Parallel Weingarten surface in Euclidean space with density e^z

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Abstract

In this study, the condition that the ruled surface of $\varphi(u, v)$ in Euclidean $3-$ space with density e^z is a Weingarten surface has been studied. While this review is being conducted, Q, F, J of $\varphi(u, v)$ are called structure functions of the ruled surface were used. Furthermore, the mean curvature H_ϕ and Gaussian curvature K_ϕ of the Weingarten surface were studied. Later on, the relationship between the mean H^r_ϕ curvature and Gauss curvature K^r_ϕ of this surface, which is the parallel surface $\varphi^r(u, v)$ of the Weingarten surface $\varphi(u, v)$ in Euclidean space with density e^z, was investigated. Weingarten surface $\varphi^r(u, v)$ condition of the parallel surface of the ruled Weingarten surface in Euclidean space with density e^z was examined.

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1 Introduction

The theory of surfaces occupies a wide place in various scientific fields such as physics, engineering, and architecture as well as mathematics. Apart from many architectures in daily life, mathematical models of various surfaces have been used in buildings including pyramids, temples, mosques, palaces, and tombs. Moreover, geometry has been also used in the interior and exterior decorations of these structures. Different types of surfaces have been used for many artifacts and materials. In the class of surfaces, one type is the developable surface that can be laid on the plane without being stretched or torn. Because of this feature, it is widely used in sheet-metal and plate-metal based industries, on the surface of a marine vessel, aircraft surfaces and architectural structures, [1]-[3]. Developable surfaces as a kind of ruled surfaces are classified into cylinders, cones or tangent surfaces of space curves, [4]. One of the important result related to the ruled surfaces is its relation with developable surfaces, although all developable surfaces are ruled ones, all ruled surfaces are not developable, [5, 6].

The ruled surfaces are very useful in many areas of sciences, for instance, Computer-Aided Manufacturing (CAM), Computer-Aided Geometric Desing (CAGD), geometric modelling and kinematics, [7]. It is possible to see many architectural structures based on ruled surfaces. For instance, at the end of the nineteenth century, the Shukhov Tower has been built by Vladimir Shukhov which is an example of the ruled surface.

The inner metric of a ruled surface determines the Gaussian curvature, therefore all the lengths and angles on the surface remain invariant in manufacturing. Hence, the ruled surfaces have been paid attention in engineering, architecture, and design, etc [8]-[11]. It is Gaspard Monge who pioneered the emergence of all these surfaces. Today, a great variety of studies have been devoted

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to investigating the classifications of ruled surfaces. In [12], Izumiya classified the ruled surfaces into five classes such as developable surfaces, principal and binormal surfaces, Darboux developable surfaces, rectifying developable surfaces and focus surfaces of space curves.

Two parallel surfaces x and y have an identical distribution of normal vectors; i.e. their Gauss maps are indistinguishable. Thus, a family of parallel surfaces can be produced by translating a surface in the direction of its normal vectors by an equal amount everywhere on the surface.

If x and y are parallel surfaces, the formulae of the Gaussian and mean curvature of them yields the following interesting consequences: Firstly, the curvatures of surfaces parallel to minimal surfaces regarding a Bonnet transformation remain unamended. Secondly, increasing of the Gaussian curvature of a surface parallel to a minimal surface has been observed. This means that the minimal surface has a larger area than parallel surfaces, [13]. Park and Kim also showed that in $3-$dimensional Euclidean space the parallel surfaces of non-developable ruled surfaces are not ruled surfaces, but the parallel surface of a developable ruled surface is a developable ruled surface [14].

Weingarten surfaces were introduced by Weingarten in 1861 in the context of the problem of finding all surfaces isometric to given surfaces of revolution, [15]. Weingarten improved his theory of $w-$surfaces in 1863, for this theory to be a Weingarten surface or $w-$surface of space in $3-$dimensional Euclidean space, the principal curvature of the surface or the Gauss curvature K and the mean curvature H are described as linearly dependent with each other. The five classes of Weingarten surfaces are translation surfaces, channel surfaces of a curve which is a constant from its principal curvatures, helicoidal surfaces, constant mean curvature surfaces, and constant Gaussian curvature surfaces. Kuhnel took the Weingarten surface classification a step further by giving a condition as the dimensions of the structure functions $Q,\ F,\ J$ of the surface is constantly based on the condition of being a Weingarten surface [16]. In differential geometry, curves and space structures in density spaces are among the subjects studied recently. Morgan and Corwin studied the density spaces, [17, 18]. Inspired by these studies, the structures of different surfaces in different spaces of density have been started to be studied.

