Generalized bounds for hyperbolic sine and hyperbolic cosine functions*

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Abstract

The main objective of this paper is to establish several new lower and upper bounds for the functions $\sinh x/x$ and $\cosh x$. Following the simple approach, our results give refinements and generalizations of some known inequalities involving these functions.

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1 Introduction

Inequalities involving hyperbolic functions are as much important as inequalities involving trigonometric functions. Recently many researchers established hyperbolic inequalities(see e.g. [3], [4], [5], [6], [7], [8], [11], [12], [13], [14], [15], [16] and references therein). We start by giving a brief summary of already proved results pertaining to the main results of this paper.

The inequalities

$$1+\frac{x^2}{6}<\frac{\sinh x}{x}<1+\frac{x^2}{k_1};\ 0<x<1 \tag{1.1}$$

where $k_1 \approx 5.707724$ and

$$1+\frac{x^2}{2}<\cosh x<1+\frac{x^2}{k_2};\ 0<x<1 \tag{1.2}$$

where $k_2 \approx 1.841348$ are proved in [3] and [6] respectively. Recently, Christophe Chesneau and Yogesh J. Bagul [8] established the following results:

$$\left(1+\frac{x^2}{\pi^2}\right)^{\frac{\pi^2}{6}}<\frac{\sinh x}{x};\ x>0 \tag{1.3}$$

and

$$\left(1+\frac{4x^2}{\pi^2}\right)^{\frac{\pi^2}{8}}<\cosh x;\ x>0. \tag{1.4}$$

Before the establishment of above inequalities, Ling Zhu [16] in 2008, discovered some inequalities having similarity with these inequalities. Zhu's inequalities are stated as

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Statement 1. ([16]) Let $0 < x < r$. Then

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\alpha} < \frac{\sinh x}{x} < \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\beta} \tag{1.5}$$

holds if and only if $\alpha \leqslant 0$ and $\beta \geqslant \frac{r^2}{12}$.

Statement 2. ([16]) Let $0 < x < r$. Then

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\alpha} < \cosh x < \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\beta} \tag{1.6}$$

holds if and only if $\alpha \leqslant 0$ and $\beta \geqslant \frac{r^2}{4}$.

In this paper, we aim to refine and generalize all the inequalities listed above.

2 Preliminaries and lemmas

We are familiar with the following power series expansions [9, 1.411]:

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \text{ and} \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}. \tag{2.1}$$

The series expansions in (2.1) are useful for our main results. We also need the following two known results.

Lemma 2.1. ([2, p. 10]) Let $f, g : [a, b] \to \mathbb{R}$ be continuous. Moreover, let f, g be differentiable on (a, b) and $g'(x) \neq 0$, on (a, b). Let,

$$A_1(x) = \frac{f(x) - f(a)}{g(x) - g(a)}, A_2(x) = \frac{f(x) - f(b)}{g(x) - g(b)}, \ x \in (a, b).$$

(i) $A_1(\cdot)$ and $A_2(\cdot)$ are increasing(strictly increasing) on (a, b) if $f'(\cdot)/g'(\cdot)$ is increasing(strictly increasing) on (a, b).

(ii) $A_1(\cdot)$ and $A_2(\cdot)$ are decreasing(strictly decreasing) on (a, b) if $f'(\cdot)/g'(\cdot)$ is decreasing(strictly decreasing) on (a, b).

The Lemma 2.1 is known in the literature as l'Hôpital's rule of monotonicity. For Lemma 2.2 refer [1, 10].

Lemma 2.2. ([1, 10]) Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < R$, where a_n and b_n are real numbers for $n = 0, 1, 2, \cdots$ such that $b_n > 0$. If the sequence a_n/b_n is strictly increasing(or decreasing), then the function $A(x)/B(x)$ is also strictly increasing(or decreasing) on $(0, R)$.

Lemma 2.1 and Lemma 2.2 have been proved to be important tools in the field of inequalities.

3 Main results

The first result of the paper states

Theorem 3.1. Let $x \in (0, r)$ where $r \in (0, \infty)$. Then the function $F(x) = \frac{\log(\sinh x/x)}{\log(1+ax^2)}$ is strictly increasing on $(0, r)$ if $a \geqslant \frac{1}{15}$. In particular, with this fixed value of a, the best possible constants α and β such that

$$(1 + ax^2)^\alpha < \frac{\sinh x}{x} < (1 + ax^2)^\beta \tag{3.1}$$

are $\frac{1}{6a}$ and $\frac{\log(\sinh r/r)}{\log(1+ar^2)}$ respectively. i.e. the inequalities (3.1) hold if and only if $\alpha \leqslant \frac{1}{6a}$ and $\beta \geqslant \frac{\log(\sinh r/r)}{\log(1+ar^2)}$.

