On the jerk in motion along a regular surface curve

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Abstract

The jerk is the third time derivative of the position vector, and hence the time derivative of acceleration vector. In this paper, we take into consideration a particle moving on a regular surface curve, which is equipped with the Darboux frame, in three dimensional Euclidean space and resolve its jerk vector along tangential direction and two special radial directions. Moreover, the motion of a particle along a right handed circular helix is given as an illustrative example.

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1 Introduction

The force acting on a particle is concerned with its acceleration through the equality $\mathbf{F} = m\mathbf{a}$ in Newtonian physics. The jerk $\mathbf{J}$ is the time derivative of acceleration. Thus, if the mass is constant the equality $\mathbf{J} = \frac{1}{m}\frac{d\mathbf{F}}{dt}$ holds. If the time derivative of force is nonzero, the jerk vector is a nonzero vector.

For a large acceleration of short duration, the jerk vector has precious applications in the design of intermittent-motion mechanisms, e.g. cams and genevas [1, 2]. When a gymnast does gymnastic exercises or a machinist drives a high-speed train or a stock-car racer races on track, the acceleration changes suddenly. In situations like this, to estimate the lower threshold of just noticeable jerk and upper values of the jerk that can be tolerated by humans without undue discomfort has an important place [3]. These calculations were considered by Melchior in [4].

The decomposition of the jerk vector along a curve in 3-dimensional Euclidean space E^3 is well known in the literature due to the study [5] of Resal. In this decomposition, the jerk vector lies along the tangential, normal and binormal components. Recently, a new resolution of jerk vector along the tangential direction, radial direction in osculating plane and the other radial direction in rectifying plane has been presented by Özen et al. [6]. Also, more recently, the jerk of a particle moving on an analytic space curve equipped with the modified orthogonal frame has been investigated by Özen et al. [7]. In the present paper, we inspire from the studies [6, 7] and discuss the same topic according to Darboux frame for a particle moving on a regular surface curve.

In this paper, firstly, we have given a short knowledge about the Serret-Frenet frame and Darboux frame. Afterwards, for a particle moving along a regular surface curve, which is equipped with the Darboux frame, we have obtained the resolution of the jerk vector along tangential and radial directions. Moreover, an example for circular helices has been given.

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2 Preliminaries

Let E^3 be equipped with the standard scalar product

$$\mathbf{U}\cdot\mathbf{V}=u_1v_1+u_2v_2+u_3v_3$$

where $\mathbf{U}=(u_1\,,\,u_2\,,\,u_3)\,,\,\mathbf{V}=(v_1\,,\,v_2\,,\,v_3)$ are any vectors in E^3. The norm of a vector $\mathbf{U}\in E^3$ is defined as $\|\mathbf{U}\|=\sqrt{\mathbf{U}\cdot\mathbf{U}}$. A curve $\alpha=\alpha(s):I\subset R\rightarrow E^3$ is called a unit speed curve if $\|\alpha'(s)\|=1$ for all $s\in I$. In that case, s is called arc-length parameter of $\alpha(s)$.

The Serret-Frenet frame of the curve $\alpha(s)$ is denoted by $\{\mathbf{T}(s)\,,\,\mathbf{N}(s)\,,\,\mathbf{B}(s)\}$ where the unit vectors $\mathbf{T}(s)\,,\ \mathbf{N}(s)$ and $\mathbf{B}(s)$ are the unit tangent, unit principal normal and unit binormal vectors respectively. Also, the Serret-Frenet formulas are given as follows:

$$\mathbf{T}'=\kappa\mathbf{N},\qquad \mathbf{N}'=-\kappa\mathbf{T}+\tau\mathbf{B},\qquad \mathbf{B}'=-\tau\mathbf{N} \tag{2.1}$$

where $\kappa(s)=\|\mathbf{T}'(s)\|$ is the curvature function and $\tau(s)=-(\mathbf{B}'(s)\cdot\mathbf{N}(s))$ is the torsion function [8].

