

The Ultrapower Axiom UA and the number of normal measures over \aleph_1 and \aleph_2

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Abstract

We show that assuming the consistency of certain large cardinals (namely a supercompact cardinal with a measurable cardinal above it of the appropriate Mitchell order) together with the Ultrapower Axiom UA introduced by Goldberg in [3], it is possible to force and construct choiceless universes of ZF in which the first two uncountable cardinals \aleph_1 and \aleph_2 are both measurable and carry certain fixed numbers of normal measures. Specifically, in the models constructed, \aleph_1 will carry exactly one normal measure, namely $\mu_\omega = \{x \subseteq \aleph_1 \mid x \text{ contains a club set}\}$, and \aleph_2 will carry exactly τ normal measures, where $\tau = \aleph_n$ for $n = 0, 1, 2$ or $\tau = n$ for $n \geq 1$ an integer (so in particular, $\tau \leq \aleph_2$ is any nonzero finite or infinite cardinal). This complements the results of [1] in which $\tau \geq \aleph_3$ and contrasts with the well-known facts that assuming AD + DC, \aleph_1 is measurable and carries exactly one normal measure, and \aleph_2 is measurable and carries exactly two normal measures.

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1 Introduction and preliminaries

In [1], the following theorems were proven, where $\mu_\omega = \{x \subseteq \aleph_1 \mid x \text{ contains a club set}\}$.

Theorem 1.1. Let $V^* \models$ “ZFC + GCH + $\kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal above $\kappa + \tau > \lambda^+$ is a fixed but arbitrary regular cardinal”. There are then a generic extension V of V^* , a partial ordering $\mathbb{P} \in V$, and a symmetric submodel $N \subseteq V^{\mathbb{P}}$ such that $N \models$ “ZF + DC + $\kappa = \aleph_1$ and $\lambda = \aleph_2$ are measurable cardinals”. In N , the regular cardinals greater than or equal to λ are the same as in V (which has the same cardinal and cofinality structure at and above λ as V^*), \aleph_1 carries exactly one normal measure (namely μ_ω), and \aleph_2 carries exactly τ normal measures.

Theorem 1.2. Let $V^* \models$ “ZFC + GCH + $\kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal above κ ”. There are then a generic extension V of V^* , a partial ordering $\mathbb{P} \in V$, and a symmetric submodel $N \subseteq V^{\mathbb{P}}$ such that $N \models$ “ZF + DC + $\kappa = \aleph_1$ and $\lambda = \aleph_2$ are measurable cardinals”. In N , the regular cardinals greater than or equal to λ are the same as in V (so \aleph_3 is regular), \aleph_1 carries exactly one normal measure (namely μ_ω), and \aleph_2 carries exactly \aleph_3 normal measures.

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Theorems 1.1 and 1.2 show that it is possible, assuming the appropriate large cardinal hypotheses, for \aleph_1 and \aleph_2 to be simultaneously measurable and also for \aleph_2 to carry exactly $\tau \geq \aleph_3$ normal measures, where τ is an arbitrary regular cardinal. This complements the results previously known about the number of normal measures \aleph_1 and \aleph_2 carry assuming AD + DC (the Axiom of Determinacy together with the Axiom of Dependent Choice). In particular, under these assumptions, \aleph_1 and \aleph_2 are measurable cardinals, \aleph_1 carries exactly one normal measure (namely μ_ω), and \aleph_2 carries exactly two normal measures. Readers are referred to [5] for a further discussion of these facts. However, the general question of the number of normal measures \aleph_1 and \aleph_2 can carry when both of these cardinals are measurable is not answered by taking Theorems 1.1 and 1.2 or the results assuming AD + DC in aggregate. This motivates the current note, whose purpose is to show that the Ultrapower Axiom UA (introduced by Goldberg in [3]), together with the appropriate large cardinal assumptions, can provide additional answers to the aforementioned question. Specifically, we will prove the following theorem.

Theorem 1.3. Let $V \models$ “ZFC + GCH + UA + $\kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal above κ such that $o(\lambda) = \delta$ for some ordinal $\delta \leq \lambda^{++}$ ”. There is then a partial ordering $\mathbb{P} \in V$ and a symmetric submodel $N \subseteq V^{\mathbb{P}}$ such that $N \models$ “ZF + DC + $\kappa = \aleph_1$ and $\lambda = \aleph_2$ are measurable cardinals”. In N , the regular cardinals greater than or equal to λ are the same as in V , \aleph_1 carries exactly one normal measure (namely μ_ω), and \aleph_2 carries exactly $|\delta|$ normal measures.