In this study, the definition of the ruled surface $\varphi(u,v)$ in $3-$dimensional Euclidean space with density e^z has been introduced and also, the condition of being Weingarten has been examined. Some theorems and results were given by proving that the ruled surface $\varphi(u,v)$ is Weingarten. In $3-$dimensional Euclidean space with density e^z, $\varphi(u,v)$ is a surface $\varphi^r(u,v)$ parallel to the ruled Weingarten surface. Then the condition of the surface being Weingarten parallel to the ruled Weingarten surface has been examined. In these regards, the theorems, corollaries and examples about $\varphi^r(u,v)$ according to the case of the ruled surface $\varphi(u,v)$ being developable and non-developable in $3-$dimensional Euclidean space with density e^z.

2 Preliminaries

A ruled surface is a surface which can be swept out at least one 1-parameter family of lines in the space. For this reason, it has a parametrization of the form

$$\varphi(u,v) = c(u) + ve(u)$$

where c and e are called the base curve and director curve of the ruled surface, respectively. The foot of perpendicular bisector of two adjacent directrices of a ruled surface on the main directrix is called striction points. The geometric position of these points is also called the striction curve. The ruled surface is called non-developable ruled surface in case the drall, which is the ratio of the shortest distance between these two directrices to the angle between rectifiers, is non-zero. If $\varphi(u,v)$ is not a developable ruled surface such that $\langle e(u), e(u)\rangle = 1$ and $\langle e'(u), e'(u)\rangle = 1$, the directrix of $c(u)$ is a striction curve. Since $c(u)$ is the striction curve of $\varphi(u,v)$, $\langle c'(u), e'(u)\rangle = 0$, [19]. For brevity, the parameter u will not be written hereinafter.

Let $\{e, t, g\}$ be the spherical Frenet frame of spherical indicatrix vector of e. Then $t = e'$ and $g = e \times e'$. Derivative vectors for the spherical Frenet frame $\{e, t, g\}$ are given as follows:

$$\begin{aligned} e' &= t \\ t' &= -e - Jg \\ g' &= Jt \end{aligned}$$

where the vectors are said to be the central normal t and the asymptotic normal g of $\varphi(u,v)$, respectively. Moreover, here $J = \langle e'', e' \times e\rangle$ denotes the geodesic curvature κ_g of a spherical indicatrix curve e. On the other hand, the derivative of the striction curve c is given by

$$c' = Fe + Qg$$

where $F = \langle c', e\rangle$ and $Q = \langle c', e \times e'\rangle$.

The functions Q, F and J are called structure functions of a non-developable ruled surface $\varphi(u,v)$ in E^3. The structure functions F and Q of $\varphi(u,v)$ satisfy $F^2 + Q^2 = 1$ because the parameter u is also the arc-length parameter of the striction curve c of $\varphi(u,v)$, [19]. The Gaussian and mean curvatures of the ruled surface $\varphi(u,v)$ are

$$K = -\frac{Q^2}{D^4} \tag{2.1}$$

$$H = \frac{1}{2D^2}\left(Jv^2 - Q'v + Q(QJ - F)\right) \tag{2.2}$$

in terms of structure functions. By of the fact that $\varphi(u,v)$ is considered a non-developable ruled surface, K is non-zero. In that case, the function Q is non-zero everywhere, [19].

Suppose that M_1 and M_2 are two hypersurfaces in E^n and unit normal vector area of M_1 is

$$N_1 = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}, \quad a_i \in C^\infty(M, R).$$

M_2 is called the parallel hypersurface of M_1, if the function

$$\begin{aligned} f : M_1 \to &\quad M_2 \\ p \ \to &\quad f(p) = (p_1 + ra_1(p), \ \dots, p_n + ra_n(p)) \end{aligned}$$

exists for a constant $r \in R$, [20]. After that, the surface parallel to the hypersurface M will be expressed as M^r.

Theorem 2.1. Let K and H denotes the Gaussian and mean curvatures of the surface of M at the point $p \in M$, respectively, and M^r denotes a parallel surface of the surface $M \subset E^3$. Then, the Gaussian and mean curvatures of the surface M^r at the point $f(P) \in M^r$ are given by

$$K^r = \frac{K}{1 + 2rH + r^2K} \tag{2.3}$$

and

$$H^r = \frac{H + rK}{1 + 2rH + r^2 K}, \tag{2.4}$$

respectively.

