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Functoriality of modified realizability

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Abstract

We study the notion of modified realizability topos over an arbitrary Schönfinkel algebra. In particular we show that such toposes are induced by subsets of the algebra which we call right pseudo-ideals, and which generalize the right ideals (or right absorbing sets) previously considered. We also investigate the notion of compatibility with right pseudo-ideals which ensures that quasi-surjective (applicative) morphisms of Schönfinkel algebras yield geometric morphisms between these toposes.

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Introduction

As is by now well known, realizability toposes come in several 'flavours': ordinary, relative, modified, Herbrand .... In each case, what we have is a construction which takes as input a Schönfinkel algebra (also known as a partial combinatory algebra) — or, in the case of relative realizability, a pair consisting of a Schönfinkel algebra with a distinguished subalgebra — and produces a topos. Naturally, it is of interest to know in what sense these constructions are functorial: that is, to determine what are the appropriate notions of 'morphism of Schönfinkel algebra' and to show that they induce geometric morphisms (or at least functors of some sort) between the corresponding toposes.

For ordinary realizability, the position is by now well understood. The first investigation of it was made by John Longley [14], who introduced the notion of an applicative morphism (which we now call simply a morphism) of Schönfinkel algebras, and showed that such morphisms correspond exactly to regular functors between ordinary realizability toposes which preserve $\neg\neg$-sheaves. Not all such functors derive from geometric morphisms; but, thanks to more recent work of Hofstra and van Oosten [6] and of the present author [10], we now know that geometric morphisms between ordinary realizability toposes correspond exactly to morphisms in Longley's sense which satisfy an additional property known as computational density or quasi-surjectivity.

For Benno van den Berg's notion of Herbrand realizability [1], we do not have such an exact characterization; but, thanks to the identification of Herbrand realizability toposes as the Gleason covers of ordinary realizability toposes [11], we can say that the construction is functorial in the sense that quasi-surjective morphisms of Schönfinkel algebras induce geometric morphisms of Herbrand realizability toposes. In contrast, for modified realizability relatively little is known; indeed, even the question of how to define modified realizability over an arbitrary Schönfinkel algebra (as opposed to the particular algebra structure on the natural numbers first introduced by Kleene) has not yet received a unanimous answer. Our purpose in this paper is, therefore, first to study what 'modified realizability over an arbitrary algebra' could (or should) mean, and secondly to investigate how its functoriality depends on the choices one makes in defining it.

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In the rest of this section, we introduce definitions and notation and recall some results that we shall need on ordinary realizability. By a *Schönfinkel algebra* we mean a set Λ equipped with a partial binary operation (that is, a partial map $\Lambda^2 \rightharpoonup \Lambda$) and two constants K, S satisfying $\mathsf{K}xy \coloneq x$ and $\mathsf{S}xyz \coloneq xz(yz)$ for all $x, y, z \in \Lambda$. (Here, for possibly-undefined terms s and t, $s \coloneq t$ denotes 'if t is defined, then so is s and they are then equal'.) In this paper, with the exception of Lemma 1.2 below, we shall always assume that Λ is *proper*, i.e. that $\mathsf{S}xy$ is defined for all x and y; but we shall not require it to be *strict* in the sense that $\mathsf{S}xyz$ is defined only when $xz(yz)$ is defined. Of course, we say Λ is *total* if xy is defined for all x and y; when we speak of a *partial Schönfinkel algebra*, we definitely mean one which is not total. In addition to the primitive combinators K and S, we shall make use of the combinators $\mathsf{I}, \mathsf{B}, \mathsf{E}, \mathsf{D}, \mathsf{P}_1$ and P_2 satisfying $\mathsf{I}x \coloneq x$, $\mathsf{B}xyz \coloneq x(yz)$, $\mathsf{E}xy \coloneq yx$, $\mathsf{D}xyz \coloneq zyx$, $\mathsf{P}_1 x \coloneq x(\mathsf{KI})$ and $\mathsf{P}_2 x \coloneq x\mathsf{K}$; and we shall often write $\langle x, y\rangle$ for $\mathsf{D}xy$ (note that $\mathsf{P}_1\langle x, y\rangle \coloneq x$ and $\mathsf{P}_2\langle x, y\rangle \coloneq y$, so that D and the P_i serve as pairing and unpairing combinators).

We say that Λ is *decisive* if it contains an element δ (a *decider* for Λ) satisfying $\delta xx \coloneq \mathsf{K}$ for all x and $\delta xy \coloneq \mathsf{KI}$ whenever $x \neq y$. (This property is commonly called 'decidability', but 'decisiveness' seems more appropriate since it is the algebra itself which does the deciding.) The (first) *Kleene algebra*, which consists of the set of natural numbers with nm defined as the output (if any) of the nth (unary) register machine program given input m, is well known to be decisive, but no total algebra can be decisive [13].

For the basic notions of tripos theory, we refer the reader to [7] and [17]. We shall be exclusively concerned with triposes on **Set**: if $\mathbb{T}$ is such a tripos and I is a set, we write T^I for its Heyting prealgebra of I-indexed families — normally we shall assume that $\mathbb{T}$ is given in the standard form described in [7], so that these are actual functions from I to a fixed set T (and in particular the 'generic' element of $\mathbb{T}$ is the identity function $T \to T$). For the ordinary realizability tripos $\mathbb{P}\Lambda$ associated with a Schönfinkel algebra Λ, we take this fixed set to be the power-set of Λ, and we preorder $P\Lambda^I$ by

$$(f \leq g) \Leftrightarrow \bigcap_{i \in I} (f(i) \Rightarrow g(i)) \text{ is inhabited}$$

where, for subsets p, q of Λ, $(p \Rightarrow q)$ denotes

$$\{\lambda \in \Lambda \mid (\forall x \in p)(\exists y \in q)(\lambda x \coloneq y)\} \quad .$$

We often say that an element of the above intersection *uniformly realizes* the inequality $f \leq g$ (or the family of implications $(f(i) \Rightarrow g(i))$).

Given a tripos $\mathbb{T}$, we write $\mathbf{Set}\langle\mathbb{T}\rangle$ for the induced topos: its objects are pairs (A, δ) where $\delta \colon A \times A \to T$ is a '$\mathbb{T}$-valued equality predicate' satisfying axioms which say that it is symmetric and transitive in the logic of $\mathbb{T}$; and morphisms $(A, \delta) \to (B, \varepsilon)$ are equivalence classes of functions $F \colon A \times B \to T$ which are strict, extensional, single-valued and entire in this logic, two such functions F, G being equivalent iff $F \cong G$ in $T^{A \times B}$.

By a *geometric morphism* $f \colon \mathbb{S} \to \mathbb{T}$ of triposes, we mean an indexed adjunction $(f^* \colon \mathbb{T} \to \mathbb{S} \dashv f_* \colon \mathbb{S} \to \mathbb{T})$ of which the left adjoint f^* preserves finite meets. It is well known that such an adjunction determines a geometric morphism $\mathbf{Set}\langle\mathbb{S}\rangle \to \mathbf{Set}\langle\mathbb{T}\rangle$ (though, in general, not every geometric morphism $\mathbf{Set}\langle\mathbb{S}\rangle \to \mathbf{Set}\langle\mathbb{T}\rangle$ is induced by a geometric morphism of triposes), and that the induced morphism of toposes is an inclusion (resp. a surjection) provided the morphism of triposes is a reflection (resp. a coreflection). (It is thus natural to transfer the terms 'inclusion' and 'surjection' from morphisms of toposes to the morphisms of triposes which induce them.)

