The Mahowald operator in the cohomology of the Steenrod algebra

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Abstract

We study the Mahowald operator $M = \langle g_2, h_0^3, - \rangle$ in the cohomology of the Steenrod algebra. We show that the operator interacts well with the cohomology of $A(2)$, in both the classical and $\mathbb{C}$-motivic contexts. This generalizes previous work of Margolis, Priddy, and Tangora.

2000 Mathematics Subject Classification. **55T15** . 16T05, 55S30, 55Q45, 14F42.
Keywords. cohomology of the Steenrod algebra, Adams spectral sequence, Mahowald operator.

1 Introduction

The cohomology of the Steenrod algebra is an algebraic object that serves as the input to the Adams spectral sequence. Therefore, its computation is of fundamental importance to the study of the stable homotopy groups of spheres. The goal of this note is to study part of the cohomology of the Steenrod algebra that displays some regular structure. We work in the $\mathbb{C}$-motivic context. Our results have immediate classical consequences, most of which are already known [12] or can be readily deduced from the results of [12]. The ultimate goal of this study is to serve as an aid in a detailed analysis of the Adams spectral sequence [10].

Let A be the $\mathbb{C}$-motivic Steenrod algebra at the prime 2 [17] [8]. Let $\mathbb{M}_2 = \mathbb{F}_2[\tau]$ be the $\mathbb{C}$-motivic cohomology of a point with $\mathbb{F}_2$-coefficients [18]. We are interested in the algebraic object $\mathrm{Ext}_{\mathbb{C}} = \mathrm{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$ because it serves as the E_2-page for the $\mathbb{C}$-motivic Adams spectral sequence. These Ext groups are of increasingly wild complexity as the dimension increases. The May spectral sequence [13] can be used to compute them in a range. Machines can compute in an even larger range [2] [3] [4] [16]. In either case, these methods cannot determine the entire structure because it is of infinite complexity.

Nevertheless, parts of the computation display regularity. For example, Adams described a regular v_1-periodic pattern near the "top of the Adams chart", i.e., when the Adams filtration is large relative to the stem [1]. May extended this v_1-periodicity to a larger range [15]. (See also [11] for results about $\mathbb{C}$-motivic v_1-periodicity.)

The goal of this article is similar. We study the Mahowald operator $\langle g_2, h_0^3, - \rangle$, which is defined on all elements x such that $h_0^3 x = 0$. We will show that the Mahowald operator behaves regularly in a certain way.

This article is very much inspired by the work of Margolis, Priddy, and Tangora [12]. We are extending those results in two senses. First, we are working in the $\mathbb{C}$-motivic, rather than classical, context. Classical results can easily be deduced from our $\mathbb{C}$-motivic results by inverting τ. Second, we work with a larger subalgebra of the Steenrod algebra, and therefore can detect more classes.

*The author was supported by NSF grant DMS-1606290.

Tbilisi Mathematical Journal Special Issue (HomotopyTheorySpectra - 2020), pp. 183–190.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: January 31, 2020.
Accepted for publication: July 22, 2021.

TABLE 1. Some values of the map $p_*: \mathrm{Ext}_{\mathbb{C}} \to \mathrm{Ext}_B$

(s,f,w)	x	$p_*(x)$
$(53,10,28)$	MP	$Pe_0v_3^2 + Ph_1^3v_3^3$
$(56,10,29)$	$\Delta^2h_1h_3$	τPgv_3^2
$(60,9,32)$	B_4	agv_3^2
$(66,10,35)$	τB_5	$\tau h_2ngv_3^2$
$(90,12,48)$	M^2	$d_0gv_3^4 + h_1^6v_3^6$

When discussing Ext groups, we grade elements in the form (s,f,w), where s is the stem, f is the Adams filtration, and w is the motivic weight. See [8] [9] for more details about notation and for specific computations.

Recall that $A(2)$ is the $\mathbb{M}_2$-subalgebra of A generated by Sq^1, Sq^2, and Sq^4. We have a complete understanding of its cohomology, i.e., of $\mathrm{Ext}_{A(2)} = \mathrm{Ext}_{A(2)}(\mathbb{M}_2, \mathbb{M}_2)$ [7].