Let M be a regular surface and $\alpha:I\subset R\rightarrow M$ be a unit speed curve on this surface. In this case, there exists the Darboux frame which is showed with $\{\mathbf{T},\,\mathbf{Y},\,\mathbf{U}\}$ along the curve α. In this frame, $\mathbf{T}$ is the unit tangent vector of the curve α, and $\mathbf{U}$ is the unit normal vector of the surface M restricted to the curve α. On the other hand, $\mathbf{Y}$ is the unit vector which is obtained by vector product these two vectors, that is, $\mathbf{Y}=\mathbf{U}\times\mathbf{T}$. Because the unit tangent vector $\mathbf{T}$ is mutual in both Serret-Frenet frame and Darboux frame, the vectors $\mathbf{N}$, $\mathbf{B}$, $\mathbf{Y}$, $\mathbf{U}$ lie in the same plane. Thus, there is a relation between these frames as in the following:

$$\mathbf{T}=\mathbf{T},\qquad \mathbf{Y}=\cos\varphi\,\mathbf{N}-\sin\varphi\,\mathbf{B},\qquad \mathbf{U}=\sin\varphi\,\mathbf{N}+\cos\varphi\,\mathbf{B} \tag{2.2}$$

where φ is the angle between the vectors $\mathbf{U}$ and $\mathbf{B}$ (or $\mathbf{Y}$ and $\mathbf{N}$). Furthermore, the derivative formulas of the Darboux frame are given by

$$\mathbf{T}'=k_g\mathbf{Y}+k_n\mathbf{U},\qquad \mathbf{Y}'=-k_g\mathbf{T}+\tau_g\mathbf{U},\qquad \mathbf{U}'=-k_n\mathbf{T}-\tau_g\mathbf{Y} \tag{2.3}$$

where k_n is the normal curvature, k_g is the geodesic curvature and τ_g is the geodesic torsion of α. Also, k_g, k_n and τ_g satisfy the equations

$$\begin{aligned} k_g(s) &= \sqrt{{k_g}^2(s)+{k_n}^2(s)}\cos\varphi(s)\,,\\ k_n(s) &= \sqrt{{k_g}^2(s)+{k_n}^2(s)}\sin\varphi(s)\,,\\ \tau_g(s) &= \tau(s)-\frac{d\varphi(s)}{ds} \end{aligned} \tag{2.4}$$

where $\varphi(s)=\arctan\left(k_n(s)/k_g(s)\right)$ and $\sqrt{{k_g}^2(s)+{k_n}^2(s)}=\kappa(s)$ [9–12].

Let a particle P of mass $m\,(>0)$ move on a regular surface M in E^3. Choose any fixed origin O in the space and symbolize the position vector of P at time t by $\mathbf{x}$. Let C, with the arc length parameter s, be the oriented curve traced out by P where the arc length of C corresponds t. In that case, the unit tangent vector for the curve C can be written as

$$\mathbf{T}=\frac{d\mathbf{x}}{ds}. \tag{2.5}$$

By means of (2.1) and (2.5), the velocity vector $\mathbf{v} = d\mathbf{x}/dt$, the acceleration vector $\mathbf{a} = d\mathbf{v}/dt$ and the jerk vector $\mathbf{J} = d\mathbf{a}/dt$ of P at time t are given by (see [13] for more details):

$$\begin{aligned}
\mathbf{v} &= \left(\frac{ds}{dt}\right)\mathbf{T}, \\
\mathbf{a} &= \left(\frac{d^2s}{dt^2}\right)\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}, \\
\mathbf{J} &= \left[\left(\frac{d^3s}{dt^3}\right) - \left(\frac{ds}{dt}\right)^3\kappa^2\right]\mathbf{T} + \left[3\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\kappa + \left(\frac{ds}{dt}\right)^3\frac{d\kappa}{ds}\right]\mathbf{N} + \left[\kappa\tau\left(\frac{ds}{dt}\right)^3\right]\mathbf{B}.
\end{aligned}$$

On the other hand, the acceleration vector $\mathbf{a}$ of P at time t is given as

$$\mathbf{a} = \left(\frac{d^2s}{dt^2}\right)\mathbf{T} + \sqrt{{k_g}^2 + {k_n}^2}\left(\frac{ds}{dt}\right)^2\cos\varphi\,\mathbf{Y} + \sqrt{{k_g}^2 + {k_n}^2}\left(\frac{ds}{dt}\right)^2\sin\varphi\,\mathbf{U} \tag{2.6}$$

with respect to Darboux basis [14].

