Pell polynomial approach for solving a class of systems of nonlinear differential equations

Musa Cakmak[1] and Sertan Alkan[2]

[1]Department of Accounting and Tax Applications, Hatay Mustafa Kemal Univ., Hatay, Turkey
[2]Department of Computer Engineering, Iskenderun Technical University, Hatay, Turkey
E-mail: musacakmak@mku.edu.tr[1], sertan.alkan@iste.edu.tr[2]

Abstract

In this study, Pell polynomial approach is used for approximately solving a class of systems of nonlinear differential equations with initial conditions. The given problem is firstly expressed as a matrix-vector system via collocation points. Then the unknown coefficients of the approximate solution are obtained. Also, some test problems are given to demonstrate accuracy and effectiveness of the proposed method. Additionally, the calculated numerical values are compared with exact solutions of the test problems and approximate ones obtained with other methods in literature.

2010 Mathematics Subject Classification. **42C05**. 45B05.
Keywords. Nonlinear systems, differential equations, Pell polynomials.

1 Introduction

The systems of nonlinear differential equations are frequently used in astrophysics, physics engineering, scientific phenomena and their modelling. For this reason, it is very important to obtain the solution of these systems. Due to the difficulties on obtaining the analytical solutions, several methods are developed to solve those equations approximately. Some of the applied numerical methods on the approximate solutions of nonlinear differential equations are as follows: Variational iteration method [1], operational matrix method based on Bernoulli polynomials [2], optimized decomposition method [3], homotopy analysis method [4].

In [5], the Pell-Lucas collocation method is used to obtain approximate solution of linear functional differential equations. In [6], a class of linear partial differential equations with Dirichlet conditions is approximately solved using the Pell collocation method. In the paper given by [7], the approximate solutions of a class of delay Fredholm integro-differential equations is obtained using PellLucas collocation method by authors. In [8], the Pell-Lucas collocation method is used to calculate the approximate solution of a class of linear Fredholm-Volterra integro differential equations. In [9], logistic growth model and preypredator model are numerically solved by using PellLucas collocation method. In [10], some properties and definition of Pell polynomials are presented by authors. Also, in [11], the authors study identities involving Pell polynomial and Pell-Lucas polynomial.

In this paper, the Pell collocation method is developed for solving the following class of systems of nonlinear differential equation:

$$\sum_{k=0}^{2}\sum_{r=1}^{2} R_{jkr}(x)u_r^{(k)}(x) + \sum_{k=0}^{2}\sum_{r=1}^{2} Q_{jkrsp}(x)u_s^{r}(x)\,u_p^{(k)}(x) = g_j(x), \tag{1.1}$$

Advanced Studies: Euro-Tbilisi Mathematical Journal Special Issue (10 - 2022), pp. 1–15.

Tbilisi Centre for Mathematical Sciences.

Received by the editors: 12 June 2021.
Accepted for publication: 15 Nov 2021.

$$0 \leq x \leq 1,\ j,s,p = 1,2$$

with the initial conditions

$$\sum_{k=0}^{1}\left[a_{jk}u_r^{(k)}(0) + b_{jk}u_r^{(k)}(0)\right] = \delta_{jr}, \qquad j = 1,2 \tag{1.2}$$

where $u_r^{(0)}(x) = u_r(x),\ u_r^{0}(x) = 1$ and $u_r(x)$ is an unknown functions. $R_{jkr}(x)$, $Q_{jkrsp}(x)$ and $g_j(x)$ are given continuous functions on interval $[0,1]$, $a_{jk},\ b_{jk},$ and δ_{jr} are suitable constants. The aim of this study is to get the approximate solutions as the truncated Pell series defined by

$$u_r\,(x) = \sum_{n=1}^{N+1} c_{rn}P_n(x) \tag{1.3}$$

where $P_n(x)$ denotes the Pell polynomials; c_{rn} $(1 \leq rn \leq N+1)$ are unknown Pell polynomial coefficients, and N is any positive integer such that $N \geq m$.

The paper consists of six sections. In Section 2, the basic properties and definitions related to Pell polynomials are presented. In Section 3, the fundamental matrix forms of Pell collocation method by using fundamental relations of Pell polynomials are constructed to obtain the approximate solutions for the given class of systems of nonlinear differential equations. In section 4, the absolute error function is formulated. In Section 5, three test problems are presented and the method is tested using the absolute error function. Finally, conclusions are given in Section 6.

