Area, perimeter and cocyclical polygons

Jean-Christophe Léger

Lycée P.G. De Gennes, 11 rue Pirandello, Paris 75013, France
E-mail: jcleger75@gmail.com

Abstract

We give altenative proofs of results of G. Khimshiashvili and collaborators on polygonal linkages which are critical points for the area functional under various edge lengths constraints.

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1 Introduction

In his geometry treatise, [Leg41], Proposition VII, p.134, A.-M. Legendre proves the popular isoperimetric theorem for planar polygons :

Theorem 1.1. Let $N \geq 4$ be an integer and ℓ a positive real number. Among the N-gons with perimeter ℓ, the one with maximal area is the regular N-gon with perimeter ℓ.

This result is probably due to L'Huilier, one cand find it (p.108, §13) in [L'H89], and seems to date back from 1782, whose proof is given *verbatim* by Legendre.

This proof is based on a lesser-known result [1] :

Theorem 1.2. Let $N \geq 4$ be an integer and $\ell = (\ell_1, \ldots, \ell_N)$ be an N-uple of positive real numbers. Among the N-gons with successive edge lengths $\ell_1, \ldots, \ell_N$, an N-gon of maximal area is *cocyclical*, *i.e.* inscribed inside a circle.

The purpose of the present article is to present an alternative proof of a much more recent result[2] belonging to the same family and due to Khimshiashvili and Panina, [KP08], which one can state in an informal manner :

On the space of N-gons with fixed edge lengths the critical points of the oriented area are the cocyclical N-gons.

Before proving this statement, the least we can do is to give it a precise meaning within a definite setup which could be usesul for subsequent developments. This is carried out in §2. The proof itself will be developped in §3, §4.1 and §5.

The developped setup allows us to lead numerical experiments that we describe in §7 and we will also give variants of the main theorem concerning critical N-gons under various other edge lengths constraints in §6. We notice in this direction that in [KPS19], Khimshiashvili, Panina and Siersma give a full description of the critical points of the oriented area under the sole perimeter constraint, closing one of the main topic opened in §6.

[1] [L'H89], p.107, §11, [Leg41], Proposition VI, p.133, one may notice that this theorem disappears from subsequent editions of the treatise, [Leg49], whereas Legendre died in 1833

[2] The precise statement is Theorem 2.4

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2 Definitions, notations and statement

- For plane geometry matters, we identify once and for all the complex number field $\mathbb{C}$ with the oriented Euclidean plane in the usual way.
- For a complex number $z \in \mathbb{C}$, we denote $\mathrm{Re}(z)$, resp. $\mathrm{Im}(z)$, its real part, resp. imaginary part.
- We denote $\mathbb{U}$ the group of complex numbers of modulus 1, that is the unit circle in the plane. This group acts over $\mathbb{C}$ by multiplication, this action being the one of the rotation group SO_2.
- We denote by $\#S$ the cardinal of a finite set S.
- For linear algebra matters, we denote by $\mathrm{Ker}\,(L)$, resp. $\mathrm{Im}\,(L)$ the kernel, resp. the image of a linear map L. The dimension of a $\mathbb{C}$-vector space $\mathcal{A}$ will be denoted by $\dim_{\mathbb{C}} \mathcal{A}$, the dimension of a $\mathbb{R}$-vector space $\mathcal{B}$ by $\dim_{\mathbb{R}} \mathcal{B}$.
- We use the notation . to denote either the image of a vector by a linear map as in $L.v$ or to denote the composition of linear maps as in $L.M$. This notation is of course consistent with the matrix multiplication involved in the numerical computations of §7.3.

2.1 Indexation matters

Let $A_0, \ldots, A_{N-1}$ be N points in the plane $N \geq 2$. The oriented planar polygon $(A_0, \ldots, A_{N-1})$ is the sequence of these points, read in this order so that we may consider the *edges* of this polygon, $A_0 \to A_1, A_1 \to A_2, \ldots, A_{N-2} \to A_{N-1}$, and, least but not last, $A_{N-1} \to A_0$.

Indexing the points of an oriented planar polygon is cyclic by nature as well is the indexing of its edges.

For our theoretical computations we will adopt an abstract indexation in denoting (S, Σ) the cyclic oriented graph over N vertices, so that $\#S = N$ and Σ is the edge set of (S, Σ).

Whenever $n \in S$, $n-1$ is the vertex preceding n in S.

