On approximation of functions by means of Fourier trigonometric series in weighted generalized grand Lebesgue spaces

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Abstract

In the present work we investigate the approximation of the functions by the Zygmund means in the weighted generalized grand Lebesgue spaces. The estimates are obtained in terms of the best approximation and modulus of smoothness. Also, the approximation problems of Cesaro, Zygmund and Abel sums of Faber series in the generalized Smirnov classes defined on the bounded simply connected domains of complex plane are studied.

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1 Introduction and the main results

Let $\mathbb{T}$ denote the interval $[0,2\pi]$, a function ω is called a *weight* on $\mathbb{T}$ if $\omega:\mathbb{T}\to[0,\infty]$ is measurable and $\omega^{-1}(\{0,\infty\})$ has measure zero (with respect to Lebesgue measure).

Let ω be a 2π periodic weight function. We denote by $L^p_\omega(\mathbb{T}), 1<p<\infty$, the weighted Lebesgue space of all measurable functions on $\mathbb{T}$ for which the norm

$$\|f\|_p=\left(\int_{\mathbb{T}}|f(x)|^p\,\omega dx\right)^{1/p}<\infty.$$

We define a class $L^{p),\theta}_\omega(\mathbb{T}),\ \theta>0,\ 1<p<\infty$ of 2π periodic measurable functions on $\mathbb{T}$ satisfying the condition

$$\sup_{0<\varepsilon<p-1}\left\{\frac{\varepsilon^\theta}{2\pi}\int_{\mathbb{T}}|f(x)|^{p-\varepsilon}\,\omega(x)dx\right\}^{1/(p-\varepsilon)}<\infty.$$

The class $L^{p),\theta}_\omega(\mathbb{T}),\ \theta>0,\ 1<p<\infty$ is a Banach space with respect to the norm

$$\|f\|_{L^{p),\theta}_\omega(\mathbb{T})}:=\sup_{0<\varepsilon<p-1}\left\{\varepsilon^\theta\frac{1}{2\pi}\int_{\mathbb{T}}|f(x)|^{p-\varepsilon}\,\omega(x)dx\right\}^{1/(p-\varepsilon)}. \tag{1.1}$$

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The class $L_{\omega}^{p),\theta}(\mathbb{T})$ with the norm (1.1) is called *the weighted generalized grand Lebesgue space.* Note that non- weighted grand Lebesgue space $L^{p)}(\mathbb{T})$ was introduced by Iwaniec and Sbordone [21]. Information about properties of the grand and generalized grand Lebesgue spaces can be found in [14], [21], [22] and [34]. Note that the grand and generalized grand Lebesgue spaces have been applied in many fields of science (see, for example: [39], [40], [22]). Also, some problems of quasilinear operators theory and interpolation theory are investigated in these spaces (see, for example: [18], [15], [16]).

The embeddings

$$L^{p}(\mathbb{T}) \subset L^{p)}(\mathbb{T}) \subset L^{p-\varepsilon}$$

hold. According to [14] $L^{p}(\mathbb{T})$ is not dense in $L^{p)}(\mathbb{T})$. Also, if $\theta_1 < \theta_2$ and $1 < p < \infty$, for weighted generalized grand Lebesgue space, the following relations hold:

$$L_{\omega}^{p}(\mathbb{T}) \subset L_{\omega}^{p),\theta_1}(\mathbb{T}) \subset L_{\omega}^{p),\theta_2}(\mathbb{T}) \subset L_{\omega}^{p-\varepsilon}(\mathbb{T}).$$

The closure of the space $L^{p}(\mathbb{T})$, $1 < p < \infty$ by the norm of the grand Lebesgue spaces $L_{\omega}^{p),\theta}(\mathbb{T})$, $\theta > 0, 1 < p < \infty$ does not coincide with the later space. Let us denote this closure by $\widetilde{L}_{\omega}^{p),\theta}(\mathbb{T})$. It is clear that this subspace of $L_{\omega}^{p),\theta}(\mathbb{T})$ is a set of functions for which

$$\lim_{\varepsilon\to 0} \varepsilon^{\theta} \int_{\mathbb{T}} |f(t)|^{p-\varepsilon} \omega(t)\, dt = 0.$$

Let $1 < p < \infty$ and let $A_p(\mathbb{T})$ be the collection of all weights on $\mathbb{T}$ satisfying the condition

$$\sup_{I} \left(\frac{1}{|I|}\int_{I} \omega(x)^{p} dx\right)^{1/p} \left(\frac{1}{|I|}\int_{I} [\omega(x)]^{-1/(p-1)} dx\right)^{p-1} < \infty \tag{1.2}$$

where the supremum is taken over all intervals I with length $|I| \le 2\pi$. The condition (1.2) is called the *Muckenhoupt* -A_p condition and the weight functions which belong to $A_p(\mathbb{T})$, $(1 < p < \infty)$, are called the *Muckenhoupt weights.*