2 Properties of Pell polynomials

The Pell polynomials and series were studied by Horadam, A. F. and Mahon, J. M. [12–17]. The recurrence relation of those polynomials is defined by

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x) \tag{2.1}$$

For $n \geqslant 3.$, $P_1(x) = 1,\ P_2(x) = 2x$. The properties were further investigated by Horadam, A. F. and Mahon, J. M. [14]. The first few Pell polynomials are

$$\begin{aligned}
P_1(x) &= \mathbf{1}, \\
P_2(x) &= 2x, \\
P_3(x) &= \mathbf{4}x^2 + \mathbf{1}, \\
P_4(x) &= 8x^3 + 4x, \\
P_5(x) &= \mathbf{16}x^4 + \mathbf{12}x^2 + \mathbf{1}, \\
P_6(x) &= 32x^5 + 32x^3 + 6x, \\
P_7(x) &= \mathbf{64}x^6 + \mathbf{80}x^4 + \mathbf{24}x^2 + \mathbf{1}, \\
P_8(x) &= 128x^7 + 192x^5 + 80x^3 + 8x, \\
P_9(x) &= \mathbf{256}x^8 + \mathbf{448}x^6 + \mathbf{240}x^4 + \mathbf{40}x^2 + \mathbf{1}, \\
&\ \vdots
\end{aligned} \tag{2.2}$$

3 Fundamental relations

Let us assume that linear combination of Pell polynomials (1.3) is an approximate solutions of Eq (1.1). Our purpose is to determine the matrix forms of Eq (1.1) by using (1.3). Firstly, we can write Pell polynomials (2.2) in the matrix form

$$\mathbf{P}(x)=\mathbf{T}(x)\mathbf{M} \tag{3.1}$$

where $P(x)=[P_1(x)\ \ P_2(x)\cdots P_{N+1}(x)]$, $\mathbf{T}(x)=\left[1\ x\ x^2\ x^3...x^N\right]$, $\mathbf{C}_r=[c_{r1}\ c_{r2}\ \ \cdots c_{r(N+1)}]^T$ $,r=1,2$ and

$$\mathbf{M}=\begin{bmatrix} \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \mathbf{1} & \cdots \\ 0 & 2 & 0 & 4 & 0 & 6 & 0 & 8 & 0 & \cdots \\ 0 & 0 & \mathbf{4} & 0 & \mathbf{12} & 0 & \mathbf{24} & 0 & \mathbf{40} & \cdots \\ 0 & 0 & 0 & 8 & 0 & 32 & 0 & 80 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \mathbf{16} & 0 & \mathbf{80} & 0 & \mathbf{240} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 32 & 0 & 192 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{64} & 0 & \mathbf{448} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 128 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{256} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Then we set the approximate solutions defined by a truncated Pell series (1.3) in the matrix form

$$u_r(x)=\mathbf{P}(x)\mathbf{C}_r. \tag{3.2}$$

By using (3.1) and (3.2), the matrix relation is expressed as

$$\begin{aligned} u_r(x) &\cong u_{rN}(x)=\mathbf{P}(x)\mathbf{C}_r=\mathbf{T}(x)\mathbf{M}\mathbf{C}_r \\ u_r'(x) &\cong u_{rN}'(x)=\mathbf{TBMC}_r \\ u_r''(x) &\cong u_{rN}''(x)=\mathbf{T}(x)\mathbf{B}^2\mathbf{MC}_r \\ &\quad ... \\ u_r^{(k)}(x) &\cong u_{rN}^{(k)}(x)=\mathbf{T}(x)\mathbf{B}^k\mathbf{MC}_r \end{aligned} \tag{3.3}$$

where $r=1,2$. Also, the relations between the matrix $\mathbf{T}(x)$ and its derivatives, $\mathbf{T}'(x)$, $\mathbf{T}''(x)$,...,$\mathbf{T}^{(k)}(x)$ are

$$\begin{aligned} \mathbf{T}'(x) &= \mathbf{T}(x)\mathbf{B},\ \mathbf{T}''(x)=\mathbf{T}(x)\mathbf{B}^2 \\ \mathbf{T}'''(x) &= \mathbf{T}(x)\mathbf{B}^3,...,\mathbf{T}^{(k)}(x)=\mathbf{T}(x)\mathbf{B}^k \end{aligned} \tag{3.4}$$

Then we set the approximate solution defined by a truncated Pell series (1.3) in the matrix form

$$u_r(x)\cong u_{rN}(x)=\mathbf{P}(x)\mathbf{C}_r. \tag{3.5}$$

By substituting the Pell collocation points defined by

$$x_i=\frac{i}{N},\ i=0,1,...N \tag{3.6}$$

into Eq(3.3), we have

$$u_r^{(k)}(x_i) = \mathbf{T}(x_i)\,\mathbf{B}^k\mathbf{M}\mathbf{C}_r. \tag{3.7}$$

and the compact form of the relation (3.7) becomes

$$\mathbf{U}_r^{(k)} = \mathbf{T}\mathbf{B}^k\mathbf{M}\mathbf{C}_r, \qquad k = 0,1,2, \qquad r = 1,2 \tag{3.8}$$

where

$$\mathbf{U}_r^{(k)} = \begin{bmatrix} u_r^{(k)}(x_0) \\ u_r^{(k)}(x_1) \\ \vdots \\ u_r^{(k)}(x_N) \end{bmatrix}, \tag{3.9}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \mathbf{B}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}(x_0) \\ \mathbf{T}(x_1) \\ \vdots \\ \mathbf{T}(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & \ldots & x_0^N \\ 1 & x_1 & \ldots & x_1^N \\ 1 & \vdots & \ldots & \vdots \\ 1 & x_N & \ldots & x_N^N \end{bmatrix}.$$