A *morphism of Schönfinkel algebras* $t\colon \Lambda \nrightarrow \Pi$ is an entire relation (that is, one which relates each element of Λ to at least one element of Π) for which there exists an element $\tau \in \Pi$ (a *witness* for t) such that, whenever we have $t(x, y)$, $t(x', y')$ and xx' is defined, then $\tau yy'$ is defined and $t(xx', \tau yy')$. The set of all morphisms $\Lambda \nrightarrow \Pi$ is preordered by setting $s \leq t$ iff there exists $\rho \in \Pi$ such that $s(x, y)$ implies $t(x, \rho y)$. These definitions make Schönfinkel algebras into the objects of a (locally ordered) 2-category $\mathsf{Schön}$. A morphism $t\colon \Lambda \nrightarrow \Pi$ induces an indexed functor $t_+\colon \mathbb{P}\Lambda \to \mathbb{P}\Pi$ given by composing $P\Lambda$-valued functions with the mapping $(p \mapsto \{y \in \Pi \mid (\exists x \in p)(t(x, y)\})$, which may be shown to preserve finite meets; and an inequality $s \leq t$ induces an indexed natural transformation $s_+ \to t_+$ in an obvious way. We say a morphism t is *functional* if it is the graph of a function, i.e. if each element of Λ is related to exactly one element of Π, and *surjective* if t° is entire, i.e. each element of Π is a relative of at least one element of Λ. Finally, we say t is a *homomorphism* if it is functional and witnessed by I (that is if we have $t(x)t(x') = t(xx')$ for all $x, x' \in \Lambda$ — where, as we often do for functional morphisms, we are using the standard functional notation $y = t(x)$ for $t(x, y)$), and in addition t preserves the constants K and S.

In [6] it was shown that t_+ has a right adjoint (i.e., is the inverse image part of a geometric morphism of triposes) iff t satisfies a condition called computational density, and in [10] it was shown that this condition is equivalent to a simpler one called quasi-surjectivity: we say t is *quasi-surjective* if there exists a function $r\colon \Pi \to \Lambda$ and an element $\rho \in \Pi$ such that for all $y, y' \in \Pi$, if $t(r(y), y')$ holds then $\rho y' = y$. (Note that any surjective functional morphism t is quasi-surjective: we may take r to be any function whose graph is contained in t°, and ρ to be the I combinator. However, a quasi-surjective morphism need not be surjective even if it is functional; a counterexample is provided by the mapping $(x \mapsto \langle \mathsf{K}, x\rangle)$ from any Λ to itself, which is quasi-surjective because it is isomorphic to the identity.) It follows that the assignment $(\Lambda \mapsto \mathbf{Set}\langle\mathbb{P}\Lambda\rangle)$ becomes a pseudofunctor (contravariant on 1-cells) from the 2-category $\mathsf{Schön}_{qs}$ obtained by restricting to quasi-surjective 1-cells in $\mathsf{Schön}$, to the 2-category of toposes and geometric morphisms. In addition, it was shown in [10] that this 2-functor is locally an equivalence, i.e. that all geometric morphisms between ordinary realizability toposes, and all geometric transformations between them, are (up to isomorphism) of this form.

1 Right ideals and right pseudo-ideals

The basic idea behind modified realizability is that any proposition should come equipped with two sets of 'potential' and 'actual' realizers, of which the second is contained in the first and, in addition, the first contains all members of some fixed (nonempty) set Θ. An actual realizer for an implication should code a function mapping potential realizers for the first proposition to potential realizers for the second, and also mapping actual realizers for the first to actual realizers for the second. The tripos-theoretic formulation of this idea was first investigated by Grayson [5] and subsequently by Hyland and Ong [8] and van Oosten [15]: it leads to the definition of a tripos which we shall denote $\mathbb{M}_\Theta\Lambda$ (or simply $\mathbb{M}\Lambda$, if we do not need to specify Θ explicitly), whose I-indexed families are functions from I to the set

$$M_\Theta\Lambda = \{(p, a) \in P\Lambda \times P\Lambda \mid a \cup \Theta \subseteq p\}$$

and whose preorder is given by

$$(f \leq g) \Leftrightarrow \bigcap\{(f_1(i) \Rightarrow g_1(i)) \cap (f_2(i) \Rightarrow g_2(i)) \mid i \in I\} \text{ is inhabited .}$$

(Here and subsequently, we write $f_1(i)$ and $f_2(i)$ for the first and second components of $f(i)$.)

In verifying that $\mathbb{M}_\Theta\Lambda$ is indeed a tripos, for suitable choices of Θ, it is convenient to compare it with a simpler tripos $\mathbb{P}_1\Lambda$, which is simply the case $\Theta = \varnothing$ of the above. The fact that this is a tripos is easy to verify: its meet and join operations, and its quantification, are simply induced 'componentwise' by those of $\mathbb{P}\Lambda$, and its implication is given by

$$(f \Rightarrow g)(i) = ((f_1(i) \Rightarrow g_1(i)), ((f_1(i) \Rightarrow g_1(i)) \cap (f_2(i) \Rightarrow g_2(i)))) \ .$$

The resulting topos $\mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$ may be identified with the (ordinary) realizability topos over the Sierpiński topos $[\mathbf{2}, \mathbf{Set}]$ induced by the identity mapping $(\Lambda \to \Lambda)$, regarded as an internal Schönfinkel algebra in $[\mathbf{2}, \mathbf{Set}]$; we shall not prove this here, since we shall not need it, but it follows straightforwardly from the 'tripos iteration theorem' of A.M. Pitts (see [17], 2.7.1).

In the previous work of Hyland and Ong [8], the set Θ of 'universal' potential realizers was assumed to be a *right ideal* (or right absorbing set) in Λ, that is a set such that, if $\theta \in \Theta$, then θx is defined and belongs to Θ for all $x \in \Lambda$. (We may express this more simply by the inclusion $\Theta \subseteq (\Lambda \Rightarrow \Theta)$.) Clearly, $\varnothing$ is always a right ideal in Λ, as is the whole of Λ if Λ is a total algebra; but we wish to exclude these two degenerate cases, so from now on 'right ideal' will always mean 'nonempty proper right ideal'.

Grayson [5] and van Oosten [15] made a stronger assumption: working with the Kleene algebra, they assumed a coding of recursive functions such that 0 codes the constant function with value 0 (so that $\{0\}$ is a right ideal), and also that the coding of pairs is such that $\langle 0, 0\rangle = 0$, so that the right ideal is closed under pairing. The latter is impossible if pairs are coded by the combinator D as we have assumed: note that $\mathsf{D}xy(\mathsf{K}(\mathsf{K}z)) \mathrel{=} z$ for any z, so that no element of the form $\mathsf{D}xy$ can belong to a (proper) right ideal. However, we note that all elements of the form $\mathsf{E}x$ belong to $(\Theta \Rightarrow \Theta)$ for any right ideal Θ, so our unpairing combinators P_1 and P_2 map Θ to itself. (Of course, if Θ is a singleton and pairing is taken to be a bijection $\Lambda \times \Lambda \to \Lambda$, as it can be for the Kleene algebra, then closure under pairing is equivalent to closure under unpairing.)

The main reason for requiring Θ to be a right ideal is contained in the following lemma. Recall that a full subcategory of a cartesian closed category is called an *exponential ideal* if an exponential B^A belongs to the subcategory whenever B does so.