It follows from Theorem 1.3 that if $1 \leq \delta < \omega$, then \aleph_2 will carry a precise finite number of normal measures (e.g., 1, 2, 375, etc.) in N . Further, if $\delta = \omega$, κ , or λ , then \aleph_2 will carry exactly \aleph_0 , \aleph_1 , or \aleph_2 normal measures respectively in N . If $\delta = \lambda^+$ or $\delta = \lambda^{++}$, then \aleph_2 will carry exactly \aleph_3 or \aleph_4 normal measures in N , facts which automatically follow from Theorems 1.1 and 1.2. Thus, Theorems 1.1 – 1.3 and the results assuming AD + DC taken together show that when \aleph_1 and \aleph_2 are simultaneously measurable, \aleph_2 can consistently carry an arbitrary (regular cardinal) number of normal measures.

We take this opportunity to make some additional remarks concerning UA. First, we note that currently, UA is not known to hold in any model of ZFC containing very large cardinals, including models of ZFC with supercompact cardinals. However, as Goldberg has pointed out in [3, Section 1], UA is true in the usual inner models constructed at lower levels of the large cardinal hierarchy. UA has also been verified in canonical inner models built by Woodin and Neeman-Steel for finite levels of supercompactness using iteration hypotheses, thereby suggesting that UA may be expected to hold in canonical inner models containing (fully) supercompact cardinals (if they indeed exist). Further, by [3, Theorem 2.5], UA implies that the Mitchell ordering of normal measures over a measurable cardinal is linear (which is likely far weaker than the full UA and is the only consequence of UA used in the proof of Theorem 1.3). These facts make UA, as well as its implication concerning the linearity of the Mitchell ordering, plausible when assumed in models of ZFC containing very large cardinals.

Before beginning the proof of Theorem 1.3, we very briefly mention some preliminary material. Although we will largely follow the terminology of [1] and feel free to quote verbatim from [1] when appropriate, we take this opportunity to provide some additional background information. Specifically, the *Ultrapower Axiom UA*, introduced by Goldberg in [3, Definitions 2.1 – 2.3], says the following:

Suppose $V \models \text{ZFC}$ and $\mathcal{U}_0, \mathcal{U}_1 \in V$ are countably complete ultrafilters over $x_0 \in V, x_1 \in V$ respectively with $j_{\mathcal{U}_0} : V \rightarrow M_{\mathcal{U}_0}$ and $j_{\mathcal{U}_1} : V \rightarrow M_{\mathcal{U}_1}$ the associated elementary embeddings. Then there exist $\mathcal{W}_0 \in M_{\mathcal{U}_0}$ an $M_{\mathcal{U}_0}$ -countably complete ultrafilter over $y_0 \in M_{\mathcal{U}_0}$ and $\mathcal{W}_1 \in M_{\mathcal{U}_1}$ an $M_{\mathcal{U}_1}$ -countably complete ultrafilter over $y_1 \in M_{\mathcal{U}_1}$ such that:

1. For $j_{\mathcal{W}_0} : M_{\mathcal{U}_0} \rightarrow M_{\mathcal{W}_0}$ and $j_{\mathcal{W}_1} : M_{\mathcal{U}_1} \rightarrow M_{\mathcal{W}_1}$ the associated elementary embeddings, $M_{\mathcal{W}_0} = M_{\mathcal{W}_1} = M$.
2. $j_{\mathcal{W}_0} \circ j_{\mathcal{U}_0} = j_{\mathcal{W}_1} \circ j_{\mathcal{U}_1}$.

The following commutative diagram illustrates UA pictorially:

$$\begin{array}{ccc} V & \xrightarrow{j_{\mathcal{U}_0}} & M_{\mathcal{U}_0} \\ j_{\mathcal{U}_1} \downarrow & & \downarrow j_{\mathcal{W}_0} \\ M_{\mathcal{U}_1} & \xrightarrow{j_{\mathcal{W}_1}} & M \end{array}$$

The *Mitchell ordering on normal measures over a measurable cardinal κ* , introduced by Mitchell in [6], is defined by $\mathcal{U}_0 \triangleleft \mathcal{U}_1$ for normal measures $\mathcal{U}_0, \mathcal{U}_1$ over κ iff $\mathcal{U}_0 \in V^\kappa/\mathcal{U}_1$. By [4, Lemma 19.32], the Mitchell ordering is well-founded. As in [4, Definition 19.33], the *Mitchell order of the normal measure \mathcal{U}* , $o(\mathcal{U})$, is the rank of \mathcal{U} in \triangleleft , and the *Mitchell order of κ* , $o(\kappa)$, is the height of \triangleleft . Assuming GCH, the maximal value of $o(\kappa)$ is κ^{++} . Further information on the Mitchell ordering may be found in [4, pages 357 – 360].