For $f \in \widetilde{L}_{\omega}^{p),\theta}(\mathbb{T})$ and $\omega \in A_p$ we define the Steklov operator by

$$s_h(f)(x) = \frac{1}{h}\int_{x}^{x+h} f(t)dt = \frac{1}{h}\int_{0}^{h} f(u+x)du$$

and the $r-th$ *modulus of smoothness* $\Omega_r(f,\cdot,)_{p),\theta,\omega}$ $(r = 1, 2, ...)$ by

$$\Omega_r(f,\delta)_{p),\theta,\omega} = \sup_{\substack{0<h_i\le\delta \\ 1\le i\le r}} \left\| \prod_{i=1}^{r} (I - s_{h_i})(f) \right\|_{L_{\omega}^{p),\theta}(\mathbb{T})} \quad \delta > 0,\ r = 1, 2, ...$$

where I is the identity operator.

The modulus of continuity $\Omega_r(f,\cdot)_{p),\theta,\omega}$ is a non-decreasing, nonnegative function of $\delta > 0$ and

$$\lim_{\delta\to 0}\Omega_r(f,\cdot)_{p),\theta,\omega} = 0, \quad \Omega_r(f+g,\cdot)_{p),\theta,\omega} \le \Omega(f,\cdot)_{p),\theta,\omega} + \Omega(g,\cdot)_{p),\theta,\omega}$$

for $f, g \in \widetilde{L}_{\omega}^{p),\theta}(\mathbb{T})$, $\theta > 0$, $1 < p < \infty$.

Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}A_k(x,f),\ \ A_k(x,f):=a_k(f)\cos kx+b_k(f)\sin kx \tag{1.3}$$

be the Fourier series of the function $f\in L_1(\mathbb{T})$, where $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function f .

The $n-th$ *partial sums*, *Zygmund means of order* k $(k\in\mathbb{N})$ and *Abel -Poisson means* of the series (1.3) are defined, respectively as [19], [46]:

$$\begin{aligned}
S_n(x,f) &= \frac{a_0(f)}{2}+\sum_{k=1}^{n}A_k(x,f),\\
Z_{n,k}(x,f) &= \frac{a_0(f)}{2}+\sum_{\nu=1}^{n}\left(1-\frac{\nu^k}{(n+1)^k}\right)A_\nu(x,f),\ \ k=1,2,...,\ \ n=1,2,...,\\
U_s(x,f) &= \frac{1}{2\pi}\int_T P_s(x-t)\,f(t)\,dt,
\end{aligned}$$

where

$$P_s(t)=\frac{1-s^2}{1-2s\cos t+s^2},\ \ 0\le s<1$$

is the Poisson kernel.

It is clear that

$$S_0(x,f)=Z_{0,k}(x,f)=\frac{a(f)}{2}$$

The best approximation of $f\in\widetilde{L}_\omega^{p),\theta}$, $\theta>0,\ 1<p<\infty$ in the class $\prod_n$ of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{p),\theta,\omega}:=\inf\left\{\|f-T_n\|_{L_\omega^{p),\theta}(\mathbb{T})}:T_n\in\prod\nolimits_n\right\}.$$

Let G be a finite domain in the complex plane $\mathbb{C}$, bounded by a rectifiable Jordan curve Γ, and let $G^-:=ext\Gamma$. We denote

$$\mathbb{T}^*:=\{w\in\mathbb{C}:|w|=1\},\ \mathbb{D}:=int\,\mathbb{T}^*\ ,\ \mathbb{D}^-:=ext\,\mathbb{T}^*.$$

Let $w=\varphi(z)$ be the conformal mapping of G^- onto $\mathbb{D}^-$ normalized by

$$\varphi(\infty)=\infty,\ \ \lim_{z\to\infty}\frac{\varphi(z)}{z}>0$$

and let ψ denote the inverse of φ.

Let $w=\varphi_1(z)$ denote a function that maps the domain G conformally onto the disk $|w|<1$.

The inverse mapping of φ_1 will be denoted by ψ_1. Let Γ_r denote circular images in the domain G, that is, curves in G corresponding to circle $|\varphi_1(z)|=r$ under the mapping $z=\psi_1(w)$.