Lemma 1.1. $\mathbb{M}_\Theta\Lambda$ is an indexed exponential ideal in $\mathbb{P}_1\Lambda$ iff Θ is a right ideal in Λ.

Proof. Clearly, $\Theta \subseteq (\Lambda \Rightarrow \Theta)$ iff $\Theta \subseteq (p \Rightarrow q)$ whenever $\Theta \subseteq q$. So this is immediate from the definition of implication in $\mathbb{P}_1\Lambda$. □

We shall say that an element λ of a Schönfinkel algebra is *omnivorous* if $\lambda x_1 x_2 \cdots x_n$ is defined for all finite sequences $(x_1, x_2, \ldots, x_n)$, and *irreversible* if there does not exist a sequence $(x_1, x_2, \ldots, x_n)$ for which $\lambda x_1 x_2 \cdots x_n \mathrel{=} \mathsf{I}$ (equivalently, if not every element of Λ can be expressed in the form $\lambda x_1 x_2 \cdots x_n$). In a partial algebra, every omnivorous element is irreversible, since if $\lambda x_1 \cdots x_n \mathrel{=} \mathsf{I}$ and $y_1 y_2$ is undefined then $\lambda x_1 \cdots x_n y_1 y_2$ is undefined; on the other hand, in a total algebra every element is omnivorous, but not all are irreversible. It is clear that every element of a right ideal must be omnivorous and irreversible; conversely, any element which is both omnivorous and irreversible generates a right ideal. Thus the right ideals in Λ are exactly the nonempty upwards-closed subsets of the set Θ_1 of all omnivorous and irreversible elements, preordered by setting $\lambda \leq \mu$ iff there exists a sequence $(x_1, \ldots, x_n)$ such that $\lambda x_1 \cdots x_n \mathrel{=} \mu$. (In particular, if Θ_1 is nonempty then it is the unique largest right ideal.)

Any total Schönfinkel algebra contains irreversible elements: for example, the element YK, where Y is the usual fixed-point combinator, generates a singleton right ideal, since we have $\mathsf{YK}x = \mathsf{K}(\mathsf{YK})x = \mathsf{YK}$ for all x. More generally, any fixed point of the K combinator generates a singleton right ideal, even in a partial algebra. And even if K has no fixed points, any proper algebra contains an element θ which codes its own constant function, i.e. such that $\{\theta\}$ is a right ideal: specifically, if we set

$$\mathsf{X} = \mathsf{S}(\mathsf{S}(\mathsf{KS})(\mathsf{S}(\mathsf{S}(\mathsf{KK})))(\mathsf{S}(\mathsf{S}(\mathsf{KS})(\mathsf{S}(\mathsf{KK})\mathsf{I}))(\mathsf{S}(\mathsf{KK})\mathsf{I})))(\mathsf{KI})$$

and then take

$$\theta = \mathsf{XX} \simeq \mathsf{S}(\mathsf{S}(\mathsf{KK})(\mathsf{S}(\mathsf{KX})(\mathsf{KX})))\mathsf{I}\ ;$$

then for any x we have

$$\begin{array}{rcl}\theta x & \simeq & \mathsf{S}(\mathsf{KK})(\mathsf{S}(\mathsf{KX})(\mathsf{KX}))x(\mathsf{I}x) \\ & \simeq & \mathsf{KK}x(\mathsf{KX}x(\mathsf{KX}x))x \\ & \simeq & \mathsf{K}(\mathsf{XX})x \simeq \mathsf{XX} = \theta\ .\end{array}$$

However, if we do not insist on propriety, there are partial algebras which contain no omnivorous elements (and therefore no right ideals): the following lemma was stated, but not proved, in [8].

Lemma 1.2. The algebra of strongly normalizing closed λ-terms modulo closed β-equivalence has no omnivorous elements.

Proof. Supppose τ were an omnivorous element of this algebra. Then any term of the form $\tau\mathsf{KKK}\cdots\mathsf{K}$ would be strongly normalizing; and it is easy to see that if we reduce this term to normal form, then the number of λ-abstractions in it which are *not* part of subterms of the form $\lambda x\ .\ \lambda y\ .\ x$ (that is, K) must decrease strictly as the number of applications increases. So if the number of K's is sufficiently large we must reduce to a term which is simply a string of K's (possibly with some nontrivial bracketing); but from any such term it is easy to get to I, and thence to a non-strongly-normalizing term, by further applications. □

For this and other reasons which will emerge below, it seems appropriate to introduce a 'relaxation' of the notion of right ideal. In addition to the property described in 1.1, the other feature of right ideals which is of importance in the proof that $\mathbb{M}\Lambda$ is a tripos is the ability to make an arbitrary set disjoint from Θ without destroying its 'information content': given $p \subseteq \Lambda$, we may define $p^+ = \{\langle \mathsf{I}, x\rangle \mid x \in p\}$. We note that this set is indeed disjoint from Θ, that $(p \Rightarrow p^+)$ and $(p^+ \Rightarrow p)$ are uniformly realizable by DI and P_2 respectively, and that the latter also realizes $(\Theta \Rightarrow \Theta)$. Abstracting from these properties, we arrive at the definition of a right pseudo-ideal.

Definition 1.3. We define an inhabited proper subset $\Theta \subset \Lambda$ to be a *right pseudo-ideal* provided there exist elements α, β, β' and γ of Λ satisfying

(*a*) $\beta(\alpha x) \simeq x$ for all x and $\beta \in (\Theta \Rightarrow \Theta)$;

(*b*) $\beta'(\alpha x) \simeq x$ for all x and $\beta' \in (\Theta \Rightarrow (\Lambda \Rightarrow \Theta))$; and

(*c*) $\gamma x(\alpha y) \simeq \alpha(xy)$ for all x, y and $\gamma \in (\Lambda \Rightarrow (\Theta \Rightarrow \Theta))$.

Given such a set Θ, we redefine p^+ to be $\{\alpha x \mid x \in p\}$ for any $p \subseteq \Lambda$. It is clear that $(p \Rightarrow p^+)$ and $(p^+ \Rightarrow p)$ are uniformly realized by α and β respectively; but it may not be obvious that p^+ is necessarily disjoint from Θ. However, if $\alpha x \in \Theta$ for some x, then we have $\gamma(\mathsf{K}y)(\alpha x) = \alpha y \in \Theta$, and hence $\beta(\gamma(\mathsf{K}y)(\alpha x)) = y \in \Theta$, for any y; so Θ is improper. We also remark that the element β in the definition is redundant; we can construct a suitable β from β' and γ, namely $\mathsf{B}(\mathsf{EK})(\mathsf{B}\beta'(\gamma\mathsf{K}))$. (Conversely, if Θ is in fact a right ideal, then any β satisfying (a) also satisfies (b).)

Any right ideal is a right pseudo-ideal, with α taken to be DI, $\beta = \beta' = \mathsf{P}_2$ and

$$\gamma = \mathsf{B}(\mathsf{SP}_1)(\mathsf{B}(\mathsf{B}(\mathsf{DI}))(\mathsf{B}(\mathsf{SP}_1)(\mathsf{B}(\mathsf{EP}_2)\mathsf{B})))\ .$$

Given an arbitrary right pseudo-ideal Θ, we define $M_\Theta\Lambda$, as before, to be $\{(p,a) \in P\Lambda \times P\Lambda \mid a \cup \Theta \subseteq p\}$, and make it into a **Set**-indexed preorder $\mathbb{M}_\Theta\Lambda$ by taking its I-indexed families to be all functions $I \to M_\Theta\Lambda$, ordered by $f \leq g$ iff $\bigcap\{(f \Rightarrow g)_2(i) \mid i \in I\}$ is inhabited. Now we may state

Lemma 1.4. The inclusion functor $\mathbb{M}_\Theta\Lambda \to \mathbb{P}_1\Lambda$ has an indexed left adjoint which preserves finite meets.

