $(G,p)=(SO(2\ell+1),2)$ and $\mathbb{F}=\mathbb{G}/B_k$ be versal. Then $CH^*(\mathbb{F})$ is torsion free, and

$$CH^*(\mathbb{F})/2\cong S(t)/(2,c_1^2,...,c_\ell^2),\quad CH^*(R(\mathbb{G}))/2\cong \Lambda(c_1,...,c_\ell).$$

Corollary 6.2. Let $(G,p)=(SO(2\ell+1),2)$ and $\mathbb{G}$ be versal. Then $D(\mathbb{G})\cong 0$.

Proof. We have $Ker(j^+)\cong (c_1^2,...,c_\ell^2)\cong Ideal(Im(i^+))$ for $j^*: CH^*(BB_k)/2\to CH^*(\mathbb{F})/2$. Q.E.D.

7 $BSpin(n)$ for $p=2$

In this section, we study Chow rings for the cases $G=Spin(n)$, $p=2$. Recall that the $mod(2)$ cohomology is given by Quillen [Qu]

$$H^*(BSpin(n);\mathbb{Z}/2)\cong \mathbb{Z}/2[w_2,...,w_n]/J\otimes \mathbb{Z}/2[e]$$

where $e=w_{2^h}(\Delta)$ and $J=(w_2,Q_0w_2,...,Q_{h-2}w_2)$. Here w_i is the Stiefel-Whitney class for the natural covering $Spin(n)\to SO(n)$. The number 2^h is the Radon-Hurwitz number, dimension of the spin representation Δ (which is the representation $\Delta|_C\neq 0$ for the center $C\cong \mathbb{Z}/2\subset Spin(n)$). The element e is the Stiefel-Whitney class w_{2^h} of the spin representation Δ.

Hereafter this section we always assume $G=Spin(n)$ and $p=2$. For the projection $\pi: Spin(n)\to SO(n)$, the maximal torus T of $Spin(n)$ is given $\pi^{-1}(T')$ for the maximal torus T' of $SO(n)$, and $W=W_{Spin(n)}(T)\cong W_{SO(n)}(T')$. Benson-Wood [Be-Wo] determined $H^*(BT)^W$ and proved

Theorem 7.1. (Benson-Wood Corollary 8.4 in [Be-Wo]) Let $G=Spin(n)$ and $p=2$. Then $i_H^*: H^*(BG)\to H^*(BT)^W$ is surjective if and only if $n\le 10$ or $n\neq 3,4,5\ mod(8)$ (i.e., it is not the quaternion case).

Moreover, in this section, we assume $Spin(n)$ is in the real case [Qu], that is $n=8\ell-1, 8\ell+1$ (hence i_H^* is surjective and $h=4\ell-1,4\ell$ respectively).

Benson and Wood define invariants q_i, $\eta_{\ell-1}$ such that

$$(1)\quad q_1=1/2p_1,\quad q_i^2=2q_{i+1}\quad with\ |q_i|=2^{i+1},$$

$$(2)\quad \eta_{\ell-1}^2=i^*(c_{2^h}(\Delta_{\mathbb{C}}))=i^*(e^2),\quad |\eta_{\ell-1}|=2^h.$$

In fact in $H^*(BT)^W$, it is defined as $\eta_{\ell-1}=\Pi_{I\subset\{2,...,\ell\}}(q_1-(\Sigma_{i\in I}x_i))$.

Then Benson-Wood prove

Theorem 7.2. (Theorem 7.1 in [Be-Wo]) If $n=2\ell+1\ge 7$, then

$$H^*(BT)^W\cong \mathbb{Z}_{(2)}[p_2,...,p_\ell,\eta_{\ell-1}]\otimes \Lambda_{\mathbb{Z}}(q_1,...,q_{\ell-2})$$

where $\Lambda_{\mathbb{Z}}(a_1,...,a_k)$ is the free module generated by $a_1^{\varepsilon_1}...a_k^{\varepsilon_k}$ for $\varepsilon_i=0,1$.

On the other hand, by Kono [Ko], $H^*(BG;\mathbb{Z})$ has no higher 2-torsion,

$$H(H^*(BG;\mathbb{Z}/2);Q_0)\cong (H^*(BG)/Tor)\otimes \mathbb{Z}/2.$$

Benson and Wood also define $s_i\in H^*(BSO(n);\mathbb{Z}/2)$ such that

$$Q_0(s_i)=Q_i(w_2)\quad mod(s_1,...,s_{i-1})$$

and hence $s_i\in H(H^*(BG;\mathbb{Z}/2);Q_0)$. So we can identify $s_i\in H^*(BG)/Tor$.

Corollary 7.3. ([Be-Wo]) The cohomology $H^*(BG)/Tor$ is isomorphic

$$D_\ell\otimes \Lambda_{\mathbb{Z}}(s_3,...,s_\ell,e)\quad with\ D_\ell=\mathbb{Z}_{(2)}[c_4,c_6,...,c_{2\ell},c_{2^h}]$$

where $c_i=w_i^2$ are lifts in $H^*(BG;\mathbb{Z})/Tor$ of the same named elements in $H^*(BG;\mathbb{Z}/2)$.

The map i^* is given with modulo (decomposed elements)

$$c_{2i}\mapsto p_i,\quad e\mapsto \eta_{\ell-1},\quad s_i\mapsto q_{i-2}.$$

For actions of Q_i on $H^*(BG;\mathbb{Z}/2)$, we use the following lemma, which I learned from Koichi Inoue.

Lemma 7.1. Let us write $(W)=\mathbb{Z}/2[w_2,...,w_n]^+$. In $H^*(BSO(N);\mathbb{Z}/2)$. we have

$$(1)\quad Q_i(w_j)=\begin{cases} w_{j+2^i-1}\quad mod(W^2)\quad if\ j=even\\ 0\ mod(W^2)\quad j=odd.\end{cases}$$

$$(2)\quad when\ N<2^{i+1}-1+j,\ \ Q_i(w_j)=w_jw_{2^{i+1}-1}\ mod(W^3).$$

Lemma 7.2. Let $2^i<2\ell+1$. Then we can take $s_{i-1}=w_{2^i}\ mod(W^2)$. The element s_{i-1} is not in the image of the cycle map from the Chow ring.