Theorem 1.1. There exists a sub-Hopf algebra B of the $\mathbb{C}$-motivic Steenrod algebra (defined below in Definition 2.1) such that:

1. $\mathrm{Ext}_B = \mathrm{Ext}_B(\mathbb{M}_2, \mathbb{M}_2)$ is isomorphic to $\mathbb{M}_2[v_3] \otimes_{\mathbb{M}_2} \mathrm{Ext}_{A(2)}$, where v_3 has degree $(14,1,7)$.
2. the map $p_*: \mathrm{Ext}_{\mathbb{C}} \to \mathrm{Ext}_B$ takes Mx to the product $(e_0v_3^2 + h_1^3v_3^3)p_*(x)$, whenever Mx is defined. Also, p_* takes the indeterminacy in Mx to zero.

The theorem can also be stated in the classical context, in an essentially identical form.

Theorem 1.1 allows us to extract much information about the global structure of $\mathrm{Ext}_{\mathbb{C}}$. The following corollary gives partial information about the structure of $\mathrm{Ext}_{\mathbb{C}}$ in very high dimensions.

Corollary 1.2. Let x be an element of $\mathrm{Ext}_{\mathbb{C}}$ such that $h_1^3x = 0$, and let $k \geq 0$ such that the image of e_0^kx in $\mathrm{Ext}_{A(2)}$ is non-zero. Then M^kx is non-zero in $\mathrm{Ext}_{\mathbb{C}}$.

Proof. By Theorem 1.1, $p_*(M^kx)$ is non-zero in Ext_B. Therefore, M^kx must be non-zero. Q.E.D.

For example, x could be h_0, h_0^2, τh_1, h_2, or h_0h_2 with any value of k. There are many other possibilities as well. Some, but not all, cases of Corollary 1.2 are already covered by [12].

Here is an even more explicit illustration of the kind of information that can be deduced from Corollary 1.2. Consider the element h_0d_0. We have that $h_0d_0e_0^k$ is non-zero in $\mathrm{Ext}_{A(2)}$ for all $k \geq 0$. (However, $\tau^2h_0d_0e_0^k = 0$ in $\mathrm{Ext}_{A(2)}$ when $k \geq 2$). We can conclude that $M^kh_0d_0$ is non-zero in $\mathrm{Ext}_{\mathbb{C}}$ for all $k \geq 0$, even though it may be annihilated by powers of τ.

Theorem 1.1 detects additional phenomena in Ext_A that are not captured by Corollary 1.2. Namely, the theorem can be used to study classes in Ext_A whose image under p_* is non-zero. Some examples are listed in Table 1. This list is far from exhaustive.

Sometimes, analysis of a particular Adams differential requires knowledge of the algebraic structure of $\mathrm{Ext}_{\mathbb{C}}$ in much higher dimensions. If the higher dimension is not too large, then one can rely on explicit machine computations. But if the higher dimension goes beyond the current range of machine computations, then results such as Corollary 1.2 can be of great use. See [10] for specific examples of precisely this situation.

The subalgebra $A(2)$ of the $\mathbb{C}$-motivic Steenrod algebra is of particular importance because there exists a $\mathbb{C}$-motivic modular forms spectrum *mmf* [5] whose cohomology is isomorphic to $A//A(2)$. This implies that $\mathrm{Ext}_{A(2)}$ is the E_2-page of the Adams spectral sequence that converges to the homotopy groups of *mmf*.

One might hope that the sub-Hopf algebra B is similarly realizable. We will show in Theorem 6.1 that it is not. In other words, while Ext_B is useful for studying the algebraic structure of the $\mathbb{C}$-motivic Adams E_2-page $\mathrm{Ext}_{\mathbb{C}}$, it cannot be used to study Adams differentials.

2 A subalgebra of the $\mathbb{C}$-motivic Steenrod algebra

Recall that the dual $\mathbb{C}$-motivic Steenrod algebra A_* [18] [8] takes the form

$$\frac{\mathbb{M}_2[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]}{\tau_i^2 = \tau\xi_{i+1}},$$

where $\mathbb{M}_2 = \mathbb{F}_2[\tau]$. The coproduct of A_* is given by the formulas

$$\tau_i \mapsto \tau_i \otimes 1 + \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \tau_k \qquad\qquad \xi_i \mapsto \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \xi_k.$$

By convention, we let ξ_0 equal 1.