Proof. The left adjoint is induced by composition with $m_\Theta \colon P_1\Lambda \to M_\Theta\Lambda$, where $m_\Theta(p,a) = (p^+ \cup \Theta, a^+)$. In practice, we shall omit the subscript Θ when only one right pseudo-ideal is under consideration, but we shall need it in 1.10 below. The verification that m is order-preserving uses the element γ in the definition: if x realizes $f \leq g$, then γx realizes $mf \leq mg$, since if applied to an element αy of Λ^+ it yields $\alpha(xy)$, and if applied to an element of Θ it yields an element of Θ. The unit $(p,a) \leq m(p,a)$ is realized by α, and the counit $m(p,a) \leq (p,a)$ is realized, for all (p,a) with $\Theta \subseteq p$, by β.

To verify that the left adjoint preserves finite meets, it suffices by [9], A4.3.1 to verify that $\mathbb{M}_\Theta\Lambda$ is an exponential ideal 'up to isomorphism' in $\mathbb{P}_1\Lambda$, i.e. that $m((p,a) \Rightarrow (q,b)) \leq ((p,a) \Rightarrow (q,b))$ is uniformly realizable over all quadruples (a,b,p,q) with $\Theta \subseteq q$. But the element β' in the definition does exactly this; for if applied to αx, where x belongs to $(p \Rightarrow q)$ and/or $(a \Rightarrow b)$, then it yields x, and if applied to an element of Θ it yields an element of $(\Lambda \Rightarrow \Theta) \subseteq (p \Rightarrow q)$. □

Corollary 1.5. $\mathbb{M}\Lambda$ is a tripos.

Proof. This is immediate from 1.4: each $M\Lambda^I$ is a Heyting prealgebra because it is reflective and an exponential ideal in $P_1\Lambda^I$, and $\mathbb{M}\Lambda$ is complete and cocomplete as an indexed category because it is reflective in $\mathbb{P}_1\Lambda$. □

It is also clear that the inclusion $\mathbb{M}\Lambda \to \mathbb{P}_1\Lambda$ and its left adjoint form a geometric morphism of triposes, and hence induce a geometric morphism (in fact an inclusion) $\mathbf{Set}\langle\mathbb{M}\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$. To see where $\mathbf{Set}\langle\mathbb{M}\Lambda\rangle$ sits amongst the subtoposes of $\mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$, we need to consider the relationship between the latter and the ordinary realizability topos $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle$. Between the triposes $\mathbb{P}\Lambda$ and $\mathbb{P}_1\Lambda$, we have a string of five indexed adjoint functors $(f_1 \dashv f_2 \dashv f_3 \dashv f_4 \dashv f_5)$, induced respectively by composition with $(p \mapsto (p,\varnothing))$, $((p,a) \mapsto p)$, $(p \mapsto (p,p))$, $((p,a) \mapsto a)$ and $(p \mapsto (\Lambda,(\Lambda \Rightarrow p)))$. (The adjunctions are all easy to verify except for the last: if $\lambda \in (a \Rightarrow q)$, then $\mathsf{B}(\mathsf{B}\lambda)\mathsf{K} \in ((p,a) \Rightarrow (\Lambda,(\Lambda \Rightarrow q)))$, and if $\mu \in ((p,a) \Rightarrow (\Lambda,(\Lambda \Rightarrow q)))$, then $\mathsf{B}(\mathsf{EK})\mu \in (a \Rightarrow q)$.) Although the leftmost adjoint f_1 does not preserve the top element, it does preserve binary meets, and hence composition with it can be shown to induce a (full and faithful) functor $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$. Moreover, the image of this functor is easily identified: it consists of those objects (A,ε) for which

$\varepsilon_2(a,a')$ is always empty, or equivalently those which admit morphisms to the nontrivial subterminal object $U = (\{*\},\varepsilon)$ where $\varepsilon(*,*) = (\Lambda,\varnothing)$. Thus it induces an equivalence between $\mathbf{Eff}(\Lambda)$ and $\mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle/U$, which identifies the functor induced by f_1 with the forgetful functor Σ_U, and hence those induced by f_2 and f_3 are identified with U^* and Π_U respectively, i.e. they form an open geometric inclusion $u\colon \mathbf{Set}\langle\mathbb{P}\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$. In addition, f_3 and f_4 induce a connected geometric morphism $\mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}\Lambda\rangle$, left adjoint to u in Top; and this morphism has a further left adjoint $v\colon \mathbf{Set}\langle\mathbb{P}\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$ induced by f_4 and f_5, i.e. it is a local morphism in the sense of [12].

The two subtoposes u and v correspond to local operators (that is, idempotent finite-meet-preserving indexed monads) j and k on the tripos $\mathbb{P}_1\Lambda$, which are simply (the indexed functors induced by) the composites f_3f_2 and f_5f_4 respectively; i.e. the maps $(p,a)\mapsto(p,p)$ and $(p,a)\mapsto(\Lambda,\Lambda\Rightarrow a)$. Similarly, the inclusion of $\mathbf{Set}\langle\mathbb{M}\Lambda\rangle$ corresponds to the operator m of 1.4, regarded as a local operator on $\mathbb{P}_1\Lambda$. The composite jm sends (p,a) to $(p^+\cup\Theta, p^+\cup\Theta)$; since Θ is inhabited, this is isomorphic to the top element $(p,a)\mapsto(\Lambda,\Lambda)$ of the preorder $\mathbf{Lop}(\mathbb{P}_1\Lambda)$ of local operators on $\mathbb{P}_1\Lambda$. Hence the join of j and m is (isomorphic to) the top element; equivalently, $\mathbf{Set}\langle\mathbb{M}\Lambda\rangle$ is disjoint from the open subtopos u. On the other hand, the inequality $m(p,a)\le k(p,a)$ is uniformly realized by BKP_2; so $\mathbf{Set}\langle\mathbb{M}\Lambda\rangle$ contains the subtopos v.

Thus we have proved

Proposition 1.6. Let Θ be a right pseudo-ideal in a Schönfinkel algebra Λ. Then $\mathbf{Set}\langle\mathbb{M}_\Theta\Lambda\rangle$ is equivalent to a subtopos of $\mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$, disjoint from the open subtopos u but containing v; moreover, it admits a local geometric morphism to $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle$.

Proof. Only the local morphism to $\mathbf{Set}\langle\mathbb{P}\Lambda\rangle$ requires further comment. But the factorization of v through $\mathbf{Set}\langle\mathbb{M}_\Theta\Lambda\rangle$ remains (an inclusion, and) left adjoint to the composite $\mathbf{Set}\langle\mathbb{M}_\Theta\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle \to \mathbf{Set}\langle\mathbb{P}\Lambda\rangle$, so the latter is local. □

Up to this point, we have not used any particular property of the right pseudo-ideal Θ (other than nonemptiness). However, the next result does depend on the choice of Θ:

Lemma 1.7. For a right pseudo-ideal $\Theta\subset\Lambda$, the following are equivalent:

(i) There exists $\pi\in\Lambda$ such that $\pi\lambda = \mathsf{K}$ whenever $\lambda\in\Lambda^+$, and $\pi\theta = \mathsf{KI}$ whenever $\theta\in\Theta$;

(ii) The implication $((p\wedge(p^+\cup\Theta), p\wedge a^+)\Rightarrow(p,a))$ is uniformly realizable.