Proof. By Inoue's lemma,

$$Q_0(s_i)=Q_i(w_2)=w_{2^{i+1}+1}\ mod(W^2).$$

Hence $s_i=w_{2^{i+1}}\ mod(W^2)$.

Since $Q_i(x)=0$ for each class x in the $mod(2)$ Chow ring, the second statements follows from

$$Q_1(w_{2^{i+1}})=w_{2^{i+1}+3}\notin J\ mod(W^2)\quad when\ 2^i<2\ell-1.$$

For $2^i=2\ell$, we have $Q_i(w_{2^{i+1}})=w_{2^{i+1}-1}w_{2^{i+1}}\notin J\ mod(W^3)$. Q.E.D.

In our (real) case, it is known [Qu] that each maximal elementary abelian 2-group A has $rank_2A=h+1$ and $e|A=\Pi_{x\in H^1(B\bar{A};\mathbb{Z}/2)}(z+x)$. Here we identify $A\cong C\oplus\bar{A}$ and

$$H^*(BC;\mathbb{Z}/2)\cong\mathbb{Z}/2[z],\quad H^*(B\bar{A};\mathbb{Z}/2)\cong\mathbb{Z}/2[x_1,...,x_h].$$

The Dickson algebra is written as a polynomial algebra

$$\mathbb{Z}/2[x_1,....,x_h]^{GL_h(\mathbb{Z}/2)}\cong\mathbb{Z}/2[d_0,....,d_{h-1}].$$

where d_i is defined as $e|A=z^{2^h}+d_{h-1}z^{2^{h-1}}+...+d_0z$. We can also identify $d_i=w_{2^h-2^i}(\Delta)\in H^*(BG;\mathbb{Z}/2)$ [Qu].

Lemma 7.3. (Corollary 2.1 in [Sc-Ya]) We have

$$Q_{h-1}e = d_0 e \quad and \quad Q_k e = 0 \quad for\ 0 \le k \le h-2.$$

Thus we know that $e = \eta_{\ell-1}$ is not in the image from $CH^*(BG)$. Let us consider $i^*/2 : CH^*(BG) \to CH^*(BT)/2$ (but not to $CH^*(BT)^W/2$).

Conjecture 7.4. Let $G = Spin(2\ell+1)$ be of real type. Then we have

$$Im(i^*/2(CH^*(BG_k)) \cong D_\ell/2 = \mathbb{Z}/2[c_4, c_6, ..., c_{2\ell}, c_{2^h}] \subset H^*(BT)/2.$$

We will see that the above conjecture is true when $G = Spin(7), Spin(9)$, and some weaker version for $Spin(\infty)$.

We consider the motivic cohomology so that

$$CH^*(X)/2 \cong H^{2*,*}(X;\mathbb{Z}/2).$$

The degree is given $deg(w_i) = (i,i)$ and $deg(c_i) = (2i,i)$. The cohomology operation Q_i exists in the motivic cohomology with $deg(Q_i) = (2^{i+1}-1, 2^i-1)$. Hence

$$Q_iQ_0(w_2) \in H^{2*,*}(BG_k;\mathbb{Z}/2) \cong CH^*(BG_k)/2.$$

Using these facts, we can see

Theorem 7.5. ([Ya1]) The ring $CH^*(BSpin(n)_k)/2$ has a subring

$$RQ(n) = \mathbb{Z}/2[c_2, ..., c_n]/(Q_1Q_0w_2, ..., Q_{n-1}Q_0w_n) \otimes \mathbb{Z}/2[c_{2^h}(\Delta_{\mathbb{C}})]$$

where c_i is the Chern class for $Spin(n) \to SO(n) \to U(n)$ and $c_{2^h}(\Delta_{\mathbb{C}})$ is that of complex representation for Δ.

Proof. This theorem is proved in [Ya1] for $k = \bar{k}$. It is well known $K^*(BG_k) \cong K^*(BG_{\bar{k}})$. Hence we see all Chern classes in $CH^*(BG_{\bar{k}})$ can be extended for $CH^*(BG_k)$. (see Corollary 4.3.) Q.E.D.

Lemma 7.4. Let $m = 2\ell+1$ and G be real type. Then we have the isomorphism

$$i^*(RQ(m)) \cong D_\ell/2 = \mathbb{Z}/2[c_4, c_6, ..., c_{2\ell}, c_{2^h}(\Delta_{\mathbb{C}})].$$

Proof. The element $Q_0Q_jw_2$ exists as a zero element in $CH^*(BG(m))/2$. The element

$$c_{2i+1} = w_{2i+1}^2 = Q_0(w_{2i})w_{2i+1} = Q_0(w_{2i}w_{2i+1})$$

also exists in $CH^*(BG(m))$ and 2-torsion. Q.E.D.

8 $\mathbb{G}/B_k$ for $G = Spin(n)$

In this section, let $G'' = SO(2\ell+1)$ and $G = Spin(2\ell+1)$. It is well known that $G''/T'' \cong G/T$ for the maximal tori T'', T for the orthogonal and spin groups. By definition, we have the 2 covering $\pi : G \to G''$. We see that $\pi^* : H^*(G/T) \cong H^*(G''/T'')$. Let $2^t \le \ell < 2^{t+1}$, i.e. $t = [log_2 \ell]$. The mod 2 cohomology is

$$H^*(G;\mathbb{Z}/2) \cong P(y) \otimes \Lambda(x_3, x_5, ..., x_{2\ell-1}) \otimes \Lambda(z), \quad |z| = 2^{t+2} - 1$$

where $P(y) \cong P(y)''/(y_2)$ where $P(y)''$ is the $P(y)$ in $grH^*(G'';\mathbb{Z}/2)$ given in §7. That is,

$$grP(y) \cong \otimes_{2i \ne 2^j} \Lambda(y_{2i}) \cong \Lambda(y_6, y_{10}, y_{12}, ..., y_{2\bar{\ell}})$$

where $\bar{\ell} = \ell - 1$ if $\ell = 2^j$ for some j, and $\bar{\ell} = \ell$ otherwise.