Under the conditions of the above theorem, the following equalities can be written

$$K = \frac{K^r}{1 - 2rH^r + r^2 K^r}$$

$$H = \frac{H^r - rK^r}{1 - 2rH^r + r^2 K^r}.$$

If the Gaussian and mean curvatures that given by the equations (2.3) and (2.4) of the parallel surface are rearranged in terms of structure functions,

$$K^r = \frac{-Q^2}{2D^4 + 4Dr\left(Jv^2 - Q'v + Q^2 J - QF\right) - 2r^2 Q^2}$$

$$H^r = \frac{D\left(Jv^2 - Q'v + Q^2 J - QF\right) - 2rQ^2}{2D^4 + 2Dr\left(Jv^2 - Q'v + Q^2 J - QF\right) - 2r^2 Q^2}$$

are obtained. Being a Weingarten surface in E^3 means that the changes of Gaussian and mean curvatures belonging to this surface are linearly independent of each other. That is

$$\Phi\left(K,\, H\right) = 0,$$

[21].

Theorem 2.2. Let $M\left(u, v\right) \subset E^3$ be a surface and K denotes the Gaussian curvature and H denotes the mean curvature of M. If

$$K_u H_v - K_v H_u = 0$$

holds for this surface, then M is a Weingarten surface, [21].

A manifold with a positive density function ϕ used to weight the volume and the hypersurface area. In terms of the underlying Riemannian volume dV_0 and area dA_0, the new, weighted volume and area are given by $dV = \phi dV_0$ and $dA = \phi dA_0$, respectively. For more details on manifolds with density, see [17, 18], [22, 23]. One of the best examples of density surfaces is the two-dimensional Gaussian plane. The Gaussian plane is the Euclidean plane with volume and length weighted by $(2\pi)^{-1} e^{-r^2/2}$. We can generalize the curvature of a curve or the mean curvature of a surface to manifolds with density.

Let $\phi\left(x\right)$ be the linear function in the form $\phi\left(x\right) = \sum\limits_{i=1}^{n} a_i x_i$ in Euclidean space E^n. Then $e^{\phi(x)}$ is called the log-linear density. We can write the density in e^{x_n} form by selecting the appropriate coordinates. Thus, we can examine the space E^n with density $e^{\phi(x)}$ while $E^{n-1} \oplus E_\phi$ is producing $(n-1)$ −dimensional Euclidean space E^{n-1} and E_ϕ to real lines with density e^{x_n}.

Taking into account that $\nabla\phi = (0, 0, ..., 1)$, it is seen that the following equation exists;

$$\frac{d\phi}{dN} = \langle \nabla\phi, N \rangle = \|\nabla\phi\| . \|N\| . \cos\theta\left(\nabla\phi, N\right) = \cos\theta\left(\nabla\phi, N\right).$$

Here $\frac{d\phi}{dN} = \langle \nabla\phi, N \rangle$ is the cosine of the angle between N and z-axis [24]. Riemann curvature with density is defined by

$$\kappa_\phi = \kappa - \frac{d\phi}{dN}$$

where κ is Riemann curvature, [17]. Besides, for the case $n = 3$, we can give the Gaussian and mean curvatures with density,

$$K_\phi = K - \Delta\phi \tag{2.5}$$

and

$$H_\phi = H - \frac{1}{2}\frac{d\phi}{dN}, \tag{2.6}$$

respectively [17],[24]. These curvatures are also called Gaussian $\phi-$curvature and mean $\phi-$curvature. where K and H are Gaussian and mean curvatures of the ruled surface, respectively.

In view of the fact that $\nabla\phi = (0, 0, 1)$ and $\Delta\phi = 0$, Gauss and mean $\phi-$curvatures of the ruled surface in terms of structure functions are obtained as follows

$$K_\phi = -\frac{Q^2}{D^4} \tag{2.7}$$

$$H_\phi = \frac{1}{2D^3}\left(Jv^2 - Q'v + Q\left(QJ - F\right)\right) - \frac{1}{2D}\langle \nabla\phi, Qt - vg \rangle \tag{2.8}$$

with the aid of equations (2.5) and (2.6) in E^3 with density e^z where $D = \sqrt{Q^2 + v^2}$.