3 Jerk of a particle moving along a regular surface curve

In this section, the jerk vector of a particle moving along a regular surface curve is given according to Darboux basis. Moreover, the resolution of the jerk vector along tangential and two special radial directions has been found. To do so, we continue to take into consideration the aforesaid particle P.

By using (2.3), (2.4) and (2.6), we can immediately get the jerk vector $\mathbf{J}$ of P at time t with respect to Darboux basis as follows:

$$\mathbf{J} = C_{\mathbf{T}}\,\mathbf{T} + C_{\mathbf{Y}}\,\mathbf{Y} + C_{\mathbf{U}}\mathbf{U} \tag{3.1}$$

where

$$\begin{aligned}
C_{\mathbf{T}} &= \left(\frac{d^3s}{dt^3}\right) - \left(\frac{ds}{dt}\right)^3\left({k_g}^2 + {k_n}^2\right), \\
C_{\mathbf{Y}} &= \cos\varphi\left[3\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\sqrt{{k_g}^2 + {k_n}^2} + \frac{d\left(\sqrt{{k_g}^2 + {k_n}^2}\right)}{ds}\left(\frac{ds}{dt}\right)^3\right] \\
&\quad - \sin\varphi\left[\sqrt{{k_g}^2 + {k_n}^2}\left(\tau_g + \frac{d\varphi}{ds}\right)\left(\frac{ds}{dt}\right)^3\right], \\
C_{\mathbf{U}} &= \sin\varphi\left[3\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\sqrt{{k_g}^2 + {k_n}^2} + \frac{d\left(\sqrt{{k_g}^2 + {k_n}^2}\right)}{ds}\left(\frac{ds}{dt}\right)^3\right] \\
&\quad + \cos\varphi\left[\sqrt{{k_g}^2 + {k_n}^2}\left(\tau_g + \frac{d\varphi}{ds}\right)\left(\frac{ds}{dt}\right)^3\right].
\end{aligned}$$

This equation yields

$$
\begin{aligned}
\mathbf{J} \;=\; & \left[\left(\frac{d^3 s}{dt^3}\right) - \left(\frac{ds}{dt}\right)^3 \left({k_g}^2 + {k_n}^2\right)\right]\mathbf{T} \\
& + \left[3\left(\frac{ds}{dt}\right)\left(\frac{d^2 s}{dt^2}\right)\sqrt{{k_g}^2+{k_n}^2} + \frac{d\left(\sqrt{{k_g}^2+{k_n}^2}\right)}{ds}\left(\frac{ds}{dt}\right)^3\right](\cos\varphi\mathbf{Y} + \sin\varphi\mathbf{U}) \qquad (3.2) \\
& + \left[\sqrt{{k_g}^2+{k_n}^2}\left(\tau_g + \frac{d\varphi}{ds}\right)\left(\frac{ds}{dt}\right)^3\right](-\sin\varphi\,\mathbf{Y} + \cos\varphi\,\mathbf{U})\,.
\end{aligned}
$$