The Q_i operation for z is given by Nishimoto [Ni]

$$Q_0(z) = \sum_{i+j=2^{t+1}, i<j} y_{2i}y_{2j}, \quad Q_n(z) = \sum_{i+j=2^{t+1}+2^{n+1}-2, i<j} y_{2i}y_{2j} \quad for\ n \ge 1.$$

We know that

$$grH^*(G/T)/2 \cong P(y) \otimes S(t)/(2, c_2, ..., c_\ell, c_1^{2^{t+1}}).$$

Here $c_i = \pi^*(c_i)$ and $d_{2^{t+2}}(z) = c_1^{2^{t+1}}$ in the spectral sequence converging $H^*(G/T)$.

The Chow ring $CH^*(R(\mathbb{G}))/2$ is not computed yet (for general ℓ), while we have the following lemmas.

Lemma 8.1. Let $G = Spin(2\ell+1)$, $\mathbb{G}$ is versal, and $2^t \le \ell < 2^{t+1}$. Then there is a surjection

$$\Lambda(c_2, ..., c_{\bar{\ell}}) \otimes \mathbb{Z}/2[e_{2^{t+1}}] \to CH^*(R(\mathbb{G}))/2.$$

where $c_i = \pi^*(c_i)$ and $e_j = c_1^j$ in $S(t) \cong H^*(BT)$ for $\pi : G \to G'' = SO(2\ell+1)$.

Lemma 8.2. We have

$$i^*(c_{2i}) = (c_i)^2, \quad i^*(c_{2i+1}) = 0, \quad i^*(c_{2^h}(\Delta_{\mathbb{C}})) = e_{2^{t+1}}^{2^{h-t-1}}.$$

Proof. The first equation is well known (see Lemma 7.3 in [Ya4]), in fact $c_i^2 = 0$ in $CH^*(\mathbb{G}''/B_k)$ is proved using $CH^*(BG_k'')$ for $G'' = SO(n)$. The second equation follows from $P^1c_{2i} = c_{2i+1}$ and $P^1((c_i)^2) = 0$. The last equation follows from the fact Δ is spin representation(which is nonzero in the restriction on $\mathbb{Z}/2$ (recall $e_{2^{t+1}} = c_1^{2^{t+1}}$). Q.E.D.

Lemma 8.3. Let $G(n) = Spin(n)$ and $\mathbb{G}(n)$ be versal. Then given $n \ge 1$, there is $N \ge 7$ such that

$$CH^*(R(\mathbb{G}(N))/2 \cong \Lambda(c_2, ..., c_n) \quad for\ all\ * \le n.$$

Proof. Let $N = 2\ell+1$, and $2^2 \le 2^n < 2^t \le \ell < 2^{t+1}$.
We will see

$$CH^*(R(\mathbb{G}(N))/2 \cong \Lambda(c_2, ..., c_\ell) \quad for\ * < 2^n.$$

Suppose that

$$x=\sum c_{i_1}...c_{i_s}=0\in CH^*(R(\mathbb{G}))/2 \quad for \ \ 2\le i_1<...<i_s<2^n.$$

Recall $k(n)^*=\mathbb{Z}/p[v_n]$ and $k(n)^*(\bar{R}(\mathbb{G}))\cong k(n)^*\otimes P(y)$. We note that in $k(n)^*(\bar{R}(\mathbb{G})$

$$c_{i_j}=v_ny_{2m} \quad with \ \ m=2^n-1+i_j.$$

Since $2^n<m<2^{n+1}$, the number m is not a form 2^r, $r>3$. Hence y_{2m} is a generator of $grP(y)$.

Moreover recall that

$$e_{2^{t+1}}=v_ny_{2^{t+1}-2+2^n-2}y_{2^t+2}+...$$

This element is in the $ideal(v_n^2,E)$ with $E=(y_{2j}|j>2^t)$. Hence we see $c_{i,j}=v_ny_{2m}$ is also nonzero $mod(v_n^2,E)$ since $n<t$.

Thus we see that $x'=y_{2^n-2+2i_1}....y_{2^n-2+2i_s}$, which is an additive generator of $P(y)$. Hence it is also $k(n)^*$-module generators of $k(n)^*(\bar{R}(\mathbb{G}))$. We consider the element (in $k(n)^*(\bar{R}(\mathbb{G}))$)

$$x''=res(\sum c_{i_1}...c_{i_s})=\sum v_n^sx'\neq 0\in k(n)^*\otimes P(y).$$

Moreover $v_n^{s-1}x'\not\in Im(res)$, because $Im(res)$ is generated by $res(c_{j_1})...res(c_{j_r})$ and each $res(c_j)=0\ mod(v_n)$. Hence

$$x''\neq 0 \ \ in \ \ k(n)^*(R(\mathbb{G}))\otimes_{k(n)^*}\mathbb{Z}/2\cong CH^*(R(\mathbb{G}))/2.$$

This is a contradiction. Q.E.D.

Corollary 8.1. Let $G(N)=Spin(N)$ and $\mathbb{G}(N)$ be versal. Then we have

$$lim_{\infty\leftarrow N}CH^*(R(\mathbb{G}(N))/2\cong \Lambda(c_2,c_3,...,c_n,...),$$

$$lim_{\infty\leftarrow N}CH^*(\mathbb{F})/2\cong S(t)/(2,c_2^2,c_3^2,...,c_n^2,...),$$

Proof. The second isomorphism follows from the additive isomorphism

$$CH^*(\mathbb{F})/2\cong CH^*(R\mathbb{G}(N))/2\otimes S(t)/(c_2,c_3,...).$$

Q.E.D.

Corollary 8.2. We have $lim_{\infty\leftarrow N}D(\mathbb{G}(N))=0$.