Let us denote by $E^p(G)$, where $p>0$, the class of all functions $f(z)\neq 0$ that are analytic in G and have the property that the integral

$$\int\limits_{\Gamma_r} |f(z)|^p\,|dz|$$

is bounded for $0<r<1$. We shall call the E^p-class as the *Smirnov class*. If the function $f(z)$ belongs to E^p, then $f(x)$ has definite limiting values $f(z')$ almost every where on Γ, over all nontangential paths; $|f(z')|$ is summable on Γ; and

$$\lim_{r\to 1}\int\limits_{\Gamma_r} |f(z)|^p\,|dz| = \int\limits_{\Gamma} |f(z')|^p\,|dz|\,.$$

It is known that $\varphi' = E^1(G^-)$ and $\psi'\in E^1(\mathbb{D}^-)$. Note that each function $f\in E^p(G)$ has the non-tangential limit almost everywhere (a.e.) on Γ and the boundary function belongs to $L^p(\Gamma)$. The general information about Smirnov classes can be found in the books [8, pp. 168-185] and [17, pp. 438-453].

Let $|\Gamma|$ be the Lebesgue measure of Γ and ω be a weight function on Γ.By $L^p_\omega(\Gamma)$, $1<p<\infty$ we denote the set of all measurable functions $\Gamma\to\mathbb{C}$ satisfying the condition

$$\left\{\frac{1}{|\Gamma|}\int\limits_{\Gamma} |f(z)|^p\,\omega(z)\right\}^{1/p} < \infty.$$

We denote by $L^{p),\theta}_\omega(\Gamma)$, $\theta\geq 0,\ 1<p<\infty$ the set of all measurable on Γ functions f such that

$$\sup_{0<\varepsilon<p-1}\left\{\frac{\varepsilon^p}{|\Gamma|}\int\limits_{\Gamma} |f(z)|^{p-\varepsilon}\,\omega(z)\,|dz|\right\}^{1/(p-\varepsilon)} < \infty$$

and set

$$\|f\|_{L^{p),\theta}_\omega} := \sup_{0<\varepsilon<p-1}\left\{\frac{\varepsilon^p}{|\Gamma|}\int\limits_{\Gamma} |f(z)|^{p-\varepsilon}\,\omega(z)\,|dz|\right\}^{1/(p-\varepsilon)}.$$

The normed space $L^{p),\theta}_\omega(\Gamma)$, $\theta\geq 0$ is called a *weighted generalized grand Lebesgue space.* This space becomes a Banach space. We denote by $\widetilde{L_\omega}^{p),\theta}(\Gamma)$ the closure of $L^p_\omega(\Gamma)$ in the space $L^{p),\theta}_\omega(\Gamma)$, which consists of the functions f , satisfying the condition

$$\lim_{\varepsilon\to 0}\left\{\frac{\varepsilon^p}{|\Gamma|}\int\limits_{\Gamma} |f(z)|^{p-\varepsilon}\,\omega(z)\right\} = 0.$$

Let Γ be a rectifiable Jordan curve and $f \in L^1(\Gamma)$. Then the functions f^+ and f^- defined by Γ

$$f^+(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{\psi'(w)}{\psi(w) - z} f_0(w) dw, \quad z \in G$$

and

$$f^-(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{\psi'(w)}{\psi(w) - z} f_0(w) dw, \qquad z \in G^-$$

are analytic in G and G^- respectively, and $f^-(\infty) = 0$. Thus the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\limits_{\Gamma \cap \{\zeta :\ |\zeta - z| > \varepsilon\}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

exists and is finite for almost all $z \in \Gamma$.

The quantity $S_\Gamma(f)(z)$ is called the *Cauchy singular integral* of f at $z \in \Gamma$.

According to the Privalov's theorem [17, p. 431] if one of the functions $f^+(z)$ and $f^-(z)$ has a nontangential limit on Γ a.e., then $S_\Gamma(f)(z)$ exists a. e. on Γ and also the other one of the functions $f^+(z)$ and $f^-(z)$ has a nontangential limit on Γ a. e. Conversely, if $S_\Gamma(f)(z)$ exists a. e. on Γ, then the functions $f^+(z)$ and $f^-(z)$ have nontangential limits a. e. on Γ. In both cases, the formulae

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2} f(z), \quad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2} f(z)$$

and hence

$$f = f^+ - f^-$$

holds a. e. on Γ.

We define also the *generalized grand Smirnov class* $E_\omega^{p),\theta}(G)$ as

$$E_\omega^{p),\theta}(G) := \left\{ f \in E^1(G) : f \in L_\omega^{p),\theta}(\Gamma) \right\}.$$

The closure of Smirnov class $E_\omega^p(G)$ in the space $E_\omega^{p),\theta}(G)$ we denote by $\mathbb{E}_\omega^{p),\theta}(G)$.