(iii) j and m are complementary elements of $\mathbf{Lop}(\mathbb{P}_1\Lambda)$.

(iv) $\mathbf{Set}\langle\mathbb{M}_\Theta\Lambda\rangle$ is the closed subtopos of $\mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$ complementary to the open subtopos u.

Moreover, if Λ is decisive, then these conditions hold for any right pseudo-ideal.

Proof. (i) $\Rightarrow$ (ii): Given π as in (i), we claim that the element $\mathsf{S}(\mathsf{S}(\pi\mathsf{P}_2)(\mathsf{B}\beta\mathsf{P}_2))\mathsf{P}_1$ realizes the implication in (ii). For if applied to an element of the form $\langle x,\alpha y\rangle$ it yields y, and if applied to $\langle x,\theta\rangle$ with $\theta\in\Theta$ it yields x.

(ii) $\Rightarrow$ (i): Suppose ρ uniformly realizes the implication in (ii). By considering $p=\{\mathsf{KI}\}, a=\varnothing$, we see that we must have $\rho\langle\mathsf{KI},\theta\rangle = \mathsf{KI}$ for any $\theta\in\Theta$; and by considering $p=\Lambda, a=\{\mathsf{K}\}$ we see that $\rho\langle x,\alpha\mathsf{K}\rangle = \mathsf{K}$ for any x. So if we take

$$\pi = \mathsf{B}(\rho(\mathsf{D}(\mathsf{KI}))(\gamma(\mathsf{KK}))$$

then it has the property in (i), since if applied to αx it yields $\rho\langle \mathsf{KI}, \alpha K\rangle$, and if applied to $\theta \in \Theta$ it yields $\rho\langle \mathsf{KI}, \gamma(\mathsf{KK})\theta\rangle$ — but $\gamma(\mathsf{KK})\theta \in \Theta$, since $\gamma \in (\Lambda \Rightarrow (\Theta \Rightarrow \Theta))$.

(ii) $\Leftrightarrow$ (iii): (ii) asserts that the meet of the local operators j and m is the bottom element of **Lop**$(\mathbb{P}_1\Lambda)$; but we already know that their join is the top element, so this is equivalent to (iii).

(iii) $\Leftrightarrow$ (iv) is immediate from the correspondence between local operators and subtoposes.

Finally, suppose Λ has a decider δ. Then we may take π to be $\mathsf{B}(\delta(\alpha\mathsf{K}))(\gamma(\mathsf{KK}))$, since if applied to αx it yields $\delta(\alpha\mathsf{K})(\alpha\mathsf{K}) \doteq \mathsf{K}$, and if applied to $\theta \in \Theta$ it yields $\delta(\alpha\mathsf{K})(\theta') \doteq \mathsf{KI}$ for some $\theta' \in \Theta$. □

We say Θ is *decidable* if the conditions of 1.7 hold. In the particular case when Θ is a right ideal, we may replace (i) by a simpler condition: Θ is decidable iff we can decide between it and $\{\mathsf{I}\}$, i.e. there exists σ satisfying $\sigma\mathsf{I} \doteq \mathsf{K}$ and $\sigma\theta \doteq \mathsf{KI}$ for all $\theta \in \Theta$. For, if we are given π as in 1.7(i), then $\sigma = \mathsf{B}\pi(\mathsf{E}(\alpha\mathsf{K}))$ satisfies this condition; and given σ, we may take $\pi = \mathsf{B}\sigma\mathsf{P}_1$ (recall that in this case we have $\alpha x = \langle \mathsf{I}, x\rangle$).

In particular, when Λ is decisive, every possible choice of Θ yields the same topos **Set**$\langle M_\Theta\Lambda\rangle$. In general, an indecisive Schönfinkel algebra may not have any decidable right ideals, but they can exist in particular cases; the following example is due to J. van Oosten.

Example 1.8. We consider D.S. Scott's 'graph model' of the untyped lambda-calculus: this is a total (and therefore indecisive) Schönfinkel algebra structure on the set $P\mathbb{N}$ of subsets of $\mathbb{N}$, with application defined by

$$EG = \{m \in \mathbb{N} \mid (\exists n)(F_n \subseteq G) \text{ and } \langle m, n\rangle \in E\}$$

where $\langle -, -\rangle$ is a suitable pairing function on $\mathbb{N}$ and F_n denotes the nth set in a suitable enumeration of the finite subsets of $\mathbb{N}$. If we choose F_n (as we may do) to be the set of exponents appearing in the representation of n as a sum of distinct powers of 2, and $\langle m, n\rangle = 2^m(2n+1)-1$, then $\langle 0, 0\rangle = 0$ and $F_0 = \varnothing$, so we have $\{0\}G = \{0\}$ for all G, i.e. $\{\{0\}\}$ is a singleton right ideal. Moreover, the set which represents the I combinator of this algebra cannot contain 0, since $\{0\} \subseteq \mathsf{I}$ would imply $\{0\} = \{0\}\varnothing \subseteq \mathsf{I}\varnothing = \varnothing$; and $\mathsf{I} \neq \varnothing$ since it is also easily verified that $\varnothing G = \varnothing$ for all G. So if we set

$$\sigma = \{\langle m, 1\rangle \mid m \in \mathsf{KI}\} \cup \{\langle m, 2^n\rangle \mid m \in \mathsf{K}, n \in \mathsf{I}\}$$

then it satisfies the condition given after the proof of 1.7. (On the other hand, the right ideal $\{\varnothing\}$ is not decidable, since $\varnothing \subseteq \mathsf{I}$ and hence $E\varnothing \subseteq E\mathsf{I}$ for any E — but $\mathsf{KI} \not\subseteq \mathsf{K}$.)

On the other hand, every Schönfinkel algebra has a decidable right pseudo-ideal:

Example 1.9. Given Λ, we take Θ to be $\{\langle \mathsf{KI}, x\rangle \mid x \in \Lambda\}$; we set $\alpha = \mathsf{DK}$ in this case, and take $\beta = \mathsf{S}(\mathsf{SP}_1\mathsf{P}_2)\mathsf{I}$, $\beta' = \mathsf{S}(\mathsf{SP}_1\mathsf{P}_2)\mathsf{K}$ and γ to be the combinator obtained from the lambda-term

$$\lambda x\,.\,\lambda y\,.\,\mathsf{P}_1 y(\mathsf{S}(\mathsf{K}(\mathsf{DK})(\mathsf{E}(\mathsf{P}_2 y)))(\mathsf{K}y)x \ .$$

It is straightforward to verify that these elements satisfy the conditions of 1.3. The element π required for 1.7(i) is simply the unpairing combinator P_1.

When Θ is not decidable, **Set**$\langle \mathbb{M}_\Theta\Lambda\rangle$ lies somewhere strictly between the non-open subtopos v and its closure (the complement of u). For a given Λ, there may be many different possibilities, but we can at least characterize the inclusions between them:

Lemma 1.10. Let Θ and Φ be right pseudo-ideals in a Schönfinkel algebra Λ, and write α_1, α_2 for the 'encoding elements' associated with Θ, Φ respectively (and so on). The following conditions are equivalent:

(i) There exists $\tau \in \Lambda$ such that $\tau(\alpha_1 x) = \alpha_2 x$ for all x, and $\tau \in (\Theta \Rightarrow \Phi)$.