Proof. From Lemma 7.9, we have

$$lim_{\infty\leftarrow N}Ideal(i^*/2(CH^*(BG(N)_k))\supset (D_\infty/2)$$

$$=Ideal(i^*c_4,i^*c_6,...,i^*c_{2i},...)\subset CH^*(BB_k)/2.$$

We get the result from $i^*:c_{2i}\mapsto c_i^2$ and from the preceding corollary. In fact, $Ker(j^*)\cong Ideal(2,c_2^2,c_3^2,...)$. Q.E.D.

9 $Spin(7)$ for $p=2$

In this section, we assume $G=Spin(7)$ and $p=2$. Then

$$H^*(BG;\mathbb{Z}/2)\cong \mathbb{Z}/2[w_4,w_6,w_7,w_8]$$

where w_i for $i\le 7$ (resp. $i=8$) are the Stiefel-Whitney classes for the representation induced from $Spin(7)\to SO(7)$ (resp. the spin representation Δ).

Thus the integral cohomogy is written as (using $Q_0w_6=w_7$)

$$H^*(BG)\cong \mathbb{Z}_{(2)}[w_4,c_6,w_8]\otimes(\mathbb{Z}_{(2)}\{1\}\oplus\mathbb{Z}/2[w_7]\{w_7\})$$

$$\cong D\otimes\Lambda_{\mathbb{Z}}(w_4,w_8)\otimes(\mathbb{Z}_{(2)}\{1\}\oplus\mathbb{Z}/2[w_7]\{w_7\})$$

where $D=\mathbb{Z}_{(2)}[c_4,c_6,c_8]$ with $c_i=w_i^2$.

Next we consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*'}\cong H^*(BG)\otimes BP^*\Longrightarrow BP^*(BG).$$

We can compute the spectral sequence

$$grBP^*(BG)\cong D\otimes(BP^*\{1,2w_4,2w_8,2w_4w_8,v_1w_8\}$$

$$\oplus BP^*/(2,v_1,v_2)[c_7]\{c_7\}/(v_3c_7c_8)).$$

Then $BP^*(BG)\otimes_{BP^*}\mathbb{Z}_{(2)}$ is isomorphic to ([Ko-Ya])

$$D\{1,2w_4,2w_8,2w_4w_8,v_1w_8\}/(2v_1w_8)\oplus D/2[c_7]\{c_7\}.$$

On the other hand, the Chow ring of $BG_{\mathbb{C}}$ is given by Guillot ([Gu],[Ya2])

Theorem 9.1. Let $k=\bar{k}$. Then we have isomorphisms

$$CH^*(BG_k)\cong BP^*(BG_k)\otimes_{BP^*}\mathbb{Z}_{(2)}$$

$$\cong D\otimes(\mathbb{Z}_{(2)}\{1,c_2',c_4'.c_6'\}\oplus\mathbb{Z}/2\{\xi_3\}\oplus\mathbb{Z}/2[c_7]\{c_7\})$$

where $cl(c_i)=w_i^2$, $cl(c_2')=2w_4$, $cl(c_4')=2w_8$, $cl(c_6')=2w_4w_8$, and $cl(\xi_3)=0$, $|\xi_3|=6$. However $cl_\Omega(\xi_3)=v_1w_8$ in $BP^*(BT)^W$, for the cycle map cl_Ω of the algebraic cobordism.

Now we consider $CH^*(\mathbb{G}/B_k)$. Let $G=Spin(7)$ and $\mathbb{G}$ be versal. The group G is of type (I) and we can take $b_1=c_2,b_2=c_3,b_3=e_4$ with $|e_4|=8$.

The Chow ring $CH^*(\mathbb{G}/B_k)$ is given in Theorem 2.4 (in fact, G is of type (I))

$$CH^*(\mathbb{G}/B_k)\cong S(t)/((2c_2,c_2^2,c_2c_3,c_3^2,e_4),\quad S(t)=\mathbb{Z}_{(2)}[t_1,t_2,t_3].$$

Hence we have $Ker(j(\mathbb{G}))\cong(2c_2,c_2^2,c_2c_3,c_3^2,e_4)$. Recall

$$CH^*(B\bar{G}_k)/(Tor)\cong CH^*(BB_k)^W\cong D\{1,c_2'',c_4'',c_6''\}$$

where c_i'' is a Chern class of the (complex) spin representation. Note $CH^*(B\bar{G}_k)/Tor \cong CH^*(BG_k)/Tor$ from Lemma 4.3. Since $i(c_2'') = 2w_4, ...,$ we see

$$D/2 \cong Im(i^*/2 : CH^*(BG_k) \to CH^*(BT)/2).$$

We can see that the map i^* is given $c_4 \mapsto c_2^2$, $c_6 \mapsto c_3^2$, $c_8'' \mapsto e_4^2$, and

$$c_2'' \mapsto 2c_2, \quad c_4'' \mapsto 2e_4, \quad c_6'' \mapsto 2c_2e_4.$$

In particular $i^*CH^*(BG_k) = i^*CH^*(B\bar{G}_k)$. Thus we see

Proposition 9.2. Let $G = Spin(7)$ and $\mathbb{G}$ be versal. Then we have additively

$$D(\mathbb{G}) \cong \Lambda(c_2c_3, e_4)^+ \otimes S(t,c) \quad for\ S(t,c) \cong S(t)/(c_2, c_3, e_4).$$

Proof. The result follows from $Ker(j^*)/Ideal(i^+) \cong (c_2^2, c_2c_3, c_3^2, e_4)/(c_2^2, c_3^2, e_4^2)$. Q.E.D.

10 $Spin(9)$ for $p = 2$

In this section, we assume $G = Spin(9)$ and $p = 2$ and hence $h = 4$. It is well known (in fact $w_2, w_3, w_5 \in J$)

$$H^*(BG; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_4, w_6, w_7, w_8, w_{16}]$$

where w_i for $i \le 8$ (resp. $i = 16$) are the Stiefel-Whitney class for the representation induced from $Spin(9) \to SO(9)$ (resp. the spin representation Δ and hence $w_{16} = w_{16}(\Delta) = e$).