In addition, we can obtain the matrix form $\left(\hat{\mathbf{U}}_s\right)^r\ \hat{\mathbf{U}}_p^{(k)}$ which appears in the nonlinear part of Eq. (1.1), by using Eq. (3.3) as

$$\begin{aligned} \left(\hat{\mathbf{U}}_s\right)^r\ \hat{\mathbf{U}}_p^{(k)} &= \begin{bmatrix} u_s^r(x_0)\,u_p^{(k)}(x_0) \\ u_s^r(x_1)\,u_p^{(k)}(x_1) \\ \vdots \\ u_s^r(x_N)\,u_p^{(k)}(x_N) \end{bmatrix} \\ &= \begin{bmatrix} u_s(x_0) & 0 & \ldots & 0 \\ 0 & u_s(x_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_s(x_N) \end{bmatrix}^r \begin{bmatrix} u_p^{(k)}(x_0) \\ u_p^{(k)}(x_1) \\ \vdots \\ u_p^{(k)}(x_N) \end{bmatrix} \end{aligned} \tag{3.10}$$

where

$$\left(\hat{\mathbf{U}}_s\right)^r\ \hat{\mathbf{U}}_p^{(k)} = \left(\hat{\mathbf{T}}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}_r\right)^r\ \mathbf{T}\ \left(\mathbf{B}\right)^{\mathbf{k}}\mathbf{M}. \tag{3.11}$$

$$\hat{\mathbf{T}} = \begin{bmatrix} \mathbf{T}(x_0) & 0 & \ldots & 0 \\ 0 & \mathbf{T}(x_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{T}(x_N) \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & 0 & \ldots & 0 \\ 0 & \mathbf{B} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{B} \end{bmatrix},$$

$$\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{M} & 0 & \ldots & 0 \\ 0 & \mathbf{M} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{M} \end{bmatrix}, \quad \hat{\mathbf{C}}_r = \begin{bmatrix} \mathbf{C}_r & 0 & \ldots & 0 \\ 0 & \mathbf{C}_r & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{C}_r \end{bmatrix}.$$ Substituting the collocation points $(x_i = i/N, i = 0, 1, \cdots, N)$ into Eq. (1.1), gives the system of equations

$$\sum_{k=0}^{2}\sum_{r=1}^{2} R_{jkr}(x_i)u_r^{(k)}(x_i) + \sum_{k=0}^{2}\sum_{r=1}^{2} Q_{jkrsp}(x_i)u_s^r(x_i)\,u_p^{(k)}(x_i) = g_j(x_i), \ , \quad 0 \le x \le 1$$

which can be expressed with the aid of Eqs. (3.9) and (3.10) as

$$\sum_{k=0}^{2}\sum_{r=1}^{2} R_{jkr}\mathbf{U}_r^{(k)} + \sum_{k=0}^{2}\sum_{r=1}^{2} Q_{jkrsp}\left(\hat{\mathbf{U}}_s\right)^r \ \hat{\mathbf{U}}_p^{(k)} = \mathbf{G}_j \tag{3.12}$$

where

$$\begin{aligned} R_{jkr} &= diag\left[R_{jkr}(x_0) \ \ R_{jkr}(x_1) \ \ \ldots \ \ R_{jkr}(x_N)\right], \\ Q_{jkrsp} &= diag\left[Q_{jkrsp}(x_0) \ \ Q_{jkrsp}(x_1) \ \ \ldots \ \ Q_{jkrsp}(x_N)\right] \\ \text{and} \ \mathbf{G}_j &= \left[\ gj(x_0) \quad g_j(x_1) \quad \ldots \quad g_j(x_N)\ \right]^T, \ j = 1, 2. \end{aligned}$$

Substituting the relations (3.8) and (3.11) into Eq. (3.12), the fundamental matrix equation can be obtained as

$$\left\{\sum_{k=0}^{2}\sum_{r=1}^{2} R_{jkr}\mathbf{T}\mathbf{B}^k\mathbf{M} + \sum_{k=0}^{2}\sum_{r=1}^{2} Q_{jkrsp}\left(\hat{\mathbf{T}}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}_r\right)^r \mathbf{T}\ \ (\mathbf{B})^{\mathbf{k}}\,\mathbf{M}\right\}\mathbf{C}_r = \mathbf{G}_j \tag{3.13}$$

Shortly, Eq. (3.13) is also written in the following form

$$\mathbf{W}\mathbf{C} = \mathbf{G} \quad \text{or} \ \ [\mathbf{W}; \mathbf{G}] \tag{3.14}$$