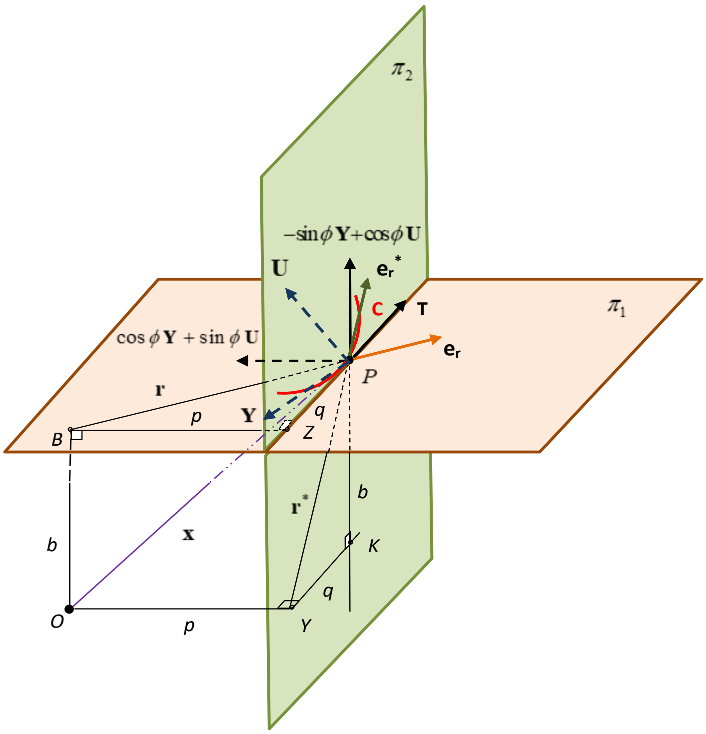

FIGURE 1. A particle P moves along a curve C on a regular surface M in 3-dimensional Euclidean space (To provide visual harmony of the figure, the surface M has been removed from the figure). B is the foot of the perpendicular which is from origin O to the plane π_1 and BZ is perpendicular to the tangent line. The position vector of P relative to B is shown with $\mathbf{r}$, and $\mathbf{e_r}$ is the unit vector in direction of BP. Y is the foot of the perpendicular which is from origin O to the plane π_2 and YK is perpendicular to the line determined by the vector $(-\sin\varphi\,\mathbf{Y} + \cos\varphi\,\mathbf{U})$. The position vector of P relative to Y is shown with $\mathbf{r}^*$, and $\mathbf{e_r}^*$ is the unit vector in direction of YP.

A particle, moving on a curve, may be seen as a point of this curve. So, P has a position vector in terms of the Darboux basis of the curve C. Suppose that the position vector of P on the Darboux basis is resolved as in the following:

$$\mathbf{x} = q\,\mathbf{T} - p\,(\cos\varphi\,\mathbf{Y} + \sin\varphi\,\mathbf{U}) + b\,(-\sin\varphi\,\mathbf{Y} + \cos\varphi\,\mathbf{U}) \tag{3.3}$$

where

$$q = \mathbf{x}\cdot\mathbf{T}, \quad -p = \mathbf{x}\cdot(\cos\varphi\,\mathbf{Y} + \sin\varphi\,\mathbf{U}), \quad b = \mathbf{x}\cdot(-\sin\varphi\,\mathbf{Y} + \cos\varphi\,\mathbf{U}). \tag{3.4}$$

Show with $\mathbf{r}$ and $\mathbf{r}^*$ the vectors

$$\mathbf{r} = q\mathbf{T} - p\,(\cos\varphi\,\mathbf{Y} + \sin\varphi\,\mathbf{U}), \quad \mathbf{r}^* = q\mathbf{T} + b\,(-\sin\varphi\,\mathbf{Y} + \cos\varphi\,\mathbf{U}) \tag{3.5}$$

lying in plane $\pi_1 = Sp\,\{\mathbf{T}, \cos\varphi\mathbf{Y} + \sin\varphi\mathbf{U}\}$ to C at P and in plane $\pi_2 = Sp\,\{\mathbf{T}, -\sin\varphi\mathbf{Y} + \cos\varphi\mathbf{U}\}$ to C at P, respectively. In that case, we get the equalities

$$r^2 = \mathbf{r}\cdot\mathbf{r} = p^2 + q^2, \quad (r^*)^2 = \mathbf{r}^*\cdot\mathbf{r}^* = q^2 + b^2 \tag{3.6}$$

where r and r^* are the lengths of the vectors $\mathbf{r}$ and $\mathbf{r}^*$, respectively (see Figure 1).

By vector product of the position vector $\mathbf{x}$ given in (3.3), and the linear momentum vector $\mathbf{p} = m\left(\frac{ds}{dt}\right)\mathbf{T}$, the angular momentum vector of P about O is obtained as follows:

$$\mathbf{H}^O = m\,b\left(\frac{ds}{dt}\right)(\cos\varphi\,\mathbf{Y} + \sin\varphi\,\mathbf{U}) + mp\left(\frac{ds}{dt}\right)(-\sin\varphi\,\mathbf{Y} + \cos\varphi\,\mathbf{U}). \tag{3.7}$$