Recall that $H^*(BG)$ has just 2-torsion by Kono. Let us write

$$D = \mathbb{Z}_{(2)}[c_4, c_6, c_8, c_{16}] \quad with\ c_i = w_i^2.$$

Then we can write

$$H^*(BG)/Tor \cong D \otimes \Lambda_{\mathbb{Z}}(w_4, w_8, w_{16}), \quad Tor \cong D \otimes \mathbb{Z}/2[w_7]^+.$$

Next we consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*'} \cong H^*(BG) \otimes BP^* \Longrightarrow BP^*(BG).$$

Using $Q_1(w_4) = w_7, Q_2(w_7) = c_7$, $Q_2(w_8) = w_7w_8$ and $Q_3(w_7w_8) = c_7c_8$, we can compute the spectral sequence (page 796, (6.14) in [Ko-Ya]). Let us write $D' = \mathbb{Z}_{(2)}[c_4, c_6, c_8]$ and $D'' = \mathbb{Z}_{(2)}[c_4, c_6, c_{16}]$. Then the infinite term is given

$$E_\infty = grBP^*(BG)$$

$$\cong D' \otimes (BP^*\{1, 2w_4, 2w_8, 2w_4w_8, v_1w_8\} \oplus BP^*/(2, v_1, v_2)[c_7]^+/(v_3c_7c_8))$$

$$\oplus D'' \otimes (BP^*\{2w_4w_{16}, 2w_{16}, v_1w_{16}, v_2w_{16}\} \oplus BP^*/(2, v_1, v_2)[c_7]\{c_7c_{16}\})$$

$$\oplus D \otimes (BP^*\{2w_8, 2w_4w_8, v_1w_8\}\{w_{16}\} \oplus BP^*/(2, v_1, v_2, v_3, v_4)[c_7]\{c_7c_8c_{16}\}).$$

However $BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(2)}$ is not so complicated, and it is isomorphic to

$$BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(2)} \ \cong\ D\{1\} \oplus D \otimes 2\Lambda_{\mathbb{Z}}(w_4, w_8, w_{16})^+$$

$$\oplus D/2\{v_1w_8, v_1w_{16}, v_1w_8w_{16}, v_2w_{16}\} \oplus D/2[c_7]^+.$$

The elements in $BP^*(BG)$ corresponding to $v_1w_8, ..., v_2w_{16}$ are all torsion free elements. However they are 2-torsion in $BP^*(BG)\otimes_{BP^*}\mathbb{Z}_{(2)}$, e.g.,

$$2v_2w_{16} \in v_2BP^*(BG), \quad \textit{since } 2w_{16} \in BP^*(BG).$$

We will prove the following lemma.

Lemma 10.1. Each element in $2\Lambda_{\mathbb{Z}}(w_4, w_8, w_{16})$ is represented by a sum of products of Chern classes.

Hence $\tilde{c}l/Tor$ is surjective. So from Lemma 4.1, we have

Theorem 10.1. We have the isomorphism

$$CH^*(B\bar{G}_k)/(Tor) \cong (BP^*(BG)\otimes_{BP^*}\mathbb{Z}_{(2)})/(Tor)$$

$$\cong D\{1, c_2'', c_4'', c_6'', c_8'', c_{10}'', c_{12}'', c_{14}''\}$$

where c_i (resp. c_j'') is the Chern class of the usual (resp. complex spin) representation.

Let us write by $Grif \subset CH^*(B\bar{G}_k)$ be the ideal of Griffiths elements, that is $Grf = Ker(cl : CH^*(B\bar{G}_k) \to H^*(BG))$.

Corollary 10.2. We have $Tor/Grif \cong D/2[c_7]^+$.

Remark. Note that $v_1w_8 \in Grif$, but we can not see v_1w_{16}, v_2w_{16} are in $CH^*(B\bar{G}_k)$ or not, i.e., we do not see $\bar{c}l$ is surjective or not.

To prove the above lemma, we recall the complex representation ring

$$R(Spin(2\ell+1)) \cong \mathbb{Z}[\lambda_1, ..., \lambda_{\ell-1}, \Delta_{\mathbb{C}}]$$

Here λ_i is the i-th elementary symmetric function in variables $z_1^2+z_1^{-2}$,...,$z_\ell^2+z_\ell^{-2}$ in $R(T) \cong \mathbb{Z}[z_1, z_1^{-1}, ..., z_\ell, z_\ell^{-1}]$ for the maximal torus T. The representation $\Delta_{\mathbb{C}}$ is defined

$$\sum z_1^{\varepsilon_1}...z_\ell^{\varepsilon_\ell} \quad \varepsilon_i = 1 \textit{ or } -1.$$

Consider the restriction $R(S^1) \cong \mathbb{Z}[z_1, z_1^{-1}]$ (i.e., $z_i = 1$ for $i \geq 2$). Since

$$\lambda_1 = z_1^2 + z_1^{-2} + ... + z_4^2 + z_4^{-2}, \quad \textit{so} \ \ \lambda_1|S^1 = z_1^2 + z_1^{-2} + 6.$$

Thus for $H^*(BS^1) \cong \mathbb{Z}[u]$, $|u| = 2$, we have

$$Res_{BS^1}(c(\lambda_1)) = (1-2u)(1+2u) = 1-4u^2.$$

From this we see $c_2(\lambda_1)|_{S^1} = -4u^2 \neq 0$.

Recall that $H^4(BG)_{(2)} \cong \mathbb{Z}_{(2)}\{w_4\}$. Note $Res_{S^1}(w_4) = 0$ in $H^*(BS^1;\mathbb{Z}/2)$, and w_4 is not represented by a Chern class (in fact, it does not exist in $BP^*(BG)$). Using these facts, we see

$$Res_{BS^1}(w_4) = 2u^2 \quad \textit{and so} \quad Res_{BS^1}(2w_4) = 4u^2$$

which is represented by Chern classes.