(ii) The implication $(m_\Theta(p, a) \Rightarrow m_\Phi(p, a))$ is uniformly realizable.

(iii) $\mathbf{Set}\langle \mathbb{M}_\Phi \Lambda \rangle \leq \mathbf{Set}\langle \mathbb{M}_\Theta \Lambda \rangle$ as subtoposes of $\mathbf{Set}\langle \mathbb{P}_1 \Lambda \rangle$.

Proof. Clearly, any τ as in (i) will realize the implication in (ii). The fact that a realizer for (ii) has to have the properties in (i) may be seen by considering the cases when a is a singleton, and when $p = \varnothing$. The equivalence of (ii) and (iii) is immediate from the correspondence between (isomorphism classes of) local operators on $\mathbb{P}_1\Lambda$ and subtoposes of $\mathbf{Set}\langle \mathbb{P}_1 \Lambda \rangle$. □

We write $\Theta \preceq \Phi$ if the equivalent conditions of 1.10 hold; clearly, this defines a preordering on the right pseudo-ideals of Λ. They always hold when Θ is decidable; this is of course immediate from condition (iii) of 1.10, but can also be established by observing that if π satisfies condition (i) of 1.7 then $\mathsf{S}(\mathsf{S}\pi(\alpha_2\beta_1))(\mathsf{K}\varphi)$ satisfies condition (i) of 1.10, for any $\varphi \in \Phi$.

As before, the condition of 1.10(i) may be replaced by a simpler one in the case when Φ is a right ideal; it is equivalent to the existence of an element ν satisfying $\nu(\alpha_1 x) = \mathsf{I}$ for all x and $\nu \in (\Theta \Rightarrow \Phi)$. If Θ is also a right ideal, we may further simplify this to $\nu\mathsf{I} = \mathsf{I}$ and $\nu \in (\Theta \Rightarrow \Phi)$. Note that this latter condition holds in particular if $\Theta \subseteq \Phi$, since we may take ν to be I. (Thus the set of all omnivorous and irreversible elements of Λ is a $\preceq$-greatest right ideal.) In general, even if we restrict our attention to right ideals, the preorder $\preceq$ can be quite nontrivial.

Example 1.11. Let Λ be the free (total) Schönfinkel algebra on a set X of generators. As we observed earlier, the elements of X are all irreversible, so any nonempty subset $S \subseteq X$ generates a right ideal which we shall denote $S\Lambda$ (though this is a slight abuse of notation: the elements of $S\Lambda$ are all multiple products $x\lambda_1\lambda_2 \ldots \lambda_n$ with $x \in S$, not just those of the form $x\lambda$). Clearly $S \subseteq T$ implies $S\Lambda \subseteq T\Lambda$ and hence $S\Lambda \preceq T\Lambda$; we claim that these are the only cases in which the relation $\preceq$ holds between such ideals. To prove this, it suffices to show that if $x \notin T$ then $\{x\}\Lambda \not\preceq T\Lambda$. Suppose we had an element ν witnessing this inequality; then $\nu\mathsf{I} = \mathsf{I}$ and νx is expressible as a word beginning with a member of T. But we have a homomorphism $f \colon \Lambda \to \Lambda$ sending x to I and all other generators to themselves; so we have

$$\mathsf{I} = f(\mathsf{I}) = f(\nu\mathsf{I}) = f(\nu)\mathsf{I} = f(\nu)f(x) = f(\nu x) \ ,$$

which is impossible since the right-hand side is a word beginning with a member of T. Hence the toposes $\mathbf{Set}\langle \mathbb{M}_{S\Lambda}\Lambda \rangle$, for $\varnothing \neq S \subseteq X$, are all distinct as subtoposes of $\mathbf{Set}\langle \mathbb{P}_1 \Lambda \rangle$.

Similarly, one may show that if $x_1, x_2, x_3, \ldots$ are distinct generators, then the elements $x_1, x_1x_2, x_1x_2x_3, \ldots$ generate a strictly descending sequence of right ideals $\Theta_1, \Theta_2, \Theta_3, \ldots$ relative to the preorder $\preceq$. The intersection of these ideals is clearly empty; and it seems likely that the sequence has no lower bound relative to $\preceq$ among the right ideals of Λ. Indeed, if there were such a lower bound, we could take it to be the right ideal generated by an irreversible element of the subalgebra Λ_0 of closed terms. For if Φ were any lower bound, we should have 'witnesses' ν_n to the inequality $J \preceq I_n$ for each n. Now pick $t \in \Phi$, and suppose the variables occurring in t are contained in

$\{x_1, \ldots, x_n\}$. Consider the homomorphism $f\colon \Lambda \to \Lambda$ sending x_m to I if $m \leq n$, and to x_{m-n} otherwise; then $f(t)$ is a closed term, and if it were reversible we could find a sequence of closed terms $u_1, \ldots, u_k$ such that $f(t)u_1 \cdots u_k = f(tu_1 \cdots u_k) = \mathsf{I}$, and hence $f(\nu_m(tu_1 \cdots u_k)) = \mathsf{I}$ for all $m > n$. But $tu_1 \cdots u_k \in \Phi$, so $\nu_{n+1}(tu_1 \cdots u_k)$ is a term beginning with $x_1 x_2 \ldots x_{n+1}$, so its image under f is a term beginning with x_1. Hence $f(t)$ must be irreversible; but now $f(\nu_m)$ witnesses the inequality $f(t)\Lambda \preceq I_{m-n}$ for each m.

Similar arguments apply to the algebra of β-equivalence classes of λ-terms with free variables in X, and to the free extensional Schönfinkel algebra on X (the quotient of the free Schönfinkel algebra obtained by adding the equations $\mathsf{S}(\mathsf{S}(\mathsf{KK})x)y = x$, $\mathsf{S}(\mathsf{S}(\mathsf{S}(\mathsf{KS})x)y)z = \mathsf{S}(\mathsf{S}xz)(\mathsf{S}yz)$, $\mathsf{K}(xy) = \mathsf{S}(\mathsf{K}x)(\mathsf{K}y)$ and $\mathsf{S}(\mathsf{K}x)\mathsf{I} = x$, cf. [4]).