By using the equation (3.5), let us express the vector $(\cos\varphi\,\mathbf{Y} + \sin\varphi\,\mathbf{U})$ in terms of $\mathbf{r}$ and $\mathbf{T}$ and express the vector $(-\sin\varphi\,\mathbf{Y} + \cos\varphi\,\mathbf{U})$ in terms of $\mathbf{r}^*$ and $\mathbf{T}$ to resolve the jerk vector $\mathbf{J}$ in (3.2) along the tangential direction, radial direction BP in the plane π_1, and radial direction YP in the plane π_2. But, these are possible if and only if $p \neq 0$ and $b \neq 0$. If we make the physical assumption that the components of the angular momentum along the vectors $(\cos\varphi\,\mathbf{Y} + \sin\varphi\,\mathbf{U})$ and $(-\sin\varphi\,\mathbf{Y} + \cos\varphi\,\mathbf{U})$ never vanish, we can ensure that each of p and b are not equal to zero. Then, we can write the equations

$$\cos\varphi\,\mathbf{Y} + \sin\varphi\,\mathbf{U} = \frac{1}{p}\,(-\mathbf{r} + q\mathbf{T}), \quad -\sin\varphi\,\mathbf{Y} + \cos\varphi\,\mathbf{U} = \frac{1}{b}\,(-q\mathbf{T} + \mathbf{r}^*). \tag{3.8}$$

Also, from the aforesaid physical assumption and the equation (3.6), $r \neq 0$ and $r^* \neq 0$. Therefore, the unit vectors $\mathbf{e_r}$ and $\mathbf{e_r}^*$ can be defined as

$$\mathbf{e_r} = \frac{1}{r}\mathbf{r}, \quad \mathbf{e_r}^* = \frac{1}{r^*}\mathbf{r}^*. \tag{3.9}$$

By means of (3.8) and (3.9), we get

$$\cos\varphi\mathbf{Y} + \sin\varphi\mathbf{U} = \frac{(-r\,\mathbf{e_r} + q\mathbf{T})}{p}, \quad -\sin\varphi\mathbf{Y} + \cos\varphi\mathbf{U} = \frac{(-q\mathbf{T} + r^*\mathbf{e_r}^*)}{b}. \tag{3.10}$$

Now, if we substitute the equation (3.10) into the equation (3.2), we obtain the jerk vector $\mathbf{J}$ as in the following:

$$\begin{aligned}
\mathbf{J} &= \left[\begin{array}{c} \left(\frac{d^3s}{dt^3}\right) - \left(\frac{ds}{dt}\right)^3\left({k_g}^2+{k_n}^2\right) + 3\frac{q}{p}\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\sqrt{{k_g}^2+{k_n}^2} \\ \\ +\frac{q}{p}\frac{d\left(\sqrt{{k_g}^2+{k_n}^2}\right)}{ds}\left(\frac{ds}{dt}\right)^3 - \frac{q}{b}\left(\frac{ds}{dt}\right)^3\sqrt{{k_g}^2+{k_n}^2}\left(\tau_g+\frac{d\varphi}{ds}\right) \end{array}\right]\mathbf{T} \\
&\quad + \left[-3\frac{r}{p}\left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\sqrt{{k_g}^2+{k_n}^2} - \frac{r}{p}\frac{d\left(\sqrt{{k_g}^2+{k_n}^2}\right)}{ds}\left(\frac{ds}{dt}\right)^3\right]\mathbf{e_r} \qquad (3.11) \\
&\quad + \left[\frac{r^*}{b}\left(\frac{ds}{dt}\right)^3\sqrt{{k_g}^2+{k_n}^2}\left(\tau_g+\frac{d\varphi}{ds}\right)\right]\mathbf{e_r}^* \\
&= T_t\,\mathbf{T} + T_r\,\mathbf{e_r} + T_{r^*}\,\mathbf{e_r}^*.
\end{aligned}$$

Here, T_r, T_{r^*} and T_t are the first radial, second radial and tangential components of the jerk, respectively.

By taking into consideration the expressions and equations which are given above, we can state the following theorem:

Theorem 3.1. Let a particle P of mass $m\,(>0)$ move on a regular surface curve C in 3-dimensional Euclidean space E^3, and suppose that each of the components of angular momentum along the unit vector $(\cos\varphi\,\mathbf{Y}+\sin\varphi\,\mathbf{U})$ and along the unit vector $(-\sin\varphi\,\mathbf{Y}+\cos\varphi\,\mathbf{U})$ never vanish. In that case, the jerk vector $\mathbf{J}$ of the particle P can be expressed as in (3.11).