3 Ruled surfaces in E^3 with density e^z

In this section, the ruled Weingarten surfaces in 3$-$dimensional Euclidean space E^3 with density e^z and parallel surfaces of these surfaces will be examined.

3.1 Ruled Weingarten surfaces in Eculidean space with density e^z

First, we investigate the state of a ruled surface to be Weingarten by using the Gaussian and mean $\phi-$ curvatures of the ruled surface $\varphi\left(u, v\right)$ in the Euclidean space with density.

If there exists the Jacobi equation $\Phi\left(H_\phi, K_\phi\right) = 0$ between the Gaussian and the mean $\phi-$ curvatures in the Euclidean space with density e^z, it is called Weingarten surface.

Theorem 3.1. Let K_ϕ and H_ϕ denote the Gaussian and mean $\phi-$curvatures of the ruled surface $\varphi\left(u, v\right)$ in the 3$-$dimensional Euclidean space with density e^z, respectively. If the following equality

$$\left(K_\phi\right)_u \left(H_\phi\right)_v - \left(K_\phi\right)_v \left(H_\phi\right)_u = 0 \tag{3.1}$$

holds, the ruled surface $\varphi\left(u, v\right)$ is the Weingarten surface.

Proof. Let us first observe the derivatives of H_ϕ and K_ϕ with respect to parameters u and v. The derivatives of the equation (2.7) with respect to parameters u and v are

$$(K_\phi)_u = \frac{2QQ'\left(Q^2+v^2\right)}{D^6} \quad , \quad (K_\phi)_v = \frac{4vQ^2}{D^6} \tag{3.2}$$

respectively. The central and asymptotic normals of the ruled surface $\varphi(u,v)$ are vectors $t = (t_1,t_2,t_3)$ and $g=(g_1,g_2,g_3)$, respectively. If one takes the derivative of the equation (2.6) with respect to u and v by considering t, g and $\nabla\phi = (0,0,1)$, the following have been obtained

$$(H_\phi)_u = \frac{S_u}{2D^5} \quad , \quad (H_\phi)_v = \frac{S_v}{2D^5}. \tag{3.3}$$

The equalities of the expression of S_u and S_v can be obtained by basic calculations, left to the reader.

Let us arrange the equation (3.1) that is provided by the equations (3.2) and (3.3). This gives

$$(K_\phi)_u(H_\phi)_v - (K_\phi)_v(H_\phi)_u = \frac{\left(2QQ'v^2+2Q'Q^3\right)S_v - 4vQ^2S_u}{2D^{11}}.$$

So that this surface is a Weingarten surface in the Euclidean space with density e^z, the right side of the equation must be zero. Since the Gaussian curvature of K given by the equation (2.1) is different from zero, $Q'=F'=J'=0$ and ${t'}_3={g'}_3=0$ must be held in the neighbourhood of any point satisfying the condition $Q\neq 0$. In this case, the values Q, F, J, g_3 and t_3 must be constant. Moreover, $J=F=0$ when $H=0$. In this case, the surface is helicoidal surface. Q.E.D.

Hereunder this theory, we can reach the following result.

Result 3.2. For a non-developable $\varphi(u,v)$ ruled surface in the $3-$dimensional Euclidean space with density e^z, the following conditions hold;

i) φ is a Weingarten surface.

ii) The values Q,F,J,g_3,t_3 are constants.

Example 3.3. Let φ be a ruled surface in $3-$dimensional Euclidean space with density e^z defined as

$$\varphi(u,v) = (-\cos(u)+v\sin(u), -\sin(u)-v\cos(u), u).$$

The Gaussian and mean curvatures of the ruled surface $\varphi(u,v)$ are

$$K = -\frac{1}{\left(1+v^2\right)^2} \quad , \quad H = -\frac{1}{2\left(1+v^2\right)^{3/2}} \tag{3.4}$$

respectively. This non-developable helicoidal surface in Euclidean space is a ruled Weingarten surface. Let us show that this surface is a Weingarten surface in Euclidean space E^3 with density e^z.