Definition 2.1. Let B_* be the quotient

$$\frac{\mathbb{M}_2[\tau_0, \tau_1, \tau_2, \tau_3, \xi_1, \xi_2]}{\tau_0^2 + \tau\xi_1, \xi_1^4, \tau_1^2 + \tau\xi_2, \xi_2^2, \tau_2^2, \tau_3^2}$$

of A_*. Let B be the dual subobject of A.

Remark 2.2. The classical dual Steenrod algebra A_*^{cl} takes the form $\mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$, which is the result of inverting τ in A_*, where τ_i and ξ_{i+1} correspond to ζ_{i+1} and ζ_{i+1}^2 respectively. The classical analogue of B_* is the quotient

$$\frac{\mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \zeta_4]}{\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4^2}.$$

Lemma 2.3. The quotient B_* is a Hopf algebra that splits as

$$\frac{\mathbb{M}_2[\tau_0, \tau_1, \tau_2, \xi_1, \xi_2]}{\tau_0^2 + \tau\xi_1, \xi_1^4, \tau_1^2 + \tau\xi_2, \xi_2^2, \tau_2^2} \otimes_{\mathbb{M}_2} \frac{\mathbb{M}_2[\tau_3]}{\tau_3^2}.$$

Proof. To check that B_* is a Hopf algebra, one must verify that the coproduct is well-defined. In other words, if x is an element of A_* that maps to zero in B_*, then the coproduct of x in A_* also maps to zero in $B_* \otimes B_*$. This follows from direct computation.

The splitting also follows from direct computation. Namely, the coproduct of τ_3 in A_* maps to $1 \otimes \tau_3 + \tau_3 \otimes 1$ in $B_* \otimes B_*$. Q.E.D.

Proposition 2.4. $\mathrm{Ext}_B = \mathrm{Ext}_B(\mathbb{M}_2, \mathbb{M}_2)$ is isomorphic to $\mathbb{M}_2[v_3] \underset{\mathbb{M}_2}{\otimes} \mathrm{Ext}_{A(2)}$, where v_3 has degree $(14, 1, 7)$.

Proof. This follows immediately from the splitting of Lemma 2.3, together with the observation that the cohomology of an exterior algebra is a polynomial algebra. Q.E.D.

The projection map $p : A_* \to B_*$ induces a map $p_* : \mathrm{Ext}_\mathbb{C} \to \mathrm{Ext}_B$. We will use the map p_* to detect some structural phenomena in $\mathrm{Ext}_\mathbb{C}$.

3 Massey products in Ext_B

The map p_* is induced by a map $\tilde{p} : C^*(A) \to C^*(B)$ of cobar complexes. Note that $\tilde{p}$ is a map of differential graded algebras. In particular, $C^*(B)$ is a right $C^*(A)$-module, and therefore Ext_B is a right $\mathrm{Ext}_\mathbb{C}$-module. By definition, $p_*(x)$ equals $1 \cdot x$, where 1 is the identity element of Ext_B.

Moreover, the map $\tilde{p}$ makes Ext_B into a "Toda module" over $\mathrm{Ext}_\mathbb{C}$, in the following sense. For all x in Ext_B and all a and b in $\mathrm{Ext}_\mathbb{C}$ such that $x \cdot a$ and ab are both zero, there is a bracket $\langle x, a, b\rangle$ in Ext_B. These brackets satisfy the usual properties. Later in the proof of Proposition 4.2, we will use the shuffling relation

$$\langle x, a, b\rangle \cdot c = x \cdot \langle a, b, c\rangle,$$

for x in Ext_B and a, b, and c in $\mathrm{Ext}_\mathbb{C}$.

For later use, we compute one particular bracket. In $\mathrm{Ext}_\mathbb{C}$, there is an element g_2 of degree $(44, 4, 24)$ that is detected by b_{22}^2 in the May spectral sequence. This element satisfies the relation $h_0^3 g_2 = 0$.