Proof of Lemma 10.1. We consider the Chern classes $c_i(\Delta_{\mathbb{C}})|_{BS^1}$. Consider the restriction $\Delta_{\mathbb{C}}|_{S^1} = 2^3(z_1+z_1^{-1})$. Hence

$$Res_{BS^1}(c(\Delta_{\mathbb{C}})) = (1-u^2)^8 = 1 - \binom{8}{1}u^2 + \binom{8}{2}u^4 + ... + u^{16}.$$

Recall $q_3|_{BS^1} = w_4|_{BS^1} = 2u^2$. Since $q_4^2 = 2q_3$, we see $w_8|_{BS^1} = q_4|_{BS^1} = 2u^4$. We also know $e|_{BS^1} = u^8$ (in fact $e = w_{16}$ is defined using Δ). Therefore $2w_8|_{BS^1} = 4u^4$ and $2e|_{BS^1} = 2u^8$ are represented by Chern classes. Similarly we can see that each element in $2\Lambda_{\mathbb{Z}}(w_4, w_8, w_{16})$ is represented by Chern class. For example

$$2w_4w_8w_{16}|_{S^1} = 2(2u^2)(2u^4)u^8 = 2^3u^{14} = \binom{8}{7}u^{14}$$

which is represented by a Chern class. Q.E.D.

Let $G = Spin(9)$ and $\mathbb{G}$ be versal. The Chow ring of the flag variety is given in §6 and

$$Ker(j^*(\mathbb{G})) = (c_2^2, c_2c_3, c_3^2, e_8, c_4) \subset S(t)/2,$$

The Chow ring of BG is still unknown. But we see from the preceding theorem $CH^*(BG_k)/Tor \cong D\{1, c_2'', c_4'', c_6'', c_8'', c_{10}'', c_{12}'', c_{14}''\}$. Since $i^*(c_2'') = 2w_4, i^*(c_4'') = 2w_8, ...,$ we see Conjecture 7.7 for $G = Spin(9)$.

Theorem 10.3. Let $G = Spin(9)$. Then for $D = \mathbb{Z}_{(2)}[c_4, c_6, c_8, c_{16}'']$, we have

$$D/2 \cong Im(i^*/2 : CH^*(BG_k) \to CH^*(BB_k)/2).$$

We can see the map i^* is given

$$c_4 \mapsto c_2^2, \quad c_6 \mapsto c_3^2, \quad c_8 \mapsto e_8, \quad c_{16}'' \mapsto (c_4)^4,$$

$$c_2'' \mapsto 2c_2, \quad c_4'' \mapsto 2c_4, \quad c_6'' \mapsto 2c_2c_4, \quad c_8'' \mapsto 2c_4^2, \quad c_{10}'' \mapsto 2c_2c_4^2,$$

$$c_{12}'' \mapsto 2c_4e_8, \quad c_{14}'' \mapsto 2c_2c_4e_8.$$

Here $c_i'' = c_i(\Delta_{\mathbb{C}})$ for the complex spin representation.
From Theorem 10.2, we have

Proposition 10.4. Let $G = Spin(9)$ and $\mathbb{G}$ be versal, Then we have

$$D(\mathbb{G}) = D_{CH/2}(\mathbb{G}) \cong (\mathbb{Z}/2\{1, c_2c_3\} \otimes \mathbb{Z}/2[c_4]/(c_4^4))^+ \otimes S(t,c).$$

11 The ordinary cohomology for F_4

In this and next sections, we assume $(G,p) = (F_4, 3)$. For ease of notation, the classifying space BG means the topological space $BG(\mathbb{C})$ (or the variety $BG_{\bar{k}}$). . Toda computed the $mod(3)$ cohomology of BF_4. (For details see [Tod1].)

Theorem 11.1. (Toda [Tod1]) We have additively $H^*(BG;\mathbb{Z}/3)\cong C\otimes D$,

$$where\quad C=F\{1,x_{20},x_{20}^2\}+\mathbb{Z}/3[x_{26}]\otimes\Lambda(x_9)\otimes\mathbb{Z}/3\{1,x_{20},x_{21},x_{26}\}$$

$$and\quad D=\mathbb{Z}_{(3)}[x_{36},x_{48}],\quad F=\mathbb{Z}_{(3)}[x_4,x_8].$$

Here the suffix means its degree.

Remark. The multiplicative structure is also given completely by Toda [Tod1], e.g., $x_{21}x_8+x_{20}x_9=0$.

Note that $H^*(BG)$ has no higher 3-torsion and $Q_0x_8=x_9$, $Q_0x_{20}=x_{21}$. So $x_8,x_{20}\notin H^*(BG)$. From $Q_0x_{25}=x_{26}$, we can see $x_{26}=Q_2Q_1(x_4)$. Using these we have

Corollary 11.2. ([Tod1], [Ka-Mi]) We have isomorphisms

$$H^*(BT;Z/3)^W\cong H^{even}(BG;\mathbb{Z}/3)/(Q_2Q_1x_4)\cong D/3\otimes F\{1,x_{20},x_{20}^2\}.$$

$$H^*(BT)^W\cong H^*(BG)/Tor\cong D\otimes(\mathbb{Z}_{(3)}\{1,x_4\}\oplus E)$$

where $D=\mathbb{Z}_{(3)}[x_{36},x_{48}]$, $F=\mathbb{Z}_{(3)}[x_4,x_8]$, and $E=F\{ab|a,b\in\{x_4,x_8,x_{20}\}\}$.

Note that $E\oplus\mathbb{Z}_{(3)}\{1,x_4,x_8,x_{20}\}\cong\mathbb{Z}_{(3)}[x_4,x_8,x_{20}]/(x_{20}^3)$.