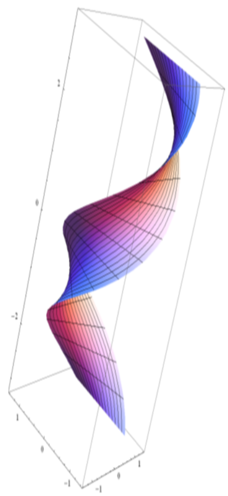

FIGURE 1. Non-developable ruled surfaces $\varphi(u,v)$

Let us substitute (3.4) in (2.7) and (2.8) such that $\nabla\phi = (0,0,1)$. So, the Gaussian $\phi-$curvature and mean $\phi-$curvature of the ruled surface $\varphi(u,v)$ in Euclidean space with density e^z are as follows:

$$K_\phi = -\frac{1}{(1+v^2)^2} \quad , \quad H_\phi = \frac{-1+v+v^3}{2(1+v^2)^{3/2}}.$$

If we calculate the partial derivative of the above equations according to the parameters u and v, and replace them in the equation (3.1), the equation

$$(K_\phi)_u (H_\phi)_v - (K_\phi)_v (H_\phi)_u = 0$$

is obtained. So the non-developable ruled surface $\varphi(u,v)$ in 3−dimensional Euclidean space with density e^z is a Weingarten surface.

Result 3.4. Let $\varphi(u,v)$ be a ruled Weingarten surface in 3−dimensional Euclidean space. Then, the ruled surface $\varphi(u,v)$ is also Weingarten in the 3−dimensional Euclidean space with density e^z.

3.2 Parallel surfaces of ruled Weingarten surfaces in Euclidean space with density e^z

Let $\varphi^r(u,v)$ be the parallel surface of the ruled Weingarten surface $\varphi(u,v)$ in Euclidean space with density e^z. Let us examine whether the parallel surface is Weingarten surface using Gaussian $\phi-$curvature and mean $\phi-$curvature.

Let the coefficients of the first and the second fundamental forms of parallel surface $\varphi^r(u,v)$ in Euclidean space with density e^z be E^r_ϕ, F^r_ϕ, G^r_ϕ and L^r_ϕ, M^r_ϕ, N^r_ϕ. In this case, there are the following equations,

$$\begin{aligned} {E_\phi}^r &= E_\phi - 2rL_\phi + r^2\langle N_u, N_u\rangle \\ {F_\phi}^r &= F_\phi - 2rM_\phi + r^2\langle N_u, N_v\rangle \\ {G_\phi}^r &= G_\phi + r^2\langle N_v, N_v\rangle \end{aligned}$$

and

$$\begin{aligned} {L_\phi}^r &= L_\phi - r\left\langle N_u, N_u \right\rangle \\ {M_\phi}^r &= M_\phi - r\left\langle N_u, N_v \right\rangle \\ {N_\phi}^r &= -r\left\langle N_v, N_v \right\rangle . \end{aligned}$$

By considering $\nabla\phi = (0,0,1)$ and $\Delta\phi = 0$, the Gaussian $\phi-$curvature and mean $\phi-$curvature of the parallel surface $\varphi^r(u,v)$ in Euclidean space with density e^z, are

$$K_\phi^r = \frac{-Q^2}{2D^4 + 4Dr\left(Jv^2 - Q'v + Q^2J - QF\right) - 2r^2Q^2}, \tag{3.5}$$

and

$$H_\phi^r = \frac{D\left(Jv^2 - Q'v + Q^2J - QF\right) - 2rQ^2}{2D^4 + 2Dr\left(Jv^2 - Q'v + Q^2J - QF\right) - 2r^2Q^2} - \frac{Qt_3 - vg_3}{2D}, \tag{3.6}$$

respectively.

Theorem 3.5. Let $\varphi^r(u,v)$ be the parallel surface of a non-developable ruled surface $\varphi(u,v) = \alpha(u) + vX(u)$ in the Euclidean space with density e^z. If the equations

$$\|X\| = \|X'\| = 1\ ,\quad \langle \alpha', X' \rangle = 0, \quad K \neq 0,$$

$$F = \langle \alpha', X \rangle\ ,\quad Q = \det(\alpha', X, X')\ ,\quad J = \det(X'', X', X)$$

exist, the following conditions are equivalent. If $\varphi(u,v)$ is a Weingarten surface, then $\varphi^r(u,v)$ is a Weingarten surface.