Let Γ be rectifiable Jordan curve. Γ is called a *regular curve* if there exists a constant $c(\Gamma) > 0$ depending on only Γ such that for every $r > 0$, $\sup \{|\Gamma \cap D(z, r)| : z \in \Gamma\} \le c(\Gamma) r$, where $|\Gamma \cap D(z, r)|$ is the length of the set $\Gamma \cap D(z, r)$.

We denote by S the set of all *regular* Jordan curves in the complex plane $\mathbb{C}$.

Let $1 < p < \infty$, $1/p + 1//q = 1$. and let ω be weight function on $\Gamma \in \dot{S}$. A weight function ω belongs Muckenhoupt class $A_p(\Gamma)$ if the condition

$$\sup_{z \in \Gamma} \sup_{\varepsilon > 0} \left(\frac{1}{\varepsilon} \int_{\Gamma(z,\varepsilon)} [\omega(\tau)]^p \, |d\tau| \right)^{1/p} \left(\frac{1}{\varepsilon} \int_{\Gamma(z,\varepsilon)} [\omega(\tau)]^{-q} \, |d\tau| \right)^{1/q'} < \infty$$

holds, where $\Gamma(z, \varepsilon) := \{\tau \in \Gamma : |\tau - z| < \varepsilon\}$

For $f \in L^{p),\theta}_{\omega}(\Gamma)$ and $\omega \in A_p$ we define the function

$$\begin{aligned} f_0(t) &:= f[\psi(t)](\psi'(t))^{\frac{1}{p-\varepsilon}}, \ t \in \mathbb{T}, \\ \omega_0(t) &= \omega(\psi(t)). \end{aligned}$$

Note that if $f \in L^{p),\theta}_{\omega}(\Gamma)$ then $f_0 \in L^{p),\theta}_{\omega_0}(\mathbb{T})$ [28].

For $f \in L^{p),\theta}_{\omega}(\Gamma)$ we define the Cauchy type integral

$$f_0^+(t) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0(z)}{z-t}dz$$

which is analytic in $\mathbb{D}$. Also, we define *rth mean moduli of smoothness* for $f \in \mathbb{E}^{p),\theta}_{\omega}(G)$ by

$$\Omega_r(f,\delta)_{G,\ p),\theta,\omega} := \Omega_r(f_0^+,\delta)_{\ p),\theta,\omega}, \ \delta > 0, \ r = 1,2,...$$

The best approximation of $f \in \mathbb{E}^{p),\theta}_{\omega}(G)$ in the class $\prod_n$ of the algebraic polynomials of degree not exceeding n is defined by

$$E_n(f)_{\mathbb{E}^{p),\theta}_{\omega}(G)} := \inf\left\{\|f-P_n\|_{L^{p),\theta}_{\omega}(\Gamma)} : P_n \in \prod\nolimits_n\right\}.$$

Let $\varphi_{k,p-\varepsilon}(z)$, $k = 0,1,2,...(1 < p < \infty$ and $0 < \varepsilon < p-1)$ be the $p-\varepsilon$ Faber polynomials for $\overline{G}$ [28] . The Faber polynomials $\varphi_{k,p-\varepsilon}(z)$, associated with $G \cup \Gamma$, are defined through the expansion

$$\frac{\psi'(t)}{\psi(t)-z} = \sum_{k=0}^{\infty} \frac{\varphi_{k,p-\varepsilon}(z)}{t^{k+1}}, \ z \in G, \ t \in \mathbb{D} \tag{1.4}$$

and the equality

$$\varphi_{k,p-\varepsilon}(z) = \frac{1}{2\pi i}\int\limits_{|t|=R} \frac{t^k \psi'(t)^{1-1/(p-\varepsilon)}}{\psi(t)-z}dt \ \ z \in G$$

holds [28], [38].

Let $f \in \mathbb{E}^{p),\theta}_{\omega}(G)$; $\omega \in A_p(\Gamma)$, $\varepsilon > 0$ and $0 < \varepsilon < p-1$. Then we have

$$f(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{f(s)}{s-z}ds = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f_0(t)\left[\psi'(t)\right]^{1-1/(p-\varepsilon)}}{\psi(t)-z}dt,$$

for every $z \in G$. Considering this formula and expansion (1.4), we can associate with f the formal series

$$f(z) \sim \sum_{k=0}^{\infty} c_k(f)\varphi_{k,p-\varepsilon}(z), \tag{1.5}$$

where

$$c_k(f) := \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{f_0(t)}{t^{k+1}}dt, \quad k = 0, 1, 2, ...$$

The series (1.5) is called the $p-\varepsilon$ *Faber series* expansion of f, and the coefficients $c_k(f),\ k = 0, 1, 2, ...$ are said to be the $p-\varepsilon$ *Faber coefficients* of f.