where

$$
\begin{aligned}
\mathbf{W} &= \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix} \\
W_{11} &= \sum_{k=0}^{2}\sum_{r=1}^{1} R_{jkr}\mathbf{T}\mathbf{B}^{k}\mathbf{M} + \sum_{k=0}^{2}\sum_{r=1}^{1} Q_{jkrsp}\left(\hat{\mathbf{T}}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}_r\right)^{r}\mathbf{T}\ \ (\mathbf{B})^{\mathbf{k}}\mathbf{M} \quad \textbf{for}\ \ \mathbf{j}=\mathbf{1} \\
W_{12} &= \sum_{k=0}^{2}\sum_{r=2}^{2} R_{jkr}\mathbf{T}\mathbf{B}^{k}\mathbf{M} + \sum_{k=0}^{2}\sum_{r=2}^{2} Q_{jkrsp}\left(\hat{\mathbf{T}}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}_r\right)^{r}\mathbf{T}\ \ (\mathbf{B})^{\mathbf{k}}\mathbf{M} \quad \textbf{for}\ \ \mathbf{j}=\mathbf{1} \\
W_{21} &= \sum_{k=0}^{2}\sum_{r=1}^{1} R_{jkr}\mathbf{T}\mathbf{B}^{k}\mathbf{M} + \sum_{k=0}^{2}\sum_{r=1}^{1} Q_{jkrsp}\left(\hat{\mathbf{T}}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}_r\right)^{r}\mathbf{T}\ \ (\mathbf{B})^{\mathbf{k}}\mathbf{M} \quad \textbf{for}\ \ \mathbf{j}=\mathbf{2} \\
W_{12} &= \sum_{k=0}^{2}\sum_{r=2}^{2} R_{jkr}\mathbf{T}\mathbf{B}^{k}\mathbf{M} + \sum_{k=0}^{2}\sum_{r=2}^{2} Q_{jkrsp}\left(\hat{\mathbf{T}}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}_r\right)^{r}\mathbf{T}\ \ (\mathbf{B})^{\mathbf{k}}\mathbf{M} \quad \textbf{for}\ \ \mathbf{j}=\mathbf{2}.
\end{aligned}
$$

Here, Eq. (3.14) is a system containing $(N+1)$ linear algebraic equations with the $(N+1)$ unknown Pell coefficients $c_{rn},\ n=1,2,...,N+1$. Using Eq. (3.8) at the point 0, the matrix representation of the initial conditions in Eq. (1.2) is given by

$$\left\{\sum_{k=0}^{m-1}\left[a_{jk}\mathbf{T}(0)+b_{jk}\mathbf{T}(0)\right](\mathbf{B})^{(k)}\mathbf{M}\right\}\mathbf{C}_r=\delta_{jr},\ j=0,1,2,...,m-1$$

or briefly

$$\mathbf{V}_{jr}\ \mathbf{C}_r=[\delta_{jr}] \qquad \text{or} \qquad [\mathbf{V}_{jr};\delta_{jr}]\,; \quad j=0,1,2,...,m-1 \tag{3.15}$$

where

$$\mathbf{V}_{jr}=\sum_{k=0}^{m-1}\left[a_{jk}\mathbf{T}(0)+b_{jk}\mathbf{T}(0)\right](\mathbf{B})^{(k)}\mathbf{M}=\left[v_{jo}\ \ v_{j1}\ \ v_{j2}\ ...\ v_{jN}\right].$$

Therefore, by replacing the row matrices in (3.15) by the last m rows of the augmented matrix (3.14), the new augmented matrix becomes

$$\hat{\mathbf{W}}\ \mathbf{C}=\hat{\mathbf{G}} \quad \text{or} \quad \left[\hat{\mathbf{W}};\hat{\mathbf{G}}\right]$$

which is an algebraic system. Here,

$$\left[\hat{\mathbf{W}};\hat{\mathbf{G}}\right]=\begin{bmatrix}\hat{W}_{11} & \hat{W}_{12}\\ \hat{W}_{21} & \hat{W}_{22}\end{bmatrix} \tag{3.16}$$

where

$$\left[\hat{W}_{11}\right] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1N+1} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2N+1} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{(N+1-m)1} & w_{(N+1-m)2} & w_{(N+1-m)3} & \cdots & w_{(N+1-m)N+1} \\ v_{11} & v_{12} & v_{13} & \cdots & v_{1N+1} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2N+1} \\ v_{31} & v_{32} & v_{33} & \cdots & v_{3N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{(m-1)1} & v_{(m-1)2} & v_{(m-1)3} & \cdots & v_{(m-1)N+1} \end{bmatrix}$$

$$\left[\hat{W}_{12}\right] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1N+1} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2N+1} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{(N+1-m)1} & w_{(N+1-m)2} & w_{(N+1-m)3} & \cdots & w_{(N+1-m)N+1} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\left[\hat{W}_{21}\right] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1N+1} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2N+1} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{(N+1-m)1} & w_{(N+1-m)2} & w_{(N+1-m)3} & \cdots & w_{(N+1-m)N+1} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\left[\hat{W}_{22}\right] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1N+1} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2N+1} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{(N+1-m)1} & w_{(N+1-m)2} & w_{(N+1-m)3} & \cdots & w_{(N+1-m)N+1} \\ v_{11} & v_{12} & v_{13} & \cdots & v_{1N+1} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2N+1} \\ v_{31} & v_{32} & v_{33} & \cdots & v_{3N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{(m-1)1} & v_{(m-1)2} & v_{(m-1)3} & \cdots & v_{(m-1)N+1} \end{bmatrix}$$

$$\hat{\mathbf{G}} = \begin{bmatrix} \hat{\mathbf{G}}_1 \\ \hat{\mathbf{G}}_2 \end{bmatrix}$$

where

$$\begin{aligned} \hat{\mathbf{G}}_1 &= \begin{bmatrix} g_1(x_0) & g_1(x_1) & \cdots & g_1(x_{N+1-m}) & \delta_{10} & \delta_{11} & \delta_{12} & \cdots & \delta_{1m-1} \end{bmatrix}^T \\ \hat{\mathbf{G}}_2 &= \begin{bmatrix} g_2(x_0) & g_2(x_1) & \cdots & g_2(x_{N+1-m}) & \delta_{20} & \delta_{21} & \delta_{22} & \cdots & \delta_{2m-1} \end{bmatrix}^T. \end{aligned}$$