Proposition 3.1. The bracket $\langle 1, g_2, h_0^3\rangle$ in Ext_B in degree $(45, 6, 24)$ equals $e_0 v_3^2 + h_1^3 v_3^3$, with no indeterminacy.

Proof. First, we should verify that the bracket is well-defined. We need that $1 \cdot g_2$ equals zero in Ext_B in degree $(44, 4, 24)$. But Ext_B is zero in that degree, so $1 \cdot g_2$ must be zero.

Next, we compute the indeterminacy. By inspection, the only possible indeterminacy is generated by $1 \cdot h_0 h_5 d_0$. But this expression is zero because $1 \cdot h_5$ is zero in Ext_B for degree reasons.

The map $\tilde{p}$ induces a map of May spectral sequences [13] [8]. The May E_1-page that converges to $\mathrm{Ext}_\mathbb{C}$ has generators of the form h_{ij} with $i \geq 1$ and $j \geq 0$. On the other hand, the May E_1-page that converges to Ext_B has generators h_0, h_1, h_2, h_{20}, h_{21}, h_{30}, and h_{40}. The map of May spectral sequences takes h_{ij} to the element of the same name, or to zero if the element is not present in the May E_1-page for Ext_B.

We will compute the bracket $\langle 1, g_2, h_0^3\rangle$ in Ext_B using the May Convergence Theorem [14] [8]. Beware that this theorem has a technical hypothesis involving the behavior of higher "crossing" differentials. In our specific case, this technical hypothesis is satisfied for degree reasons.

The key point is that there is a May differential

$$d_6((b_{21}b_{40} + b_{30}b_{31})h_0(1)) = h_0^3 g_2.$$

Therefore, $\langle 1, g_2, h_0^3\rangle$ is detected by the image of $(b_{21}b_{40} + b_{30}b_{31})h_0(1)$ in the May spectral sequence for Ext_B. By inspection, this image equals $h_{40}^2 b_{21} h_0(1)$.

Finally, we must determine the elements of Ext_B that are detected by $h_{40}^2 b_{21} h_0(1)$ in the May spectral sequence. Note that e_0 is detected by $b_{21}h_0(1)$ and v_3 is detected by h_{40}. However, beware that the element $h_1^3 v_3^3$ is detected by $h_1^3 h_{40}^3$ in lower May filtration. Consequently, $h_{40}^2 b_{21} h_0(1)$ detects both $e_0 v_3^2$ and $e_0 v_3^2 + h_1^3 v_3^3$.

We have now shown that $\langle 1, g_2, h_0^3 \rangle$ equals either $e_0 v_3^2$ or $e_0 v_3^2 + h_1^3 v_3^3$. Finally, we must distinguish between these two cases.

Recall from [6, p. 4729] that there is a relation $Mh_1^6 = e_0^3 + d_0 \cdot e_0 g$ in $\mathrm{Ext}_{\mathbb{C}}$. Apply p_* to obtain a relation in Ext_B. We have that

$$p_*(Mh_1^6) = 1 \cdot \langle g_2, h_0^3, h_1^6 \rangle = \langle 1, g_2, h_0^3 \rangle h_1^6.$$

Here, we are using the well-known Massey product

$$Mh_1^6 = \langle g_2, h_0^3, h_1 \rangle h_1^5$$

(see, for example, [8, Table 16]). So the possible values for $p_*(Mh_1^6)$ are $h_1^6 e_0 v_3^2$ and $h_1^6 e_0 v_3^2 + h_1^9 v_3^3$.

The possible values for $p_*(e_0)$ are 0, e_0, $h_1^3 v_3$, and $e_0 + h_1^3 v_3$, so the possible values of $p_*(e_0^3)$ are 0, e_0^3, $h_1^9 v_3^3$, and $e_0^3 + h_1^3 e_0^2 v_3 + h_1^6 e_0 v_3^2 + h_1^9 v_3^3$.