To show the above theorem, Toda uses the following fibering

$$\Pi\to BSpin(9)\to BF_4$$

where $\Pi=F_4/Spin(9)$ is the Cayley plane. Let T be the maximal torus of $Spin(9)\subset F_4$, and $W(G)$ be the Weyl group of G. Let us write $H^*(BT;\mathbb{Z}/3)\cong\mathbb{Z}/3[t_1,...,t_4]$. It is well known

$$H^*(BSpin(9);\mathbb{Z}/3)\cong H^*(BT;\mathbb{Z}/3)^{W(Spin(9))}\cong\mathbb{Z}/3[p_1,...,p_4]$$

where p_i is the i-th Pontrjagin class which is the i-th elementary symmetric function on variable t_j^2. The Weyl group $W(F_4)$ is generated by elements in $W(Spin(9))$ and by R with $R(u_i)=u_i-(u_1+...+u_4)$. The invariant ring of $G=F_4$ is also computed by Toda

Theorem 11.3. There is a ring isomorphism

$$H^*(BT;\mathbb{Z}/3)^{W(G)}\cong\mathbb{Z}/3[p_1,\bar p_2,\bar p_5,\bar p_9,\bar p_{12}]/(r_{15})\subset\mathbb{Z}/3[p_1,...,p_4]$$

$$where\quad \bar p_2=p_2-p_1^2,\quad \bar p_5=p_4p_1+p_3\bar p_2,\quad \bar p_9=p_3^3\ mod(I),$$

$$\bar p_{12}=p_4^3\ mod(I),\quad r_{15}=\bar p_5^3,\quad with\ I=Ideal(p_1,\bar p_2).$$

Let us write $i:T\subset F_4$. The above elements correspond even degree generator (except for x_{26}).

Corollary 11.4. We have

$$i^*(x_4)=p_1,\quad i^*(x_8)=\bar p_2,\quad i^*(x_{20})=\bar p_5,\quad i^*(x_{36})=\bar p_9,\quad i^*(x_{48})=\bar p_{12}.$$

By using this corollary, we can write the reduced power actions.

Lemma 11.1. ([Tod1]) We have

$$P^1(x_4) = -x_8 + x_1^2, \quad P^1(x_8) = x_4x_8, \quad P^1(x_{20}) = 0,$$

$$P^3(x_4) = 0, \quad P^3(x_8) = x_{20} - x_4x_8^2, \quad P^3(x_{20}) = x_{20}x_4(-x_8 + x_4^2),$$

$$P^3(x_{36}) = x_{48} \ mod(x_4, x_8).$$

Recall that the $mod(3)$ cohomology of F_4 is

$$H^*(G;\mathbb{Z}/3) \cong \mathbb{Z}/3[y_8]/(y_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}).$$

Here suffices mean their degree. Recall the cohomology of the flag variety

$$H^*(G/T;\mathbb{Z}/3) \cong P(y) \otimes S(t)/(b_1, ..., b_4)$$

and so $b_1 = p_1, b_2 = \bar{p}_2, b_3 = p_3, b_4 = p_4$. Define $D_{H/3}(G) = Ker(j^+)/(Im(i^+)$ for

$$H^*(BG;\mathbb{Z}/3) \xrightarrow{i^*} H^*(BT;\mathbb{Z}/3) \xrightarrow{j^*} H^*(G/T;\mathbb{Z}/3).$$

Proposition 11.5. We have additively

$$D_{H/3}(G) \cong \mathbb{Z}/3[p_3, p_4]^+/(p_3^3, p_4^3) \otimes S(t,p) \quad for\ S(t,p) \cong S(t)/(p_1, ..., p_4).$$

Proof. First note that $i^*(x_4) = p_1$, $i^*(x_8) = \bar{p}_2$ and $p_1, \bar{p}_2$ are zero in $D_{H/3}(G)$.

Since $i^*(x_{36}) = \bar{p}_9 = p_3^3 \ mod(I)$, we see $p_3^3 = 0 \in \bar{D}_{H/3}(G)$. Similarly, we see $p_4^3 = 0 \in D_{H/3}(G)$ from $i^*(x_{48}) = \bar{p}_{12}$. Q.E.D.

12 BP^*-theory and Chow ring for $(F_4, 3)$

We consider the Atiyah-Hirzebruch spectral sequence [Ko-Ya]

$$E_2^{*,*'} \cong H^*(BG) \otimes BP^* \Longrightarrow BP^*(BG).$$

Its differentials have forms of $d_{2p^n-1}(x) = v_n \otimes Q_n(x)$. Using $Q_1(x_4) = x_9, Q_1(x_{20}) = x_{25}, Q_1(x_{21}) = x_{26}$ and $Q_2x_9 = x_{26}$, we can compute ([Ko-Ya])

$$E_\infty^{*,*'} \cong D \otimes (BP^* \otimes (\mathbb{Z}_{(3)}\{1, 3x_4\} \oplus E) \oplus BP^*/(3, v_1, v_2)[x_{26}]^+).$$

Hence we have

Theorem 12.1. ([Ko-Ya], [Ya2]) We have the isomorphism

$$BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(3)} \cong D \otimes (\mathbb{Z}_{(3)}\{1, 3x_4\} \oplus E \oplus \mathbb{Z}/3[x_{26}]^+).$$

Lemma 12.1. ([Ya2]) We see $x_{26} \in Im(cl)$.

Proof. From Lemma 4.3 in [Ya2], (see also [Ka-Ya]) if $x \in H^4(X(\mathbb{C})$ and $px \in Im(cl)$, then there is $x' \in H^{4,3}(X;\mathbb{Z}/p)$ such that $cl(x') = x \ mod(p)$. Note

$$y = Q_2Q_1(x') \in H^{26.13}(X:\mathbb{Z}/3) \cong CH^{13}(X)/3.$$

Hence we have the lemma from $x_{26} = cl(y)$. Q.E.D.

Let $Grif \subset Tor \subset CH^*(X|_{\bar{k}})$ be the ideal generated by Griffiths elements i.e., $Grif = Ker(t_{\mathbb{C}})$ for $t_{\mathbb{C}} : CH^*(X|_{\bar{k}}) \to H^*(X)$.

Corollary 12.2. We have $Tor/Grif \cong D \otimes \mathbb{Z}/3[x_{26}]^+$ and

$$CH^*(BG_{\bar{k}})/Tor \subset D \otimes (\mathbb{Z}_{(3)}\{1, 3x_4\} \oplus E) \subset H^*(BG)/Tor.$$

If Totaro's conjecture is correct, then $Grif = \{0\}$ and the first inclusion is an isomorphism.