Proof. Consider

$$F(x) = \frac{\log\left(\frac{\sinh x}{x}\right)}{\log\left(1 + ax^2\right)} = \frac{F_1(x)}{F_2(x)}$$

where $F_1(x) = \log\left(\frac{\sinh x}{x}\right)$ and $F_2(x) = \log\left(1 + ax^2\right)$ with $F_1(0+) = 0 = F_2(0)$. After differentiating

$$\begin{aligned}\frac{F_1'(x)}{F_2'(x)} &= \frac{1}{2a}(1 + ax^2)\left(\frac{x\cosh x - \sinh x}{x^2 \sinh x}\right)\\ &= \frac{1}{2a}\left(\frac{\coth x}{x} - \frac{1}{x^2} + ax\coth x - a\right)\\ &= \frac{1}{2a}F_3(x).\end{aligned}$$

Now $F_3(x)$ is increasing if and only if $F_3'(x) > 0$. Consequently, by using Lemma 2.1 we can conclude that $F(x)$ will be increasing if $F_3'(x) > 0$. This means

$$\left[2 - \left(\frac{x}{\sinh x}\right)^2 - \frac{x}{\tanh x}\right] > ax^2\left[\left(\frac{x}{\sinh x}\right)^2 - \frac{x}{\tanh x}\right]$$

which is equivalent to

$$\frac{\left[2 - \left(\frac{x}{\sinh x}\right)^2 - x\coth x\right]}{x^2\left[\left(\frac{x}{\sinh x}\right)^2 - x\coth x\right]} < a,$$

due to well known relation $\left(\frac{x}{\sinh x}\right)^2 < 1 < \frac{x}{\tanh x}$. Thus $F_4(x) < a$, where

$$\begin{aligned}F_4(x) &= \frac{\left[2 - \left(\frac{x}{\sinh x}\right)^2 - x\coth x\right]}{x^2\left[\left(\frac{x}{\sinh x}\right)^2 - x\coth x\right]} = \frac{2\sinh^2 x - x^2 - x\sinh x\cosh x}{x^2(x^2 - x\sinh x\cosh x)}\\ &= \frac{2\cosh 2x - 2 - 2x^2 - x\sinh 2x}{2x^4 - x^3\sinh 2x}.\end{aligned}$$

Utilizing (2.1) we get

$$\begin{aligned}F_4(x) &= \frac{\sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n)!}x^{2n} - 2 - 2x^2 - \sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}x^{2n+2}}{2x^4 - \sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}x^{2n+4}} \\ &= \frac{\sum_{n=2}^{\infty}\frac{2^{2n+1}}{(2n)!}x^{2n} - \sum_{n=2}^{\infty}\frac{2^{2n-1}}{(2n-1)!}x^{2n}}{-\sum_{n=3}^{\infty}\frac{2^{2n-3}}{(2n-3)!}x^{2n}} \\ &= \frac{\sum_{n=3}^{\infty}\left[\frac{2^{2n-1}}{(2n-1)!} - \frac{2^{2n+1}}{(2n)!}\right]x^{2n}}{\sum_{n=3}^{\infty}\frac{2^{2n-3}}{(2n-3)!}x^{2n}} = \frac{\sum_{n=0}^{\infty}a_n x^{2n}}{\sum_{n=0}^{\infty}b_n x^{2n}} = \frac{A(x)}{B(x)}.\end{aligned}$$