Proof. Let $\varphi(u,v)$ is a Weingarten surface. In this condition, the values of Q , F , J , t_3 and g_3 are constants. If partial derivatives of the Gaussian $\phi-$curvature of the parallel surface given by the equation (3.5) are taken according to parameters u and v, the following equations are obtained

$$\left(K_\phi^r\right)_u = \frac{\psi_u}{D\left(Q^4 + v^4 + 2Q^2v^2 + Dr\left(Jv^2 - Q'v + Q^2J - QF\right) - 2Q^2r^2\right)} \tag{3.7}$$

and

$$\left(K_\phi^r\right)_v = \frac{\psi_v}{D\left(Q^4 + v^4 + 2Q^2v^2 + Dr\left(Jv^2 - Q'v + Q^2J - QF\right) - 2Q^2r^2\right)}, \tag{3.8}$$

respectively. The equalities of the expression of ψ_u and ψ_v can be obtained by basic calculations, left to the reader.

If the partial derivatives of the mean $\phi-$curvature of the parallel surface given by the equation (3.6) are taken with respect to the parameters u and v, the following equations are obtained

$$\left(H_\phi^r\right)_u = \frac{\mu_u}{D(2D^4 + 2Dr\left(Jv^2 - Q'v + Q^2J - QF\right) - 2Q^2r)^2} - \frac{\varpi_u}{4D^3} \tag{3.9}$$

and

$$\left(H_\phi^r\right)_v = \frac{\mu_v}{D(2D^4 + 2Dr\left(Jv^2 - Q'v + Q^2J - QF\right) - 2Q^2r)^2} + \frac{Dg_3}{v}, \tag{3.10}$$

respectively. The equalities of the expression of ϖ_u, μ_u and μ_v can be obtained by basic calculations, left to the reader. If the values found in equations (3.7)-(3.10) are substituted in the equation $\left(K_\phi^r\right)_u\left(H_\phi^r\right)_v - \left(K_\phi^r\right)_v\left(H_\phi^r\right)_u$ and the conditions of ii) in the Result 3.2 are taken into account, we have the following equation

$$\left(K_\phi^r\right)_u\left(H_\phi^r\right)_v - \left(K_\phi^r\right)_v\left(H_\phi^r\right)_u = 0.$$

This completes the proof.

Q.E.D.

Theorem 3.6. Let $\varphi^r(u,v)$ be the parallel surface of the developable ruled Weingarten surface $\varphi(u,v)$ in the Euclidean space with density e^z. Then $\varphi^r(u,v)$ is a developable ruled Weingarten surface.

Proof. Let $\varphi(u,v) = \alpha(u) + vX(u)$ be a developable ruled Weingarten surface in Euclidean space with density e^z. The normal of the developable ruled surface is

$$N_1 = \alpha_u(u) \wedge X(u) + v\left(X_u(u) \wedge X(u)\right).$$

Since the vectors $\vec{\alpha}_u(u) \wedge X(u)$ and $X_u(u) \wedge X(u)$ are linearly dependent, then

$$\alpha_u(u) \wedge X(u) = \varepsilon\left(X_u(u) \wedge X(u)\right) \tag{3.11}$$

can be written, [14].

If one substitutes the equation (3.11) in the equation of N_1, the following equality is obtained:

$$N_1 = (\varepsilon + v)\left(X_u \wedge X\right).$$

Let the unit normal of a surface in Euclidean space with density e^z be N, we found that

$$N = \frac{N_1}{\|N_1\|} = X_u \wedge X.$$

The parallel surface of developable Weingarten surface $\varphi(u,v)$ in Euclidean space with density e^z is

$$\varphi^r(u,v) = \alpha(u) + r\left(X_u(u) \wedge X(u)\right) + vX(u)$$

and the drall of $\varphi^r(u,v)$ is

$$\lambda = \langle \alpha_u, X_u \wedge X \rangle + r\langle X_{uu} \wedge X, X_u \wedge X \rangle.$$

From these calculations, it is seen that λ is zero. So $\varphi^r(u,v)$ is the developable ruled Weingarten surface in Euclidean space with density e^z. Thus the proof is completed. Q.E.D.