The *nth partial sums*, *Cesaro sums* and *Zygmund sums* of the $p-\varepsilon$ Faber series (1.5) are defined by

$$\begin{aligned} S_n(z,f) &= \sum_{k=0}^{n} c_k(f)\varphi_{k,p-\varepsilon}(z), \\ \sigma_n(z,f) &= \frac{1}{n+1}\sum_{k=0}^{n} S_k(z,f), \end{aligned}$$

$$Z_{n,k}(z,f) = \sum_{\nu=0}^{n}(1-\frac{\nu^k}{(n+1)^k})c_k(f)\varphi_{k,p-\varepsilon}(z).$$

respectively. The *Abel sum* of the $p-\varepsilon$ Faber series (1.5) is defined by

$$U_s(z,f) \sim \sum_{k=0}^{\infty} s^k c_k(f)\varphi_{k,p-\varepsilon}(z)$$

where $0 \le s < 1$.

Let $P :=$ {all algebraic polynomials (with no restrictions on the degree) }, and let $P(\mathbb{D})$ be the set of traces of members of P on $\mathbb{D}$. We define the operator

$$T_{p-\varepsilon}:\ P(\mathbb{D}) \to E_{\omega}^{p),\theta}(G)$$

as

$$T_{p-\varepsilon}(P)(z) := \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{P(w)\left[\psi'(w)\right]^{1-1/(p-\varepsilon)}}{\psi(w)-z}dw, \quad z \in G.$$

Then using (1.4) we have

$$T_{p-\varepsilon}\left(\sum_{k=0}^{\infty}\alpha_k w^k\right) = \sum_{k=0}^{\infty}\alpha_k\varphi_{k,p-\varepsilon}(z),$$

where $\varphi_{k,p-\varepsilon}(z),\ k \in \mathbb{N}$, are the $p-\varepsilon$ Faber polynomials of $\overline{G}$.

We shall use the $c, c_1, c_2, ...$ to denote constants (in general, different in different relations) depending only on quantities that are not important for the questions of interest.

The approximation problems in non-weighted, weighted grand and generalized Lebesgue spaces were studied in [6], [9]-[14], [27]-[29], [32], [35] and [42].

In this study we investigate the approximation of the functions by Zygmund means of Fourier trigonometric series in the weighted generalized grand Lebesgue spaces $\widetilde{L}_{\omega}^{p),\theta}(\mathbb{T}), 1<p<\infty,\ \theta>0$. Note that estimates in this study are obtained in terms of the best approximation $E_n(f)_{p),\theta,\omega}$ and modulus of smoothness. Also, the approximation problems of Cesaro, Zygmund and Abel sums of Faber series in the weighted generalized grand Smirnov classes $\mathbb{E}_{\omega}^{p),\theta}(G)$,defined on the bounded simply connected domains of complex plane $\mathbb{C}$ are studied. Similar problems of the approximation theory in the different spaces have been studied by several authors (see, for example, [1]-[4], [7], [19], [20], [23]-[26], [30], [31], [33], [36]-[38], [41], [43]-[46]).

Note that for the proof of the new results obtained in the weighted generalized grand Lebesgue spaces we apply the method developed in [19], [28] and [43].

Our main results are the following.

Theorem 1.1. Let $1<p<\infty,\ \theta>0,\ \ f\in\widetilde{L}_{\omega}^{p),\theta},\ \omega\in A_p(\mathbb{T})\ ,\ r\in Z_+,\ k\in N$ and let the series

$$\sum_{k=1}^{\infty}k^{r-1}\ E_{k-1}(f)_{p),\theta,\omega}$$

converges. Then f is equivalent (equal almost everywhere) to a $2\pi-$periodic absolutely continuous function $\psi\in AC(\mathbb{T})$ and the inequality

$$\begin{aligned}&\left\|\psi^{(r)}-Z_{n,k}\left(\psi^{(r)}\right)\right\|_{L_{\omega}^{p),\theta}(\mathbb{T})}\\ \leq\ & c_1(k,r)\left(\sum_{\nu=n+1}^{\infty}\nu^{r-1}E_{\nu-1}(f)_{p),\theta,\omega}+n^{-k}\sum_{\nu=1}^{n}\nu^{k+r-1}E_{\nu-1}(f)_{p),\theta,\omega}\right),\ n\in\mathbb{N}\end{aligned}$$

holds.

Theorem 1.2. Let $1<p<\infty,\ \theta>0,\ \ f\in\widetilde{L}_{\omega}^{p),\theta},\ \omega\in A_p(\mathbb{T}),\ k\in N$. Then the estimate

$$\Omega_l\left(f,\frac{1}{n}\right)_{p),\theta,\omega}\leq c_2\left\|f-Z_{n,k}(f)\right\|_{L_{\omega}^{p),\theta}(\mathbb{T})} \tag{1.6}$$

holds, where $l=\{k, k\text{-}oven,\ k+1, k-odd\}$.