In this way, by solving the linear equation system in (3.16), the unknown Pell coefficients $c_{rn},\ n = 1, 2, ..., N + 1$ are calculated and substituted into (1.3), and the approximate solution is obtained.

4 Error Estimation

In this section, to test the accuracy of the proposed method, it is presented that the error function $E_{i,N}(x)$ for $i = 1, 2$. The function $E_{i,N}(x)$ is given by

$$E_{i,N}(x) = |u_{i,N}(x) - u_i(x)| \ \text{ for } \ i = 1, 2 \tag{4.1}$$

where $u_{i,N}(x)$ and $u_i(x)$ are the approximate and exact solutions of Eq.(1.1), respectively.

5 Numerical examples

In this section, three numerical examples are given to show the accuracy of the proposed method. On these problems, the method is tested by using the error function given by (4.1). The obtained numerical results are presented with tables and graphics.

Example 1. Consider the nonlinear differential equation system

$$\begin{aligned} (x+1)\, u_1^{'}(x) + (x-1)\, u_2^{'}(x) + u_2(x) + u_1^{'}(x)\, u_2(x) &= g_1(x) \\ 2u_1^{'}(x) + x u_2^{'}(x) - u_1(x) + u_1^2(x) &= g_2(x) \end{aligned} \tag{5.1}$$

with initial conditions

$$u_1(0) = 0,\ u_2(0) = 1$$

and the exact solutions $u_1(x) = x^2 + 2x,\ u_2(x) = x^2 + 1$. The approximate the solution $u_r(x)$ by the Pell polynomials is

$$u_r(x) = \sum_{n=1}^{N+1} c_{rn} P_n(x)$$

where $N = 2$, $R_{111}(x) = x + 1$, $R_{112}(x) = x - 1$, $R_{102}(x) = 1, Q_{10112}(x) = 1$, $g_1(x) = 2x^3 + 7x^2 + 4x + 5$ and $R_{211}(x) = 2$, $R_{212}(x) = x$, $R_{201}(x) = -1$, $, Q_{20111}(x) = 1$, $g_2(x) = x^4 + 4x^3 + 5x^2 + 2x + 4$. Hence, the set of collocation points (3.6) for $N = 2$ is computed as

$$x_0 = 0, \ \ x_1 = \frac{1}{2},\ x_2 = 1$$

From Eq. (3.13), the fundamental matrix equation of the problem is

$$\begin{aligned}
\mathbf{G_1} &= \left\{\mathbf{R}_{111}\mathbf{TBM} + \mathbf{Q}_{10111}\hat{\mathbf{T}}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}_2\mathbf{T}\ \mathbf{BM}\right\}\mathbf{C}_1 \\
&+ \left\{\mathbf{R}_{112}\mathbf{TBM} + \mathbf{R}_{102}\mathbf{TM}\right\}\mathbf{C}_2 \\
\mathbf{G_2} &= \left\{\mathbf{R}_{211}\mathbf{TBM} + \mathbf{R}_{201}\mathbf{TM} + \mathbf{Q}_{20111}\hat{\mathbf{T}}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}_1\mathbf{T}\ \mathbf{M}\right\}\mathbf{C}_1 \\
&+ \left\{\mathbf{R}_{212}\mathbf{TBM}\right\}\mathbf{C}_2
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{W}_{11} &= \mathbf{R}_{111}\mathbf{TBM} + \mathbf{Q}_{10111}\hat{\mathbf{T}}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}_2\mathbf{T}\ \mathbf{BM} \\
\mathbf{W}_{12} &= \mathbf{R}_{112}\mathbf{TBM} + \mathbf{R}_{102}\mathbf{TM} \\
\mathbf{W}_{21} &= \mathbf{R}_{211}\mathbf{TBM} + \mathbf{R}_{201}\mathbf{TM} + \mathbf{Q}_{20111}\hat{\mathbf{T}}\ \hat{\mathbf{M}}\ \hat{\mathbf{C}}_1\mathbf{T}\ \mathbf{M} \\
\mathbf{W}_{22} &= \mathbf{R}_{212}\mathbf{TBM}
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{111} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}, \mathbf{R}_{112} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{R}_{102} = \mathbf{Q}_{10112} = \mathbf{Q}_{20111} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
\mathbf{R}_{211} &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \mathbf{R}_{201} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{R}_{212} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
\mathbf{T} &= \begin{bmatrix} \mathbf{T}(0) \\ \mathbf{T}\left(\frac{1}{2}\right) \\ \mathbf{T}(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix},\ \mathbf{M} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \\
\hat{\mathbf{T}} &= \begin{bmatrix} \mathbf{T}(x_0) & 0 & 0 \\ 0 & \mathbf{T}(x_1) & 0 \\ 0 & 0 & \mathbf{T}(x_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \\
\hat{\mathbf{M}} &= \begin{bmatrix} \mathbf{M} & 0 & \ldots & 0 \\ 0 & \mathbf{M} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{M} \end{bmatrix},\ \hat{\mathbf{C}}_r = \begin{bmatrix} \mathbf{C}_r & 0 & \ldots & 0 \\ 0 & \mathbf{C}_r & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{C}_r \end{bmatrix} \\
\mathbf{W} &= \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},\ \mathbf{C}_1 = \begin{bmatrix} a & b & c \end{bmatrix}^T, \mathbf{C}_2 = \begin{bmatrix} k & l & m \end{bmatrix}^T,\ \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} \\
\mathbf{G} &= \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix}, \mathbf{G}_1 = \begin{bmatrix} 5 & 9 & 18 \end{bmatrix}^T, \mathbf{G}_2 = \begin{bmatrix} 4 & \frac{109}{16} & 16 \end{bmatrix}^T, \\
W_{11} &= \begin{bmatrix} 0 & 2k+2m+2 & 0 \\ 0 & 2k+2l+4m+3 & 4k+4l+8m+6 \\ 0 & 2k+4l+10m+4 & 8k+16l+40m+16 \end{bmatrix}, W_{12} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 5 \end{bmatrix} \\
, W_{21} &= \begin{bmatrix} a+c-1 & 4 & a+c-1 \\ a+b+2c-1 & a+b+2c+3 & 2a+2b+4c+6 \\ a+2b+5c-1 & 2a+4b+10c+2 & 5a+10b+25c+11 \end{bmatrix}, W_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix}
\end{aligned}$$