From Lemma 3.1-3.4 in [Ya2], we see $x_{36}, 3x_4, x_4^3, ...$ are represented by Chern classes. Moreover we still know

Lemma 12.2. ([Ya2]) Let RP be the subalgebra of the $mod(3)$ Steenrod algebra A_3 generated by reduced powers. Then $(BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(3)})/(Tor, 3)$ is generated as an RP-module by

$$x_4^2, \quad x_8^2, \quad and \ \ products \ of \ some \ Chern \ classes.$$

Here we consider the (algebraic) K-theory with the coefficient $K^* = \mathbb{Z}_{(p)}[v_1, v_1^{-1}]$ such that

$$BP^*(BG) \otimes_{BP^*} K^* \cong K^*(BG).$$

Recall that $gr^*_{geo}(X)$ is the graded associated ring defined by the geometric filtration of $K^0(X)$ (that is isomorphic to the infinite term $E_\infty^{2*,*,0}$ of the motivic Atiyah-Hirzebruch spectral sequence). Then it is well known that we have the surjection $CH^*(X) \to gr^*_{geo}(X)$.

Lemma 12.3. We see $x_4^2 \in Im(cl)$.

Proof. Suppose that $x_4^2 \notin CH^*(BG_k)$. However x_4^2 exists in $K^*(BG_k) \cong K^*(BG)$, because it exists in $BP^*(BG)$. Since $CH^*(X) \to gr^*_{geo}(X)$ is surjective, there is an element

$$c \in CH^*(BG_k) \quad such \ that \ \ c = v_1^s x_4^2 \ for \ s \geq 1.$$

By dimensional reason, this $s = 1$ and $|c| = 4$. But by Totaro

$$CH^2(BG) \cong (BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(p)})^4,$$

which is a contradiction. Q.E.D.

We can not see $x_8^2 \in Im(cl)$ or not [Ya2], still in this paper.

Proposition 12.3. ([Ya1]) Let $(G, p) = (F_4, 3)$. Suppose $x_8^2 \in Im(cl)$. Then the modified cycle map $\bar{cl} : CH^*(BG_k) \to BP^*(BG) \otimes_{BP^*} \mathbb{Z}_{(3)}$ is surjective. Moreover, we have

$$Im(\bar{cl}) \cong Im(cl) \cong D \otimes (\mathbb{Z}_{(3)}\{1, 3x_4\} \oplus E \oplus \mathbb{Z}/3[x_{26}]^+).$$

From Theorem 2.3, we have

$$CH^*(\mathbb{G}/B_k)/3 \cong S(t)/(p_i p_j | 1 \leq i, j \leq 4).$$

Hence, we have $(p_i p_j) \supset Ideal(i^* CH^*(BG_k))$ e.g. $i^*(x_4^2) = p_1^2$, $i^*(x_4 x_8) = p_1 p_2$,...

Suppose that $x_8^2 \notin CH^*(BG_k)$. However x_8^2 exists in $K^*(BG_k) \cong K^*(BG)$, because it exists in $BP^*(BG)$. Since $CH^*(X) \to gr^*_{geo}(X)$ is surjective, there is an element

$$c \in CH^*(BG_k) \quad such \ that \ \ c = v_1^s x_8^2 \ for \ s \geq 1.$$

This c is torsion element in $CH^*(BG)$ since $3x_8^2 \in Im(cl)$.

Proposition 12.4. If $x_8^2 \notin Im(cl)$, then there is a non zero element $c \in Tor$ with $|c| = 16 - 4s$ for $s = 1$ or 2.

We consider the following ideals in $CH^*(BB_k)$

$$Ker(j^*) = (3p_1, p_1^2, p_1\bar{p}_2, 3p_3, \bar{p}_2^2, ...) \supset (3x_4, x_4^2, x_4x_8, x_4^3, \lambda x_8^2, ..) = Ideal(Im(i^*)),$$

for $\lambda \in \mathbb{Z}_{(3)}$. We note that

$$i^*(3x_4) = 3p_1, \quad i^*(x_4^2) = p_1^2, \quad i^*(x_4^3) = 3p_3, \quad i^*(x_4x_8) = p_1p_2$$

where we used $p_1^3 = 3p_3 \ mod(p_1p_2)$. Note that $\lambda \neq 0$ implies $i^*(x_8^2) = p_2^2$.

Proposition 12.5. The map $\tilde{cl}$ is surjective if and only if $D^*(\mathbb{G}) = 0$ for $* \leq 16$.

Proposition 12.6. The ring $\tilde{D}(\mathbb{G})$ is isomorphic to a quotient of

$$D(F_4)' = \mathbb{Z}/3\{p_1^{i_i}p_2^{i_2}p_3^{i_3}p_4^{i_4} | 2 \leq i_1 + ... + i_4\}/(p_1^2, p_1p_2, p_3^3, p_4^3).$$

13 E_6,E_7 for $p = 3$

The groups E_6, E_7 for $p = 3$ are of type (I). Hence

$$Kerj^+(\mathbb{G}) \cong Ideal(b_ib_j, b_k | 1 \leq i, j \leq 4, \ 5 \leq k \leq \ell) \subset S(t)/3.$$

By Kameko [Ka], there is a representation $\rho_\ell : E_\ell \to U(N)$ such that

$$i_\ell^* c_{18}(\rho_\ell) = x_{36} \quad for \ i_\ell : F_4 \to E_\ell.$$

Hence $i_\ell^*(P^3c_{18}) = x_{48}$. Thus

$$p_3^3 = i^*(c_{18}), \quad p_4^3 = i^*(P^3c_{18}).$$

Proposition 13.1. Let $G = E_\ell$ for $\ell = 6$ or 7. Then there is a surjection

$$((\mathbb{Z}/3\{1\} \otimes D(F_4)') \otimes \mathbb{Z}/3[b_5, ..., b_\ell])^+ \to \tilde{D}(\mathbb{G}).$$

Proof. From the proof of Lemma 12.5, we see $p_1^2 \in Im(i^*)$. Since $P^1(p_1^2) = p_1\bar{p}_2$, we see $p_1p_2 \in Im(i^*)$ also for E_6, E_7. Q.E.D.

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