The only possible value for $p_*(d_0)$ is d_0. The possible values for $p_*(e_0 g)$ are 0, $e_0 g$, $h_1^3 g v_3$, and $e_0 g + h_1^3 g v_3$. (Recall that $e_0 g$ is an indecomposable element in $\mathrm{Ext}_{\mathbb{C}}$.) Therefore, the possible values for $p_*(d_0 \cdot e_0 g)$ are 0, e_0^3, $h_1^3 e_0^2 v_3$, and $e_0^3 + h_1^3 e_0^2 v_3$. Here, we are using the relation $e_0^2 = d_0 g$ in $\mathrm{Ext}_{A(2)}$.

By inspection, the only consistent possibilities are that $p_*(e_0) = e_0 + h_1^3 v_3$, $p_*(e_0 g) = e_0 g + h_1^3 g v_3$, and $p_*(Mh_1^6) = h_1^6 e_0 v_3^2 + h_1^9 v_3^3$. Q.E.D.

Remark 3.2. The May spectral sequence argument in the proof of Proposition 3.1 is much the same as the corresponding proof in [12]. However, the complications involving $h_1^3 v_3^3$ are new.

Remark 3.3. The careful reader may wonder about the definitional distinction between e_0 and $e_0 + h_1^3 v_3$. Cannot e_0 in Ext_B be defined to be the value of $p_*(e_0)$? The answer lies in the multiplicative structure of $\mathrm{Ext}_{A(2)}$. There is a relation $h_1^2 e_0 = c_0 u$ in $\mathrm{Ext}_{A(2)}$. From the formula $p_*(e_0) = e_0 + h_1^3 v_3$, it follows that $p_*(h_1^2 e_0)$ is not divisible by c_0 or u in Ext_B. This multiplicative fact is not consistent with the possibility that $p_*(e_0) = e_0$ under a different choice of basis.

4 The Mahowald operator

Definition 4.1. Let x be an element of $\mathrm{Ext}_{\mathbb{C}}$ such that $h_0^3 x = 0$. Define Mx to be the Massey product $\langle g_2, h_0^3, x \rangle$.

As always, the Massey product Mx can have indeterminacy. In other words, Mx may be a set of elements, not just a single well-defined element.

If $h_0^3 x = 0$, then the iterated Massey products $M^k x = M(M^{k-1} x)$ are defined for all $k \geq 1$. This follows from the computation that

$$h_0^3 \langle g_2, h_0^3, x \rangle = \langle h_0^3, g_2, h_0^3 \rangle x = 0$$

because $\langle h_0^3, g_2, h_0^3 \rangle = 0$.

Proposition 4.2. Let x be an element of $\mathrm{Ext}_{\mathbb{C}}$ such that $h_0^3 x = 0$. Then $p_*(Mx)$ equals $(e_0 v_3^2 + h_1^3 v_3^3) p_*(x)$ in Ext_B.

In particular, Proposition 4.2 implies that $p_*(Mx)$ always consists of a single element, even if Mx has indeterminacy.

Proof. Consider the shuffling relation

$$p_*(Mx) = 1 \cdot \langle g_2, h_0^3, x\rangle = \langle 1, g_2, h_0^3\rangle \cdot x.$$

Proposition 3.1 computes the second bracket. Q.E.D.

Remark 4.3. We have stated our results in terms of Massey products of the form $\langle g_2, h_0^3, x\rangle$. However, they also apply to Massey products of the form $\langle h_0 g_2, h_0^2, x\rangle$ and $\langle h_0^2 g_2, h_0, x\rangle$, using the shuffling relations

$$\langle h_0^2 g_2, h_0, x\rangle \subseteq \langle h_0 g_2, h_0^2, x\rangle \subseteq \langle g_2, h_0^3, x\rangle.$$

5 h_1-periodic Ext

In the spirit of [6], one ought to study the h_1-periodic maps

$$\mathrm{Ext}_{\mathbb{C}}[h_1^{-1}] \xrightarrow{p_*} \mathrm{Ext}_B[h_1^{-1}] \longrightarrow \mathrm{Ext}_{A(2)}[h_1^{-1}].$$

Computationally, this diagram is

$$\mathbb{F}_2[h_1^{\pm 1}][v_1^4, v_2, v_3, \ldots] \xrightarrow{p_*} \mathbb{F}_2[h_1^{\pm 1}][v_1^4, v_2, v_3, u] \longrightarrow \mathbb{F}_2[h_1^{\pm 1}][v_1^4, v_2, u].$$

Both maps take v_1^4 to v_1^4. Moreover, the composition takes v_n to $v_2 u^{2^{n-2}-1}$ [6, Conjecture 5.5 and Proposition 6.4] [5]. In this formula and throughout this section, we suppress all multiples of h_1 since it is a unit.