Table 1: Numerical comparison of the functions u_1 and u_2 for *Example 2*

x	Adomian decomposition method [18]		Runge-Kutta method [18]		The proposed method	
	u_1	u_2	u_1	u_2	u_1	u_2
0.1	0.9004640155	1.004524209	0.9004639852	1.004524129	0.9004639428	1.0045242796
0.2	0.8034482877	1.016374098	0.8034482895	1.016373909	0.8034482353	1.0163741569
0.3	0.7108239629	1.033327532	0.7108241556	1.033327161	0.7108240368	1.0333275272
0.4	0.6238900641	1.053576241	0.6238926950	1.053574362	0.6238924878	1.0535748457
0.5	0.5434799759	1.075665396	0.5435048184	1.075649444	0.5435044969	1.0756500526
0.6	0.4699911064	1.098485574	0.4701502231	1.098381159	0.4701497676	1.0983818994
0.7	0.4032626837	1.121371970	0.4040245971	1.120858494	0.4040239951	1.1208593704

$$\left[\hat{\mathbf{W}}\right] = \begin{bmatrix} \hat{W}_{11} & \hat{W}_{12} \\ \hat{W}_{21} & \hat{W}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 2k+2m+2 & 0 & 1 & -2 & 1 \\ 0 & 2k+2l+4m+3 & 4k+4l+8m+6 & 1 & 0 & 0 \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ a+c-1 & 4 & a+c-1 & 0 & 0 & 0 \\ a+b+2c-1 & a+b+2c+3 & 2a+2b+4c+6 & 0 & 1 & 2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix};$$

$$\hat{\mathbf{G}} = \begin{bmatrix} 5 & 9 & \mathbf{0} & 4 & \frac{109}{16} & \mathbf{1} \end{bmatrix}^T$$

From Eq. (3.15), the matrix form for initial condition is

$$[\mathbf{V}_{11};\delta_{11}] = \begin{bmatrix} 1 & 0 & 1 & ; & \mathbf{0} \end{bmatrix},\ [\mathbf{V}_{12};\delta_{12}] = \begin{bmatrix} 1 & 0 & 1 & ; & \mathbf{1} \end{bmatrix}.$$

Consequently, by solving the system $[\hat{\mathbf{W}};\hat{\mathbf{G}}]$, the Pell coefficients matrix are obtained

$$\mathbf{C} = \begin{bmatrix} \frac{-1}{4} & 1 & \frac{1}{4} & \frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix}^T$$

where

$$\mathbf{C}_1 = \begin{bmatrix} \frac{-1}{4} & 1 & \frac{1}{4} \end{bmatrix}^T,\ \mathbf{C}_2 = \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix}^T.$$

The approximate solutions for $N = 2$ in terms of the Pell polynomials is obtained as

$$u_1(x) = x^2 + 2x \text{ and } u_2(x) = x^2 + 1.$$

Example 2. [18] Assume that the following differential equation system

$$\begin{cases} u_1'(x) = -2u_1(x) + u_1^2(x)u_2(x) \\ u_2'(x) = u_1(x) - u_1^2(x)u_2(x) \\ u_1(0) = 1, u_2(0) = 1 \end{cases} \tag{5.2}$$

Table 1 presents a numerical comparison of the proposed method with Adomian decomposition method and Runge-Kutta method for Eq.(5.2).