Theorem 3.7. The principal curvatures $k_{1\phi}$ and $k_{2\phi}$ of a surface $\varphi(u,v)$ in E^3 with density e^z are the roots of the following quadratic equation:

$$\lambda^2 - \lambda\left(2H_\phi - \frac{d\phi}{dN}\right) + \left(K_\phi - 2H_\phi\frac{d\phi}{dN} + \nabla\phi\right) = 0.$$

Here the principal curvatures of $k_{1\phi}$ and $k_{2\phi}$ are obtained as follows:

$$k_{1\phi} = \frac{\left(2H_\phi - \frac{d\phi}{dN}\right) + \sqrt{4{H_\phi}^2 + \left(\frac{d\phi}{dN}\right)^2 - 4K_\phi + 4H_\phi \frac{d\phi}{dN} - 4\nabla\phi}}{2}$$

$$k_{2\phi} = \frac{\left(2H_\phi - \frac{d\phi}{dN}\right) - \sqrt{4{H_\phi}^2 + \left(\frac{d\phi}{dN}\right)^2 - 4K_\phi + 4H_\phi \frac{d\phi}{dN} - 4\nabla\phi}}{2}.$$

Proof. Let $\varphi(u,v)$ be a ruled surface in Euclidean space with the density of e^z and $k_{1\phi}$, $k_{2\phi}$ denote the principal curvatures of this surface. The equation $\det(A - \lambda \mathrm{I}) = 0$ provides us a characteristic equation such that $A = \begin{pmatrix} k_{1\phi} & 0 \\ 0 & k_{2\phi} \end{pmatrix}$ is a matrix and $\lambda$is a constant.

To find the roots of this equation, let's calculate Gaussian $\phi-$curvature and mean $\phi-$curvature. The Gaussian ϕ - curvature in terms of principle curvatures $k_{1\phi}$ and $k_{2\phi}$ is

$$\begin{aligned} K_\phi &= k_1 k_2 - \nabla\phi = \left(k_{1\phi} + \frac{d\phi}{dN}\right)\left(k_{2\phi} + \frac{d\phi}{dN}\right) - \nabla\phi \\ &= k_{1\phi}k_{2\phi} + \frac{d\phi}{dN}\left(2H_\phi - \frac{d\phi}{dN}\right) + \left(\frac{d\phi}{dN}\right)^2 - \nabla\phi \end{aligned}$$

and then the product of roots is

$$k_{1\phi}k_{2\phi} = K_\phi - 2H_\phi \frac{d\phi}{dN} + \nabla\phi.$$

The mean $\phi-$curvature in terms of principle curvatures $k_{1\phi}$ and $k_{2\phi}$ is

$$\begin{aligned} H_\phi &= \frac{k_1 + k_2 - \frac{d\phi}{dN}}{2} = \frac{\left(k_{1\phi} + \frac{d\phi}{dN}\right) + \left(k_{2\phi} + \frac{d\phi}{dN}\right) - \frac{d\phi}{dN}}{2} \\ &= \frac{1}{2}\left(k_{1\phi} + k_{2\phi} + \frac{d\phi}{dN}\right) \end{aligned}$$

and then the sum of roots is found as

$$k_{1\phi} + k_{2\phi} = 2H_\phi - \frac{d\phi}{dN}.$$

By considering the relations for $k_{1\phi}k_{2\phi}$ and $k_{1\phi} + k_{2\phi}$ the equation $\det(A - \lambda \mathrm{I}) = 0$ give us

$$\begin{aligned} \mathrm{P}_s(\lambda) &= \det\left(\begin{pmatrix} k_{1\phi} & 0 \\ 0 & k_{2\phi} \end{pmatrix} - \lambda . \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= k_{1\phi}k_{2\phi} - \lambda\left(k_{1\phi} + k_{2\phi}\right) + \lambda^2 \\ &= \lambda^2 - \lambda\left(2H_\phi - \frac{d\phi}{dN}\right) + \left(K_\phi - 2H_\phi \frac{d\phi}{dN} + \nabla\phi\right) = 0. \end{aligned}$$

The roots of this quadratic equation are

$$k_{1\phi} = \frac{\left(2H_\phi - \frac{d\phi}{dN}\right) + \sqrt{4{H_\phi}^2 + \left(\frac{d\phi}{dN}\right)^2 - 4K_\phi + 4H_\phi \frac{d\phi}{dN} - 4\nabla\phi}}{2}$$

and

$$k_{2\phi} = \frac{\left(2H_\phi - \frac{d\phi}{dN}\right) - \sqrt{4{H_\phi}^2 + \left(\frac{d\phi}{dN}\right)^2 - 4K_\phi + 4H_\phi \frac{d\phi}{dN} - 4\nabla\phi}}{2},$$

respectively.

Q.E.D.

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