For degree reasons, p_* takes v_2 to v_2. The computations of $p_*(e_0)$ and $p_*(e_0 g)$ at the end of the proof of Proposition 3.1 imply that $p_*(v_3) = v_3 + v_2 u$ and that $p_*(v_4) = v_3 u^2 + v_2 u^3$. We suspect that $p_*(v_n) = v_3 u^{2^{n-2}-2} + v_2 u^{2^{n-2}-1}$ in general, although we have not actually computed this formula.

On the other hand, the map

$$\mathrm{Ext}_B[h_1^{-1}] \to \mathrm{Ext}_{A(2)}[h_1^{-1}]$$

takes v_2 and u to the elements of the same name in the target, and it must take v_3 to 0.

This information can be used to study h_1-periodic values of the Mahowald operator. In the notation of this section, the element $e_0 v_3^2 + h_1^3 v_3^3$ of Ext_B maps to $(v_3 + v_2 u) v_3^2$ in $\mathrm{Ext}_B[h_1^{-1}]$. Therefore, for x in $\mathrm{Ext}_{\mathbb{C}}[h_1^{-1}]$, we have that Mx maps to $(v_3 + v_2 u) v_3^2 p_*(x)$ in $\mathrm{Ext}_B[h_1^{-1}]$.

6 Non-Realizability

The purpose of Theorem 6.1 is that Ext_B is not the E_2-page of an Adams E_2-page. In other words, while Ext_B is useful for studying the algebraic structure of the $\mathbb{C}$-motivic Adams E_2-page $\mathrm{Ext}_{\mathbb{C}}$, it cannot be used to study Adams differentials.

Theorem 6.1. There does not exist a $\mathbb{C}$-motivic ring spectrum E equipped with ring map $f : E \to mmf$ such that the $\mathbb{F}_2$-motivic cohomology of E is $A//B$, and such that f induces the projection

$$A//A(2) \to A//B$$

in cohomology.

Proof. Suppose that E exists. The unit map $S^{0,0} \to E$ induces a map of Adams spectral sequences. On E_2-pages, this map is $p_* : \mathrm{Ext}_{\mathbb{C}} \to \mathrm{Ext}_B$. By Theorem 1.1, the element Mh_1 of $\mathrm{Ext}_{\mathbb{C}}$ maps to $h_1 e_0 v_3^3 + h_1^4 v_3^3$ in Ext_B. Since Mh_1 is a permanent cycle in the Adams spectral sequence for the $\mathbb{C}$-motivic sphere spectrum [8], it follows by naturality that $h_1 e_0 v_3^3 + h_1^4 v_3^3$ is a permanent cycle in the Adams spectral sequence for E.

On the other hand, the map f also induces a map of Adams spectral sequences. On E_2-pages, this map takes the form $\mathrm{Ext}_B \to \mathrm{Ext}_{A(2)}$. The element v_3 must be a permanent cycle for degree reasons. Also, $d_2(e_0) = h_1^2 d_0$ in the Adams spectral sequence for mmf. By naturality of f, it follows that $d_2(h_1 e_0 v_3^3 + h_1^4 v_3^3) = h_1^3 d_0 v_3^3$.

This contradiction shows that E cannot exist. Q.E.D.

Remark 6.2. One can also pose an analogous question about a classical spectrum whose cohomology is $A^{\mathrm{cl}}//B^{\mathrm{cl}}$. Such a classical spectrum also does not exist, for essentially the same reasons. However, one must use the non-zero classical differential $d_3(e_0) = Pc_0$ in the Adams spectral sequence for *tmf*.

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