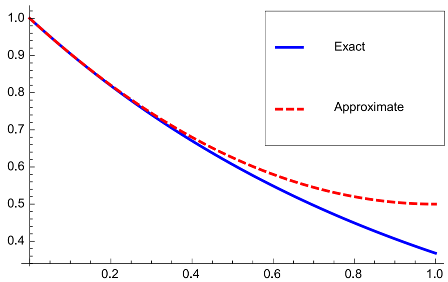
Table 2: Numerical comparison of the error functions $E_{1,N}$ and $E_{2,N}$ at the different values of N for *Example 3*

x	u_1			u_2		
	$E_{1,3}$	$E_{1,5}$	$E_{1,8}$	$E_{2,3}$	$E_{2,5}$	$E_{2,8}$
0.2	1.24824×10^{-4}	6.19185×10^{-7}	8.43282×10^{-11}	1.67152×10^{-4}	1.12256×10^{-6}	1.46487×10^{-10}
0.4	5.24571×10^{-4}	1.15094×10^{-6}	2.09448×10^{-10}	7.34585×10^{-4}	2.36088×10^{-6}	3.44978×10^{-10}
0.6	2.88945×10^{-4}	1.69739×10^{-6}	3.63609×10^{-10}	2.68779×10^{-4}	3.92165×10^{-6}	5.9967×10^{-10}
0.8	2.57203×10^{-3}	1.71744×10^{-6}	5.28537×10^{-10}	5.06667×10^{-3}	2.61007×10^{-6}	9.02742×10^{-10}
1	1.09323×10^{-2}	6.07564×10^{-5}	8.14292×10^{-9}	2.20430×10^{-2}	1.33261×10^{-4}	1.96085×10^{-8}

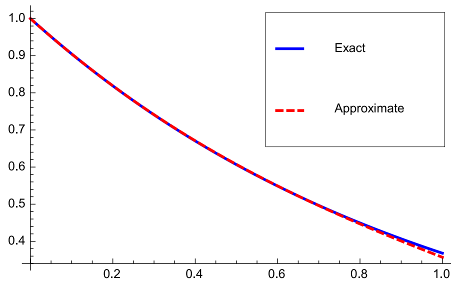
Example 3. Consider that the following differential equation system

$$\begin{cases} u_1''(x)(1+u_2^2(x))+u_2'(x)(1+u_1(x)) & =g_1(x) \\ u_2''(x)(1+u_1^2(x))+u_1'(x)(1+u_2(x)) & =g_2(x) \\ u_1(0)=1, u_2(0)=1 \\ u_1'(0)=-1, u_2'(0)=1 \end{cases} \tag{5.3}$$

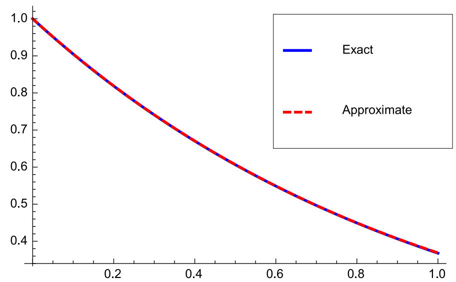
The exact solution of Eq.(5.3) is given by $u_1(x) = e^{-x}$, $u_2(x) = e^{x}$. Here, $g_1(x) = e^{x} + e^{-x} + 2$, $g_2(x) = e^{x} - e^{-x}$. Table 2 presents the numerical values of error function given in Eq.(4.1) for Eq.(5.3) when $N = 3, 5$ and 8. In Figure 1 and Figure 2, it is shown that the graphical comparison of the approximate and exact solutions obtained by the proposed method for u_1 and u_2 when $N = 2, 3$ and 4.