Theorem 1.3. Let $\Gamma\in S$ and $\omega\in A_p(\Gamma)$. Then for $f\in\mathbb{E}_{\omega}^{p),\theta}(G),\ 1<p<\infty,\ \theta>0$ the ineguality

$$\left\|f(\cdot)-\sigma_n(\cdot,f)\right\|_{L_{\omega}^{p),\theta}(\Gamma)}\leq c_3\Omega_r(\frac{1}{n+1},f)_{G,p),\theta,\omega}$$

holts with a constant $c_3>0,$ not depend on n.

Theorem 1.4. If the conditions of Theorem 1.3 are satisfed, then for $f\in\mathbb{E}_{\omega}^{p),\theta}(G),\ 1<p<\infty,\ \theta>0$ there exists a constant $c_4>0,$ not depend on n, such that

$$\left\|f(\cdot)-Z_{n,k}(\cdot,f)\right\|_{L_{\omega}^{p),\theta}(\Gamma)}\leq c_4\Omega_r(\frac{1}{n+1},f)_{G,p),\theta,\omega}$$

Theorem 1.5. If the conditions of Theorem 1.3 hold, then there exists a constant $c_5 > 0$, not depend on s, such that the estimate

$$\|f(\cdot) - U_s(\cdot, f)\|_{L_\omega^{p),\theta}(\Gamma)} \leq c_5 \Omega_r(1-s, f)_{G,p),\theta,\omega}$$

holds for $f \in \mathbb{E}_\omega^{p),\theta}(G)$, $1 < p < \infty$, $\theta > 0$.

We need the following results in the proof of the main results.

Theorem 1.6 ([10])**.** Let $1 < p < \infty$ and $\theta > 0$. Let $\omega \in A_p$.Then there exists a positive constant c_6 such that for arbitrary $f \in \widetilde{L}_\omega^{p),\theta}(\mathbb{T})$ and n the following estimate

$$\|f(\cdot) - \sigma_n(\cdot, f)\|_{L_\omega^{p),\theta}(\mathbb{T})} \leq c_6 \Omega_r(\frac{1}{n+1}, f)_{p),\theta,\omega}$$

holds.

Theorem 1.7 ([10])**.** Let $1 < p < \infty$ and $\theta > 0$. Let $\omega \in A_p$.Then there exists a positive constant c_7 for arbitrary $f \in \widetilde{L}_\omega^{p,\theta}(\mathbb{T})$ and n the following inequality

$$\|f(\cdot) - Z_{n,k}(\cdot, f)\|_{L_\omega^{p),\theta}(\mathbb{T})} \leq c_7 \Omega_r(\frac{1}{n+1}, f)_{p),\theta,\omega}$$

holds.

Theorem 1.8 ([10])**.** Let $1 < p < \infty$ and $\theta > 0$. Let $\omega \in A_p$.Then for arbitrary $f \in \widetilde{L}_\omega^{p),\theta}(\mathbb{T})$ we have

$$\|f(\cdot) - U_s(\cdot, f)\|_{L_\omega^{p),\theta}(\mathbb{T})} \leq c_8 \Omega_r(\frac{1}{n+1}, f, 1-s)_{p),\theta,\omega},$$

with a constant $c_8 > 0$, not depend on s, $0 < s < 1$.

2 Proofs of theorems

Proof of Theorem 1.1. Let the series

$$\sum_{k=0}^{\infty} (k+1)^{r-1} E_k(f)_{p),\theta,\omega}$$

converge. In this case, using the proof method developed in the study S. S. Volosivets [43] we can prove that f is equivalent (equal almost everywhere) to a $2\pi-$periodic absolutely continuous function $\psi \in AC(\mathbb{T})$ and the following inequality holds:

$$E_n(\psi^{(r)})_{p),\theta,\omega} \leq c_9 \left((n+1)^r E_n(f)_{p),\theta,\omega} + \sum_{k=n+1}^{\infty} k^{r-1} E_k(f)_{p),\theta,\omega} \right). \tag{2.1}$$