2 The proof of Theorem 1.3

We turn now to the proof of Theorem 1.3.

Proof. Suppose $V \models \text{“ZFC} + \text{GCH} + \kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal above κ such that $o(\lambda) = \delta$ for some ordinal $\delta \leq \lambda^{++}$ ”. We begin with the following simple fact, which will be key to the proof of Theorem 1.3.

Proposition 2.1. Assume UA. Let $\gamma = |\delta|$. If λ is a measurable cardinal such that $o(\lambda) = \delta$, then the number of normal measures λ carries is γ .

Proof. As we have already stated in Section 1, because UA holds, the Mitchell ordering over any measurable cardinal must be linear (and in fact, must be a well-ordering, since the Mitchell ordering is well-founded). Let \mathbb{U} be the set of normal measures over λ . The function $f : \delta \rightarrow \mathbb{U}$ given by

$$f(\alpha) = \text{The unique normal measure over } \lambda \text{ of Mitchell order } \alpha$$

therefore is well-defined and is a bijection between δ and \mathbb{U} . Thus, the number of normal measures λ carries is γ . This completes the proof of Proposition 2.1. Q.E.D.

We will now follow the exposition found in [1]. In particular, we begin by describing the symmetric inner model N which will witness the conclusions of Theorem 1.3. What we are about to present is almost completely dependent on the discussion of the proof of [2, Theorem 1]. As in [1], since this material is quite complicated, we will not duplicate it here, but will refer readers to [2] for any missing details.

The forcing conditions \mathbb{P} to be used are $\text{SC}(\kappa, \lambda) \times \text{Coll}(\omega, < \kappa)$, where $\text{SC}(\kappa, \lambda)$ is supercompact Radin forcing as described in [2], and $\text{Coll}(\omega, < \kappa)$ is the usual Lévy collapse of κ to \aleph_1 . Let G be V -generic over \mathbb{P} . Take \mathcal{G} as the set of restrictions of G described in [2, page 595], which code collapses of cardinals in the open interval (ω, κ) to \aleph_0 and collapses of cardinals in the open interval (κ, λ) to κ . N is then given by $\text{HVD}^{V[G]}(\mathcal{G})$, the class of all sets hereditarily V -definable in $V[G]$ from an element of the set \mathcal{G} .

As in [1], standard arguments show that $N \models \text{ZF}$. By [2, Lemmas 1.1 - 1.5], the intervening remarks of [2], and [1, Proposition 3], $N \models$ “ $\text{DC} + \kappa = \aleph_1 + \lambda = \kappa^+ = \aleph_2 + \text{For any normal measure } \mathcal{U} \in V \text{ over } \lambda, \mathcal{U}' = \{x \subseteq \lambda \mid \exists y \subseteq x [y \in \mathcal{U}]\}$ is a normal measure over $\lambda + \aleph_1$ is measurable and carries exactly one normal measure (namely μ_ω)”.

The following facts were established in [1].

Lemma 2.2. (Lemma 5 of [1]) Suppose $\mathcal{U}^* \in N$ is a normal measure over λ . Then for some normal measure $\mathcal{U} \in V$ over λ , $\mathcal{U}^* = \{x \subseteq \lambda \mid \exists y \subseteq x [y \in \mathcal{U}]\}$.

Lemma 2.3. (Lemma 6 of [1]) In N , the regular cardinals greater than or equal to λ are the same as in V .

Since $V \models$ “ $\text{UA} + o(\lambda) = \delta$ ”, by Proposition 2.1, in V , the number of normal measures λ carries is $|\delta|$. By Lemma 2.2 and the remarks immediately preceding its statement, the normal measures over λ in N are in a 1-1 correspondence with the normal measures over λ in V which is definable in N . Because $V \subseteq N$, this means that in N , the number of normal measures $\lambda = \aleph_2$ carries is precisely $|\delta|^N$. By Lemma 2.3, the cardinal and cofinality structure in N is as stated in the hypotheses of Theorem 1.3. This completes the proof of Theorem 1.3. Q.E.D.

We note that in Theorems 1.1 – 1.3, there is only one normal measure over \aleph_1 . We therefore conclude by asking how many normal measures \aleph_1 can have when \aleph_1 and \aleph_2 are simultaneously measurable. As in [1], we conjecture that both \aleph_1 and \aleph_2 can carry an arbitrary number of normal measures under these circumstances.

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