(a) $N = 2$

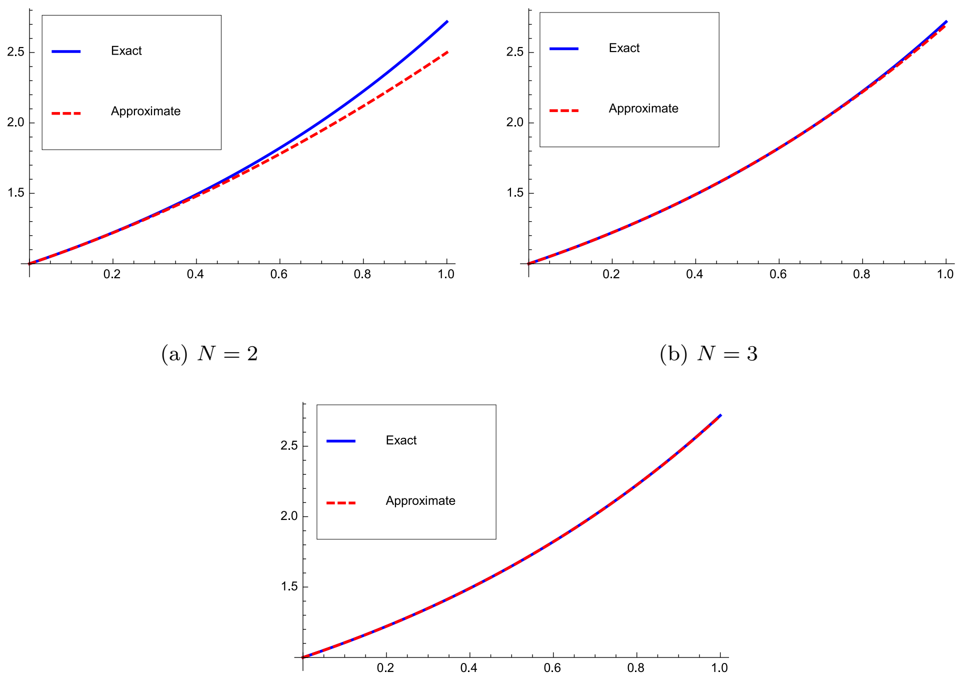


(b) $N = 3$



(c) $N = 4$

Figure 1: Graphical comparison of the exact and approximate solutions for u_1 when $N = 2, 3, 4$ for Example 3



(a) $N = 2$

(b) $N = 3$

(c) $N = 4$

Figure 2: Graphical comparison of the exact and approximate solutions for u_2 when $N = 2, 3, 4$ for Example 3

6 Conclusions

In this study, the Pell polynomial approach was used to obtain the approximate solution of a class of systems of nonlinear differential equations. The accuracy and efficiency of the method with three different example are illustrated. The obtained approximate solutions are compared with ones obtained with Adomian decomposition method and Runge-Kutta method. These comparisons reveal that the method is effective and applicable to obtain approximate solution of systems of nonlinear differential equations.

References

[1] J. H. He, H. Latifizadeh, *A general numerical algorithm for nonlinear differential equations by the variational iteration method*, International Journal of Numerical Methods for Heat & Fluid Flow, 2020, 30(11),4797-4810.

[2] D. Rani, V. Mishra, *Numerical inverse Laplace transform based on Bernoulli polynomials operational matrix for solving nonlinear differential equations*, Results in Physics, 2020, 16, 102836.

[3] Z. Odibat, *An optimized decomposition method for nonlinear ordinary and partial differential equations*, Physica A: Statistical Mechanics and its Applications, 2020, 541, 123323.

[4] Z. Odibat, *An improved optimal homotopy analysis algorithm for nonlinear differential equations*, Journal of Mathematical Analysis and Applications, 2020, 488(2), 124089.

[5] M. ahin, M. Sezer, *Pell-Lucas collocation method for solving high-order functional differential equations with hybrid delays*, Celal Bayar niversitesi Fen Bilimleri Dergisi, (2018), 14(2), 141-149.

[6] S. Cayan, M. Sezer, *Pell polynomial approach for Dirichlet problem related to partial differential equations*, Journal of Science and Arts, (2019), 19(3), 613-628.

[7] D. D. Demir, . K. Krk, M. Sezer, *PellLucas series approach for a class of Fredholm-type delay integro-differential equations with variable delays*, Mathematical Sciences, (2021), 15(1), 55-64.

[8] . Yzbasi, G. Yildirim, *Pell-Lucas collocation method to solve high-order linear Fredholm-Volterra integro-differential equations and residual correction*, Turkish Journal of Mathematics, (2020), 44(4), 1065-1091.

[9] . Yzbasi, G. Yildirim, *PellLucas collocation method for numerical solutions of two population models and residual correction*, Journal of Taibah University for Science, (2020), 14(1), 1262-1278.

[10] S. H. Aziz, S. Shihab, M. Rasheed, *On Some Properties of Pell Polynomials*, Journal of Pure Science, (2021), 26(1), 39-54.

[11] D. Jhala, G. P. S. Rathore, K. Sisodiya, *Some Determinantal Identities Involving Pell Polynomials*, International Journal of Scientific and Innovative Mathematical Research, (2014), 2(5), 481-488.

[12] M. Bicknell, *A primer on the Pell Sequence and Related Sequences*, Fibonacci Quarterly,(1975), 13(4), 345-349.

[13] A. F. Horadam, *Applications of Modified Pell Numbers to Representations*, Ulam Quarterly, (1994), 3(1), 34-53.

[14] A. F. Horadam, J. M. Mahon, *Pell and Pell-Lucas Polynomials*, The Fibonacci Quarterly,(1985), 23(1), 7-20.

[15] T. Koshy, *Pell and Pell-Lucas Numbers with Applications*, ISBN: 978-1-4614-8489-9,(2014), Springer.

[16] J. Ercolano, *Matrix generators of Pell sequences*, Fibonacci Quart.,(1979), 17(1), 71-77.

[17] Q. Mushtaq, U. Hayat, *Pell numbers, Pell–Lucas numbers and modular group*, Algebra Colloq.,(2007) 14(1), 97-102.

[18] J. Biazar, Z. Ayati, *An approximation to the solution of the Brusselator system by Adomian decomposition method and comparing the results with Runge-Kutta method*, Int. J. Contemp. Math. Sciences, (2007), 2(20), 983-989.