2 Functoriality of modified realizability

If, as in [2], we adopt the viewpoint that $\mathbf{Set}\langle \mathbb{M}\Lambda \rangle$ should always be a closed subtopos of $\mathbf{Set}\langle \mathbb{P}_1\Lambda \rangle$, then its functoriality is easily established. Given a quasi-surjective morphism $t\colon \Lambda \looparrowright \Pi$, we know that t induces a geometric morphism of triposes $\mathbb{P}\Pi \to \mathbb{P}\Lambda$; and, since the formulae defining the direct and inverse images of this morphism preserve inclusions between subsets, it is trivial to verify that the same formulae applied componentwise induce a geometric morphism $\mathbb{P}_1\Pi \to \mathbb{P}_1\Lambda$. Moreover, the inverse image of this morphism preserves the nontrivial subterminal object U (at least up to isomorphism); and so we have pullback squares of geometric morphisms

$$\begin{array}{ccc} \mathbf{Set}\langle \mathbb{P}_1\Pi \rangle/U & \longrightarrow & \mathbf{Set}\langle \mathbb{P}_1\Lambda \rangle/U \\ \downarrow & & \downarrow \\ \mathbf{Set}\langle \mathbb{P}_1\Pi \rangle & \longrightarrow & \mathbf{Set}\langle \mathbb{P}_1\Lambda \rangle \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{Set}\langle \mathbb{M}\Pi \rangle & \longrightarrow & \mathbf{Set}\langle \mathbb{M}\Lambda \rangle \\ \downarrow & & \downarrow \\ \mathbf{Set}\langle \mathbb{P}_1\Pi \rangle & \longrightarrow & \mathbf{Set}\langle \mathbb{P}_1\Lambda \rangle \end{array}$$

Here the top edge of the left-hand square is of course the geometric morphism of ordinary realizability toposes induced by t; from that of the right hand square, we deduce

Proposition 2.1. The assignment $(\Lambda \mapsto \mathbf{Set}\langle \mathbb{M}\Lambda \rangle)$ defines a 2-functor from $\mathsf{Sch\ddot{o}n}_{qs}^{\mathrm{op}}$ to the 2-category of toposes and geometric morphisms.

Proof. Only the 2-dimensional structure requires further comment. But it is easy to verify that inequalities between morphisms of Schönfinkel algebras give rise to natural transformations between inverse image functors $\mathbf{Set}\langle \mathbb{P}_1\Lambda \rangle \to \mathbf{Set}\langle \mathbb{P}_1\Pi \rangle$, and these clearly restrict to inverse image functors between the appropriate closed subtoposes. □

However, if we work with (not necessarily decidable) right (pseudo-)ideals, then the position is more complicated — even in the case when t is an identity morphism, as shown by 1.10. Let us suppose given right pseudo-ideals Θ, Φ of Λ, Π respectively, and a quasi-surjective morphism $t\colon \Lambda \looparrowright \Pi$. We recall that such a morphism is computationally dense as defined in [6]; that is, there exist a function $s\colon \Pi \to \Lambda$ and an element $\sigma \in \Pi$ such that, for all $y \in \Pi$ and $x \in \Lambda$, if yx' is defined whenever $t(x, x')$ holds, then $s(y)x$ is defined and, for all z such that $t(s(y)x, z)$, we have $\sigma z = yx'$ for some x' satisfying $t(x, x')$. The right adjoint $t^- : \mathbb{P}\Pi \to \mathbb{P}\Lambda$ of t_+ is then induced by composition with the map

$$(q \mapsto \{x \in \Lambda \mid (\forall y \in \Pi)(t(x, y) \Rightarrow \sigma y \in q)\}) \ .$$

In the case when t is functional and surjective, the formula for t^- may be simplified slightly. In this case we may take σ to be $\mathsf{S}(\tau t(\mathsf{P}_1))(\tau t(\mathsf{P}_2))$, where τ is a witness for t, and hence set

$$t^-(q) = \{x \in \Lambda \mid (\exists y \in q)(\mathsf{P}_1 t(x)(\mathsf{P}_2 t(x)) \simeq y)\} \ .$$

(However, we cannot simply take $t^-(q)$ to be the set-theoretic inverse image $t^{-1}(q)$; the latter is not order-preserving.)

As we already observed, both t_+ and t^- preserve inclusions between subsets, and so can be applied 'componentwise' to yield a geometric morphism of triposes $\mathbb{P}_1\Pi \to \mathbb{P}_1\Lambda$; we need to consider when this restricts to a geometric morphism $\mathbb{M}_\Phi\Pi \to \mathbb{M}_\Theta\Lambda$. A necessary and sufficient condition for this is that the right adjoint t^- should map $M_\Phi\Pi$ into $M_\Theta\Lambda$, in the 'up-to-isomorphism' sense that

$$(m_\Theta(t^-(q), t^-(b)) \Rightarrow (t^-(q), t^-(b)))$$

is uniformly realizable whenever $\Phi \subseteq q$; for if this holds then we obtain a left adjoint for the restriction of t^- by restricting the composite $m_\Phi t_+$ to $M_\Theta\Lambda$.

Evidently, a sufficient condition for the displayed implication above to be uniformly realizable is the existence of an element $\upsilon \in \Lambda$ satisfying $\upsilon(\alpha x) \simeq x$ for all x (where α is the 'coding' element associated with Θ) and $\upsilon \in (\Theta \Rightarrow t^-(\Phi))$. (We cannot assert that this condition is necessary as well as sufficient, since we cannot in general reduce to the case when $t^-(b)$ is a singleton.) Let us say that t is (Θ, Φ)-*compatible* if such an element υ exists; then we have shown

Lemma 2.2. If $t \colon \Lambda \looparrowright \Pi$ is (quasi-surjective and) (Θ, Φ)-compatible, then the geometric morphism $\mathbb{P}_1\Pi \to \mathbb{P}_1\Lambda$ induced by t restricts to a geometric morphism $\mathbb{M}_\Phi\Pi \to \mathbb{M}_\Theta\Lambda$. □

It is easy to see that if Θ is decidable then any quasi-surjective $t \colon \Lambda \looparrowright \Pi$ is (Θ, Φ) compatible for any Φ: we take the element υ to be $\mathsf{S}(\mathsf{S}\pi\beta)(\mathsf{K}\varphi)$, where π is as in 1.7(i), β is the 'decoding element' for Θ as in 1.3(a), and φ is any element of $t^-(\Phi)$ (note that t^-, being a right adjoint, preserves nonemptiness). Again, if Θ and Φ are right pseudo-ideals in the same algebra Λ, then the identity map $\Lambda \to \Lambda$ is (Θ, Φ)-compatible iff the relation $\Theta \preceq \Phi$ holds: for in this case we may take $(1_\Lambda)^-$ simply to be the identity map, and so if we have a witness τ for $\Theta \preceq \Phi$, as in 1.10(i), then $\mathsf{B}\overline{\beta}\tau$ will witness the compatibility, where $\overline{\beta}$ is the 'decoding element' for Φ. (The converse is immediate from 2.2 and 1.10(iii).)

Proposition 2.3. The assignment $(\Lambda, \Theta) \mapsto \mathbf{Set}\langle \mathbb{M}_\Theta\Lambda \rangle$ defines a pseudofunctor, contravariant on 1-cells, from the 2-category of Schönfinkel algebras equipped with a right pseudo-ideal, quasi-surjective morphisms compatible with the chosen right pseudo-ideals and inequalities between them to the 2-category of toposes and geometric morphisms.

Proof. Once again, only the assertion about 2-cells requires further comment. But it may be established in exactly the same way as in 2.1 □

Of course, 2.3 should not be the end of the story. Two questions immediately arise: (1) does every geometric morphism $\mathbf{Set}\langle \mathbb{M}_\Phi\Pi \rangle \to \mathbf{Set}\langle \mathbb{M}_\Theta\Lambda \rangle$ derive from a morphism of triposes $\mathbb{M}_\Phi\Pi \to \mathbb{M}_\Theta\Lambda$? and (2) does every such morphism of triposes derive from a morphism of Schönfinkel algebras $\Lambda \looparrowright \Pi$? With regard to the second, we note that if $f \colon \mathbb{M}_\Phi\Pi \to \mathbb{M}_\Theta\Lambda$ is a geometric morphism of triposes, we may assume it is induced by composition by a pair of functions (f_*, f^*) between $M_\Phi\Pi$ and $M_\Theta\Lambda$; and then by forming the composite

$$\Lambda \longrightarrow M_\Theta\Lambda \xrightarrow{f^*} M_\Phi\Pi \xrightarrow{\pi_2} P\Pi$$

where the first factor sends λ to $(\Lambda, \{\lambda\})$, we obtain an entire relation $\Lambda \nrightarrow \Pi$, which would be the morphism inducing f if it were indeed induced by a morphism of Schönfinkel algebras. However, even proving that this relation is a morphism of Schönfinkel algebras seems less than straightforward.