Clearly $A(x)$ and $B(x)$ are convergent by ratio test and

$$\frac{a_n}{b_n} = \frac{4(n-2)}{n(2n-1)(2n-2)} = c_n(\text{say}).$$

We claim that $c_n > c_{n+1}$. For if $c_n \leqslant c_{n+1}$, then it implies $2(2n-2)(n+1)(2n+1) \leqslant (n-1)(2n-1)(2n-2)$, i.e. $4n^2 - 9n - 1 \leqslant 0$ or $n \leqslant \frac{9+\sqrt{97}}{8} < \frac{19}{8} < 3$, which contradicts to $n \geqslant 3$. Therefore, a sequence $\left\{\frac{a_n}{b_n}\right\}$ is strictly decreasing and $b_n > 0, \forall n \geqslant 3$. By Lemma 2.2, $F_4(x)$ is strictly decreasing on $(0, r)$. So, $\sup\{F_4(x) : x > 0\} \leqslant a$ and $\lim_{x\to 0+} F_4(x) = \frac{1}{15}$ gives $a \geqslant \frac{1}{15}$. Now $F(x)$ is increasing and we have

$$\lim_{x\to 0+} F(x) \leqslant F(x) \leqslant \lim_{x\to r-} F(x).$$

Lastly, $\lim_{x\to 0+} F(x) = \frac{1}{6a}$ by l'Hôpital's rule and $\lim_{x\to r-} F(x) = \frac{\log(\sinh r/r)}{\log(1+ar^2)}$ finish the proof. Q.E.D.

We prove our second result without making use of power series expansions in (2.1).

Theorem 3.2. Let $x \in (0, r)$ where $r \in (0, \infty)$. Then the function $G(x) = \frac{\log(\cosh x)}{\log(1+ax^2)}$ is strictly increasing on $(0, r)$ if $a \geqslant \frac{1}{3}$. In particular, the best possible constants γ and δ such that

$$\left(1 + ax^2\right)^{\gamma} < \cosh x < \left(1 + ax^2\right)^{\delta} \tag{3.2}$$

are $\frac{1}{2a}$ and $\frac{\log(\cosh r)}{\log(1+ar^2)}$ respectively. i.e. the inequalities (3.2) hold if and only if $\gamma \leqslant \frac{1}{2a}$ and $\delta \geqslant \frac{\log(\cosh r)}{\log(1+ar^2)}$.

Proof. Consider

$$G(x) = \frac{\log(\cosh x)}{\log(1 + ax^2)} = \frac{G_1(x)}{G_2(x)}$$

where $G_1(x) = \log(\cosh x)$ and $G_2(x) = \log(1 + ax^2)$ with $G_1(0) = G_2(0) = 0$. Differentiating

$$\frac{G_1'(x)}{G_2'(x)} = \frac{1}{2a}\frac{(1 + ax^2)\sinh x}{x\cosh x} = \frac{1}{2a}G_3(x)$$

where $G_3(x) = \frac{(1+ax^2)\sinh x}{x\cosh x}$. Obviously, $\frac{G_1'(x)}{G_2'(x)}$ is increasing if and only if $G_3'(x) > 0$. By Lemma 2.1, $G(x)$ will be increasing if $G_3'(x) > 0$. That means

$$x\cosh^2 x - \sinh x\cosh x - x\sinh^2 x > ax^2\left(-\sinh x\cosh x + x\sinh^2 x - x\cosh^2 x\right).$$

Or

$$\frac{\left(x\cosh^2 x - \sinh x\cosh x - x\sinh^2 x\right)}{x^2\left(-\sinh x\cosh x + x\sinh^2 x - x\cosh^2 x\right)} = G_4(x) < a$$

due to the fact that $\tanh x < \coth x$. $G_4(x)$ can be written as

$$G_4(x) = \frac{2x - \sinh 2x}{-x^2(2x + \sinh 2x)} = \frac{G_5(x)}{G_6(x)}$$

where $G_5(x) = 2x - \sinh 2x$ and $G_6(x) = -2x^3 - x^2\sinh 2x$ satisfying $G_5(0) = G_6(0) = 0$. Differentiation gives

$$\frac{G_5'(x)}{G_6'(x)} = \frac{2 - 2\cosh 2x}{-6x^2 - 2x^2\cosh 2x - 2x\sinh 2x} = \frac{G_7(x)}{G_8(x)}$$

with $G_7(0) = 0 = G_8(0)$. Continuing the argument

$$\frac{G_7'(x)}{G_8'(x)} = \frac{2\sinh 2x}{6x + 2x^2\sinh 2x + 4x\cosh 2x + \sinh 2x} = \frac{G_9(x)}{G_{10}(x)}.$$