4 Motion of a particle along a right-handed circular helix

Suppose that a particle P travels along a right-handed circular helix which lies on a cylinder of radius R. The position vector of P is given by

$$\mathbf{x} = R\cos(\omega t)\,\mathbf{i} + R\sin(\omega t)\,\mathbf{j} + v_z t\,\mathbf{k}$$

in Cartesian coordinates where t and ω indicate the time and time independent angular frequency, respectively. Here, $\{\mathbf{i}\,,\mathbf{j}\,,\mathbf{k}\,\}$ represents a fixed right-handed orthonormal frame, v_z represents a positive constant. Let the axis $\mathbf{k}$ be the axis of the helix and α be the helix angle which satisfies $\tan\alpha = \frac{R\omega}{v_z}$. Then, we get

$$\begin{aligned}
\mathbf{v} &= -R\omega\sin(\omega t)\,\mathbf{i} + R\omega\cos(\omega t)\,\mathbf{j} + v_z\mathbf{k}, \\
\mathbf{a} &= -R\omega^2\cos(\omega t)\,\mathbf{i} - R\omega^2\sin(\omega t)\,\mathbf{j}, \\
\mathbf{J} &= R\omega^3\sin(\omega t)\,\mathbf{i} - R\omega^3\cos(\omega t)\,\mathbf{j}
\end{aligned}$$

for the particle P. Also, we can write easily the following:

$$dx = -R\omega\sin(\omega t)\,dt,\;\; dy = R\omega\cos(\omega t)\,dt,\;\; dz = v_z dt,\;\; ds = \sqrt{R^2\omega^2+{v_z}^2}dt.$$

So, the equalities

$$\frac{ds}{dt} = \sqrt{R^2\omega^2 + {v_z}^2}, \quad \frac{d^2 s}{dt^2} = 0, \quad \frac{d^3 s}{dt^3} = 0$$

hold.

As the path of the particle P, the right-handed circular helix on a cylinder of radius R is given below:

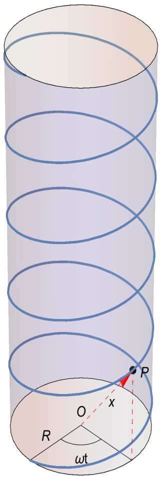

FIGURE 2.

It is easy to show that the helix can be parameterized by the arc-length $s = s(t) = t\sqrt{R^2\omega^2 + {v_z}^2}$ as

$$\gamma(s) = \left(R\cos\left(\frac{\omega s}{\sqrt{R^2\omega^2 + {v_z}^2}}\right), \; R\sin\left(\frac{\omega s}{\sqrt{R^2\omega^2 + {v_z}^2}}\right), \; \frac{v_z s}{\sqrt{R^2\omega^2 + {v_z}^2}} \right). \tag{4.1}$$

In that case, the Serret-Frenet basis for the helix can be calculated as follows:

$$\begin{aligned} \mathbf{T} &= -\sin\alpha \sin\left(\frac{\omega s}{\sqrt{R^2\omega^2 + {v_z}^2}}\right)\mathbf{i} + \sin\alpha\cos\left(\frac{\omega s}{\sqrt{R^2\omega^2 + {v_z}^2}}\right)\mathbf{j} + \cos\alpha\mathbf{k}, \\ \mathbf{N} &= -\cos\left(\frac{\omega s}{\sqrt{R^2\omega^2 + {v_z}^2}}\right)\mathbf{i} - \sin\left(\frac{\omega s}{\sqrt{R^2\omega^2 + {v_z}^2}}\right)\mathbf{j}, \\ \mathbf{B} &= \cos\alpha\sin\left(\frac{\omega s}{\sqrt{R^2\omega^2 + {v_z}^2}}\right)\mathbf{i} - \cos\alpha\cos\left(\frac{\omega s}{\sqrt{R^2\omega^2 + {v_z}^2}}\right)\mathbf{j} + \sin\alpha\mathbf{k}. \end{aligned} \tag{4.2}$$