On the other hand the inequality

$$\|g - Z_{n,k}(g)\|_{L_\omega^{p),\theta}(\mathbb{T})} \leq c_{10} n^{-k} \sum_{\nu=1}^{n} \nu^{k-1} E_{\nu-1}(g)_{p),\theta,\omega} \tag{2.2}$$

holds [10, Theorem 4 and 7]. Comparing (2.1) and (2.2) we obtain that

$$\begin{aligned}
&\left\|\psi^{(r)}-Z_{n,k}(\psi^{(r)})\right\|_{L_{\omega}^{p),\theta}(\mathbb{T})}\\
\leq\ & c_{11}n^{-k}\sum_{\nu=1}^{n}\nu^{k-1}E_{\nu-1}(\psi^{(r)})_{p),\theta,\omega}\\
\leq\ & c_{12}n^{-k}\sum_{\nu=1}^{n}\nu^{k+r-1}E_{\nu-1}(f)_{p),\theta,\omega}\\
&+c_{13}n^{-k}\sum_{\nu=1}^{n}\nu^{k-1}\sum_{\mu=\nu+1}^{n}\mu^{r-1}E_{\mu-1}(f)_{p),\theta,\omega}\\
&+c_{14}n^{-k}\sum_{\nu=1}^{n}\nu^{k-1}\sum_{\mu=n+1}^{\infty}\mu^{r-1}E_{\mu-1}(f)_{p),\theta,\omega}\\
\leq\ & c_{15}n^{-k}\sum_{\nu=1}^{n}\nu^{k+r-1}E_{\nu-1}(f)_{p),\theta,\omega}+c_{16}n^{-k}\sum_{\mu=1}^{n}\mu^{r-1}E_{\mu-1}(f)_{p),\theta,\omega}\sum_{\nu=1}^{\mu}\nu^{k-1}\\
&+c_{17}\sum_{\mu=n+1}^{\infty}\mu^{r-1}E_{\mu-1}(f)_{p),\theta,\omega}\\
\leq\ & c_{18}\left(\sum_{\nu=n+1}^{\infty}\nu^{r-1}E_{\nu-1}(f)_{p),\theta,\omega}+n^{-k}\sum_{\nu=1}^{n}\nu^{k+r-1}E_{\nu-1}(f)_{p),\theta,\omega}\right)
\end{aligned}$$

which completes the proof of Theorem 1.1. Q.E.D.

Proof of Theorem 1.2. Let $T_n(f,x)$ be a trigonometric polynomial of best approximation to f in $L_{\omega}^{p),\theta}(\mathbb{T})$. It is known that the following identity holds:

$$\begin{aligned}
&T_n(f,x)-Z_{n,k}\left(T_n(f),x\right)\\
=\ & f(x)-Z_{n,k}(f,x)+T_n(f,x)-f(x)\\
&+Z_{n,k}(f-T_n(f),x).
\end{aligned}\tag{2.3}$$

By [10] we obtain

$$\left\|Z_{n,k}(f,\cdot)\right\|_{L_{\omega}^{p),\theta}(\mathbb{T})}\leq c_{19}\left\|f\right\|_{L_{\omega}^{p),\theta}(\mathbb{T})}.\tag{2.4}$$

Consideration of (2.3) and (2.4) gives us

$$\begin{aligned}
&\left\|T_n(f,x)-Z_{n,k}\left(T_n(f),x\right)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}\\
\leq\ &\left\|f-Z_{n,k}(f)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}+\left\|T_n(f)-f\right\|_{L_\omega^{p),\theta}(\mathbb{T})}\\
&+\left\|Z_{n,k}(f-T_n(f))\right\|_{L_\omega^{p),\theta}(\mathbb{T})}\\
\leq\ &\left\|f-Z_{n,k}(f)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}+c_{20}E_n(f)_{p),\theta,\omega}\\
\leq\ &c_{21}\left\|f-Z_{n,k}(f)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}. \qquad (2.5)
\end{aligned}$$

If $k-$is an even number the following relation holds:

$$T_n(f,x)-Z_{n,k}\left(T_n(f),x\right)=(-1)^{\frac{k}{2}}\,(n+1)^{-k}\,T_n^{(k)}(f,x). \qquad (2.6)$$

Using (2.5),(2.6) and [11] we get

$$\begin{aligned}
&\Omega_k\left(f,\frac{1}{n}\right)_{p),\theta,\omega}\\
=\ &\Omega_k\left(f-T_n(f)+T_n(f),\frac{1}{n}\right)_{p),\theta,\omega}\\
\leq\ &c_{22}\Omega_k\left(f-T_n(f)\right)_{p),\theta,\omega}+c_{23}\Omega_k\left(T_n(f),\frac{1}{n}\right)_{p),\theta,\omega}\\
\leq\ &c_{24}\left\|f-T_n(f)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}+c_{25}n^{-k}\left\|T_n^{(k)}(f)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}\\
\leq\ &c_{26}E_n(f)_{p),\theta,\omega}+c_{27}\left\|T_n(f)-Z_{n,k}\left(T_n(f)\right)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}\\
\leq\ &c_{28}E_n(f)_{p),\theta,\omega}+c_{29}\left\|f-Z_{n,k}\left(f\right)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}\\
\leq\ &c_{30}\left\|f-Z_{n,k}\left(f\right)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}. \qquad (2.7)
\end{aligned}$$