Further

$$\begin{aligned}\frac{G_9'(x)}{G_{10}'(x)} &= \frac{4\cosh 2x}{6 + 12x\sinh 2x + 4x^2\cosh 2x + 6\cosh 2x}\\ &= \frac{4}{6\,\mathrm{sech}\,2x + 12x\tanh 2x + 4x^2 + 6} = \frac{2}{G_{11}(x)}\end{aligned}$$

Now $G_{11}'(x) = 6\tanh 2x(1 - \mathrm{sech}\,2x) + 12x\,\mathrm{sech}^2\,x + 4x > 0$ implies $G_{11}(x)$ is strictly increasing on $(0, r)$. By Lemma 2.1, $G_4(x)$ is also decreasing on $(0, r)$. Therefore, $\sup\{G_4(x) : x \in (0, r)\} \leqslant a$ and $\lim_{x\to 0+} G_4(x) = \frac{1}{3}$ gives $a \geqslant \frac{1}{3}$. Now $G(x)$ being increasing in $(0, r)$ for specified values of a, we have that

$$\lim_{x\to 0+} G(x) < G(x) < \lim_{x\to r-} G(x).$$

Finally, $\lim_{x\to 0+} G(x) = \frac{1}{2a}$ and $\lim_{x\to r-} G(x) = \frac{\log(\cosh r)}{\log(1+ar^2)}$ prove the desired result. Q.E.D.

4 Applications

Double inequalities in (1.1) is a particular case of Theorem 3.1 where $a = 1/6$ and $r = 1$. On a similar line inequality (1.3) can be obtained by taking $a = 1/\pi^2$ in (3.1). To get the sharpest inequality of this kind we put $a = 1/15$ in Theorem 3.1. It is stated as follows:

$$\left(1 + \frac{x^2}{15}\right)^{\frac{5}{2}} < \frac{\sinh x}{x} < \left(1 + \frac{x^2}{15}\right)^{\beta_1};\, x \in (0, r)\,\text{where}\, r \in (0, \infty) \tag{4.1}$$

and $\beta_1 = \frac{\log(\sinh r/r)}{\log(1+r^2/15)}$. Two sided inequality in (4.1) is a refinement and (or) generalization of inequalities in (1.1), (1.3) and (1.5).

On the other hand, inequalities in(1.2) and (1.4) are particular cases of Theorem 3.2 where $a = 1/2(r = 1)$ and $a = 4/\pi^2$ respectively. The sharpest inequality of this kind is obtained by putting $a = 1/3$ in Theorem 3.2 as follows:

$$\left(1+\frac{x^2}{3}\right)^{\frac{3}{2}} < \cosh x < \left(1+\frac{x^2}{3}\right)^{\delta_1} ; x \in (0,r) \, \text{where} \, r \in (0,\infty) \tag{4.2}$$

and $\delta_1 = \frac{\log(\cosh x)}{\log(1+r^2/3)}$. Again two sided inequality in (4.2) is a refinement and (or) generalization of inequalities in (1.2), (1.4) and (1.6).

At the end we state and prove the following proposition:

Proposition 4.1. Let $x > 0$. Then

$$1+\frac{x^2}{6} < \left(\frac{1+\cosh x}{2}\right)^{\frac{2}{3}}. \tag{4.3}$$

Proof. Set

$$H(x) = 2\left(1+\frac{x^2}{6}\right)^{\frac{3}{2}} - \cosh x.$$

On differentiating

$$H'(x) = x\left[\left(1+\frac{x^2}{6}\right)^{\frac{1}{2}} - \frac{\sinh x}{x}\right] < 0,$$

as $\left(1+\frac{x^2}{6}\right)^{\frac{1}{2}} < 1+\frac{x^2}{6} < \frac{\sinh x}{x}$ by Theorem 3.1. Hence $H(x)$ is strictly decreasing for $x > 0$. So $\lim_{x\to 0} H(x) = 1 > H(x)$ gives the desired inequality. Q.E.D.

Remark 4.2. Combining inequality (4.3) with the inequality [14, Thm. 2.3]

$$\left(\frac{1+\cosh x}{2}\right)^{\frac{2}{3}} < \frac{\sinh x}{x}$$

we get

$$1+\frac{x^2}{6} < \left(\frac{1+\cosh x}{2}\right)^{\frac{2}{3}} < \frac{\sinh x}{x}.$$

5 Conclusion

We obtained several inequalities involving $\sinh x/x$ and $\cosh x$, thus refined some known inequalities in the literature with a complete new approach.

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