In connection with question (1), we note that the proof of the corresponding result for ordinary realizability relies on the fact, proved in [10], that the inverse image of any geometric morphism between ordinary realizability toposes restricts to a functor between categories of assemblies (that is, $\neg\neg$-separated objects), and this in turn depends on the fact that every object in an ordinary realizability topos is a quotient of an assembly. Since the corresponding result fails for modified realizability toposes (see [15]), it seems unlikely that a similar approach will work in this case.

3 'Closing Off' a modified realizability topos

There is a construction due to van Oosten [16] which, given a Schönfinkel algebra Λ and a partial function $f\colon \Lambda \rightharpoonup \Lambda$, produces a new Schönfinkel algebra $\Lambda[f]$ (with the same underlying set, but a different notion of application which we denote by $\cdot^f$) in which f becomes trackable in the sense that there exists $\varphi \in \Lambda$ satisfying $\varphi \cdot^f x = f(x)$ for all x. Moreover, the identity function on Λ is a surjective functional morphism $\Lambda \nrightarrow \Lambda[f]$, and is universal among quasi-surjective morphisms for which (the image of) f becomes trackable, in a suitable sense. For the details, see [17], 1.7.5. (To distinguish the combinators of $\Lambda[f]$ from those of Λ, we shall attach a superscript $(-)^f$ to the former; for example, K^f may be taken to be $\tau(\mathsf{B}\tau\mathsf{K})$, where τ is a witness for the fact that the identity is a morphism $\Lambda \nrightarrow \Lambda[f]$.

As we saw in 1.7, for a right pseudo-ideal Θ the modified realizability topos $\mathbf{Set}\langle\mathbb{M}_\Theta\Lambda\rangle$ is closed in $\mathbf{Set}\langle\mathbb{P}_1\Lambda\rangle$ iff the partial function k_Θ sending all of Λ^+ to K and all of Θ to KI is trackable. It is therefore of interest, for an arbitrary Θ, to consider the effect of adjoining this particular function to our class of trackable functions. Of course, if we do this, Θ will not be a right ideal in $\Lambda[f]$ even if it was one in Λ, since the application in $\Lambda[f]$ is different from that in Λ; but it is still a right pseudo-ideal:

Lemma 3.1. Let Θ be a right pseudo-ideal in a Schönfinkel algebra Λ. Then Θ is also a right pseudo-ideal in $\Lambda[f]$, for any partial function $f\colon \Lambda \rightharpoonup \Lambda$; and the identity function is (Θ, Θ)-compatible as a morphism $\Lambda \to \Lambda[f]$. Moreover, if the function k_Θ is trackable in $\Lambda[f]$, then $\mathbf{Set}\langle\mathbb{M}_\Theta\Lambda[f]\rangle$ is a closed subtopos of $\mathbf{Set}\langle\mathbb{P}_1\Lambda[f]\rangle$.

Proof. The notion of right pseudo-ideal was defined in terms of the trackability of certain partial maps $\Lambda \rightharpoonup \Lambda$. But any partial map which is trackable in Λ remains trackable in $\Lambda[f]$, so the first assertion is immediate. For the second, we observe that we may again identify $(1_\Lambda)^-$ with the identity function, so that the decoding element β for Θ (in the sense of Λ) witnesses the (Θ, Θ)-compatibility of 1_Λ. The third assertion is immediate from the discussion above.
□

Proposition 3.2. Let Θ be a right ideal in a Schönfinkel algebra Λ. Then there is a pullback square of toposes and geometric inclusions

$$\begin{array}{ccc} \mathbf{Set}\langle \mathbb{M}_\Theta \Lambda[k_\Theta]\rangle & \longrightarrow & \mathbf{Set}\langle \mathbb{M}_\Theta \Lambda\rangle \\ \downarrow & & \downarrow \\ \mathbf{Set}\langle \mathbb{P}_1 \Lambda[k_\Theta]\rangle & \longrightarrow & \mathbf{Set}\langle \mathbb{P}_1 \Lambda\rangle \end{array}$$

in which the left vertical arrow is a closed inclusion.

Proof. The commutativity of the square, and the fact that the left vertical arrow is closed, both follow from 3.1; and the latter implies that the square is a pullback, since the pullback of $\mathbf{Set}\langle \mathbb{M}_\Theta \Lambda\rangle$ along the bottom inclusion must be disjoint from the nontrivial open subtopos of $\mathbf{Set}\langle \mathbb{P}_1 \Lambda[k_\Theta]\rangle$. □

It is not hard to see that, for two right pseudo-ideals Θ and Φ, k_Θ is trackable in $\Lambda[k_\Phi]$ iff the relation $\Theta \preceq \Phi$ of 1.10 holds. Thus we obtain

Corollary 3.3. The assignments $\Theta \mapsto \mathbf{Set}\langle \mathbb{P}_1 \Lambda[k_\Theta]\rangle$ and $\Theta \mapsto \mathbf{Set}\langle \mathbb{M}_\Theta \Lambda[k_\Theta]\rangle$ are contravariant functors from the preordered set of right pseudo-ideals of Λ to the lattice of subtoposes of $\mathbf{Set}\langle \mathbb{P}_1 \Lambda\rangle$.

Proof. If $\Theta \preceq \Phi$, then the identity morphism $\Lambda \to \Lambda[k_\Phi]$ factors through $\Lambda \to \Lambda[k_\Theta]$, by the universal property of the latter, so $\mathbf{Set}\langle \mathbb{P}_1 \Lambda[k_\Phi]\rangle \leq \mathbf{Set}\langle \mathbb{P}_1 \Lambda[k_\Theta]\rangle$ as subtoposes of $\mathbf{Set}\langle \mathbb{P}_1 \Lambda\rangle$. The second assertion follows from the first and 1.10, together with 3.2. □

It might be tempting to conjecture that $\mathbf{Set}\langle \mathbb{P}_1 \Lambda[k_\Theta]\rangle$ is the largest subtopos of $\mathbf{Set}\langle \mathbb{P}_1 \Lambda\rangle$ which $\mathbf{Set}\langle \mathbb{M}_\Theta \Lambda\rangle$ intersects in a closed subtopos, but this is not the case: it is not clear in general whether such a largest subtopos exists, but if it did it would contain the whole of the open subtopos $\mathbf{Set}\langle \mathbb{P} \Lambda\rangle$ of $\mathbf{Set}\langle \mathbb{P}_1 \Lambda\rangle$, whereas $\mathbf{Set}\langle \mathbb{P}_1 \Lambda[k_\Theta]\rangle$ intersects this in the proper subtopos $\mathbf{Set}\langle \mathbb{P} \Lambda[k_\Theta]\rangle$. It seems very probable that $\mathbf{Set}\langle \mathbb{P}_1 \Lambda[k_\Theta]\rangle$ does have some universal property, but I have not been able to find a precise formulation of it.

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