Let $\widetilde{T_n}^{(k)}(f,x)$ be a trigonometric conjugate of $T^{(k+1)}(f,x)$. If k is a odd number the relation

$$T_n(f,x)-Z_{n,k}\left(T_n(f),x\right)=(-1)^{\frac{k+3}{2}}\,(n+1)^{-k}\,\widetilde{T_n}^{(k)}(f,x). \qquad (2.8)$$

holds. Also, according to [11], [27] we obtain

$$\left\|T^{(k+1)}(f,x)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}\leq c_{31}n\left\|\widetilde{T_n}^{(k)}(f)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}. \qquad (2.9)$$

Taking into account the relations (2.8), (2.9) we have

$$
\begin{aligned}
&\Omega_{k+1}\left(f,\frac{1}{n}\right)_{p),\theta,\omega}\\
=\ &\Omega_{k+1}\left(f-T_n(f)+T_n(f),\frac{1}{n}\right)_{p),\theta,\omega}\\
\leq\ &c_{32}\Omega_{k+1}\left(f-T_n(f)\right)_{p),\theta,\omega}+c_{33}\Omega_{k+1}\left(T_n(f),\frac{1}{n}\right)_{p),\theta,\omega}\\
\leq\ &c_{34}\left\|f-T_n(f)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}+c_{35}n^{-(k+1)}\left\|T_n^{(k+1)}(f)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}\\
\leq\ &c_{36}E_n(f)_{p),\theta,\omega}+c_{37}n^{-k}\left\|\widetilde{T_n}^{(k)}(f)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}\\
\leq\ &c_{38}E_n(f)_{p),\theta,\omega}+c_{39}\left\|T_n(f)-Z_{n,k}\left(T_n(f)\right)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}\\
\leq\ &c_{40}E_n(f)_{p),\theta,\omega}+c_{41}\left\|f-Z_{n,k}\left(f\right)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}\\
\leq\ &c_{42}\left\|f-Z_{n,k}\left(f\right)\right\|_{L_\omega^{p),\theta}(\mathbb{T})}.
\end{aligned}
\tag{2.10}
$$

Consideration of (2.7) and (2.10) gives us (1.8). Thus, the proof of Theorem 1.2 is completed. Q.E.D.

Proof of Theorem 1.3. Let $f\in\mathbb{E}_\omega^{p),\theta}(G)$, $1<p<\infty$, $\theta>0$. The function f has the Faber series

$$f(z)\sim\sum_{k=0}^{\infty}c_k(f)\varphi_k(z).$$

Then by [18, Lemma 1] $f_0^+\in\mathbb{E}_\omega^{p),\theta}(\mathbb{D})$ and for the function f_0^+ the Taylor expansion

$$f_0^+(t)\sim\sum_{k=0}^{\infty}c_k(f)w^k, w\in\mathbb{U}$$

holds. It is clear that $f_0^+\in E^1(\mathbb{D})$ .Therefore, the boandary function $f_0^+\in L_\omega^{p),\theta}(\mathbb{T})$. According to [8, p.38, Theorem 3.4] the function f_0^+ has the Fourier expansion

$$f_0^+(t)\sim\sum_{k=0}^{\infty}c_k(f)e^{ikt}.$$

Using the boundedness of the operator $T_{p-\varepsilon}:\mathbb{E}_\omega^{p),\theta}(\mathbb{D})\to\mathbb{E}_\omega^{p),\theta}(G)$ [28] and Theorem 1.6 we have

$$
\begin{aligned}
&\left\|f(\cdot)-\sigma_n(\cdot,f)\right\|_{L_\omega^{p),\theta}(\Gamma)}\\
\leq\ &\left\|T_{p-\varepsilon}\left(f_0^+\right)-T_{p-\varepsilon}\left(\sigma_n\left(\cdot,f_0^+\right)\right)\right\|_{L_\omega^{p),\theta}(\Gamma)}\\
\leq\ &c_{43}\left\|T_{p-\varepsilon}\right\|\left\|f_0^+-\sigma_n\left(\cdot,f_0^+\right)\right\|_{L_{\omega_0}^{p),\theta}(\mathbb{T})}\\
\leq\ &c_{44}\Omega_r\left(f_0^+,\frac{1}{n}\right)_{p),\theta,\omega_0}=c_{45}\Omega_r\left(f,\frac{1}{n}\right)_{\Gamma,\ p),\theta,\omega}
\end{aligned}
$$

which completes the proof of theorem 1.3. Q.E.D.

The proofs of Theorem 1.4 and 1.5 follow from similar applications of Theorem 1.7 and Theorem 1.8, respectively.

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