It is not difficult to see that the curvature and the torsion are constant:

$$\kappa = \frac{R}{R^2 + ({v_z}^2/\omega^2\,)}, \qquad \tau = \frac{v_z/\omega}{R^2 + ({v_z}^2/\omega^2\,)}.$$

By means of (2.4), the equalities

$$\begin{aligned}\tau_g &= \frac{v_z/\omega}{R^2+({v_z}^2/\omega^2\ )}-\frac{d\varphi}{ds},\\ k_g &= \frac{R}{R^2+({v_z}^2/\omega^2\ )}\cos\varphi,\\ k_n &= \frac{R}{R^2+({v_z}^2/\omega^2\ )}\sin\varphi\end{aligned}$$

hold. By taking into consideration the relation (2.2) between the Serret-Frenet frame and Darboux frame and the equation (4.2), we obtain the first Darboux base as

$$\mathbf{T}=-\sin\alpha\sin\left(\frac{\omega s}{\sqrt{R^2\omega^2+{v_z}^2}}\right)\mathbf{i}+\ \sin\alpha\cos\left(\frac{\omega s}{\sqrt{R^2\omega^2+{v_z}^2}}\right)\mathbf{j}\ +\cos\alpha\,\mathbf{k}$$

second Darboux base as

$$\begin{aligned}\mathbf{Y} &= \left[-\cos\varphi\cos\left(\frac{\omega s}{\sqrt{R^2\omega^2+{v_z}^2}}\right)-\sin\varphi\cos\alpha\sin\left(\frac{\omega s}{\sqrt{R^2\omega^2+{v_z}^2}}\right)\right]\mathbf{i}\\ &\quad+\left[-\cos\varphi\sin\left(\frac{\omega s}{\sqrt{R^2\omega^2+{v_z}^2}}\right)+\sin\varphi\cos\alpha\cos\left(\frac{\omega s}{\sqrt{R^2\omega^2+{v_z}^2}}\right)\right]\mathbf{j}\\ &\quad+[-\sin\varphi\cos\alpha]\,\mathbf{k}\end{aligned}$$

and third Darboux base as

$$\begin{aligned}\mathbf{U} &= \left[-\sin\varphi\cos\left(\frac{\omega s}{\sqrt{R^2\omega^2+{v_z}^2}}\right)+\cos\varphi\cos\alpha\sin\left(\frac{\omega s}{\sqrt{R^2\omega^2+{v_z}^2}}\right)\right]\mathbf{i}\\ &\quad+\left[-\sin\varphi\sin\left(\frac{\omega s}{\sqrt{R^2\omega^2+{v_z}^2}}\right)-\cos\varphi\cos\alpha\cos\left(\frac{\omega s}{\sqrt{R^2\omega^2+{v_z}^2}}\right)\right]\mathbf{j}\\ &\quad+[\cos\varphi\sin\alpha]\,\mathbf{k}\end{aligned}$$

for the helix. From these last three equalities, (3.4) and (4.1),

$$q=\frac{sv_z\cos\alpha}{\sqrt{R^2\omega^2+{v_z}^2}}=tv_z\cos\alpha,\quad p=R,\quad b=\frac{sv_z\sin\alpha}{\sqrt{R^2\omega^2+{v_z}^2}}=tv_z\sin\alpha \tag{4.3}$$

can be written. By substituting (4.3) in (3.3), the equality

$$\mathbf{x}=(tv_z\cos\alpha)\,\mathbf{T}-(R\cos\varphi+tv_z\sin\alpha\sin\varphi)\,\mathbf{Y}+(-R\sin\varphi+tv_z\sin\alpha\cos\varphi)\,\mathbf{U}$$

is obtained. On the other hand,

$$r=\sqrt{R^2+t^2\,{v_z}^2\cos^2\alpha}\,,\qquad r^*=t\,v_z$$

can be written by using (3.6) and (4.3).

Consequently, if Theorem 3.1 is applied, the components of the jerk are obtained as follows:

$$T_t = \frac{-R^2\omega^4 - \omega^2 {v_z}^2}{\sqrt{R^2\omega^2 + {v_z}^2}}, \quad T_r = 0, \quad T_{r^*} = \omega^2 v_z$$

for the particle P.

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