Let us denote, for any vertex n in S, $n + \mathbf{\frac{1}{2}}$ the edge in Σ following the vertex n and $n - \mathbf{\frac{1}{2}}$, the edge in Σ preceding n, that is

$$n + \mathbf{\tfrac{1}{2}} := (n, n+1) \text{ and } n - \mathbf{\tfrac{1}{2}} := (n-1, n).$$

Now for an edge $\nu \in \Sigma$, $\nu = (n, n+1) = n + \mathbf{\frac{1}{2}}$, let us denote[3] $\nu - \mathbf{\frac{1}{2}}$ the source vertex n of ν, and $\nu + \mathbf{\frac{1}{2}}$ the goal vertex $n+1$ of ν and $\nu + 1 = (n+1, n+2)$ the following edge. We have thus defined an action of $\mathbb{Z}/N.\mathbb{Z}$ on Σ and we have (this is a simple scripture game),

$$\forall n \in S,\ (n - \mathbf{\tfrac{1}{2}}) - \mathbf{\tfrac{1}{2}} = n - 1,\ (n - \mathbf{\tfrac{1}{2}}) + \mathbf{\tfrac{1}{2}} = n = (n + \mathbf{\tfrac{1}{2}}) - \mathbf{\tfrac{1}{2}},\ (n + \mathbf{\tfrac{1}{2}}) + \mathbf{\tfrac{1}{2}} = n + 1$$

and

$$\forall \nu \in \Sigma,\ (\nu - \mathbf{\tfrac{1}{2}}) - \mathbf{\tfrac{1}{2}} = \nu - 1,\ (\nu - \mathbf{\tfrac{1}{2}}) + \mathbf{\tfrac{1}{2}} = \nu = (\nu + \mathbf{\tfrac{1}{2}}) - \mathbf{\tfrac{1}{2}},\ (\nu + \mathbf{\tfrac{1}{2}}) + \mathbf{\tfrac{1}{2}} = \nu + 1.$$

If we compute in the real field $\mathbb{R}$ modulo N, that is in $\mathbb{R}/N.\mathbb{Z}$, we can identify S, resp. Σ, with $\mathbb{Z}/N.\mathbb{Z}$, resp. $\mathbb{Z}/N.\mathbb{Z} + \frac{1}{2} \subset \mathbb{R}/N.\mathbb{Z}$, which allows for the identification of $n + \frac{1}{2}$ with $n + \mathbf{\frac{1}{2}}$, of $\nu - \frac{1}{2}$ with $\nu - \mathbf{\frac{1}{2}}$, and so on.

From the point of view of pure notation, elements of S and of $\mathbb{C}^S$ will be denoted, as much as possible, by latin letters, and those belonging to Σ or $\mathbb{C}^\Sigma$ by greek letters.

[3] so that $\nu = (\nu - \frac{1}{2}, \nu + \frac{1}{2})$

Definition 2.1. An oriented planar S-gon is an S-uple, that is, a sequence of N complex numbers indexed by S, that is, an element of $\mathbb{C}^S$.

We will denote generically such a polygon by $z = (z_n)_{n\in S}$. The Σ-uple of the edges of this polygon is a sequence of N complex numbers indexed by the set of the graph edges Σ and we denote will often denote it by $(\zeta_\nu)_{\nu\in\Sigma}$ with $\zeta_\nu = z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}}$.

We denote by D the linear map which, to each S-gon associate the Σ-uple of its edges.

$$\begin{array}{cccc} D: & \mathbb{C}^S & \to & \mathbb{C}^\Sigma \\ & z=(z_n)_{n\in S} & \mapsto & \zeta = D.z = (z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}})_{\nu\in\Sigma} \end{array} \tag{2.1}$$

The map D is not one-to-one, but this can fixed by setting

$$\mathbb{C}_0^S = \{z\in\mathbb{C}^S,\ \sum_n z_n = 0\} \text{ and } \mathbb{C}_0^\Sigma = \{\zeta\in\mathbb{C}^\Sigma,\ \sum_{\nu\in\Sigma}\zeta_\nu = 0\}, \tag{2.2}$$

so that, by restriction, $D:\mathbb{C}_0^S \to \mathbb{C}_0^\Sigma$ becomes an isomorphism.

2.2 Manifolds of polygons

2.2.1 From Cartesian equations...

Let $\ell = (\ell_\nu)_{\nu\in\Sigma}$ be a Σ-uple of positive real numbers. We consider the set $\mathcal{P}(\ell)$ of S-gons whose edges have lengths ℓ, that is

$$\begin{aligned} \mathcal{P}(\ell) &= \{z\in\mathbb{C}^S,\ \forall\nu\in\Sigma,\ |z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}}| = \ell_\nu\} \\ &= \{z\in\mathbb{C}^S,\ \forall\nu\in\Sigma,\ |(D.z)_\nu| = \ell_\nu\} \end{aligned} \tag{2.3}$$

It is a well-known exercise[4] about the triangle inequality that $\mathcal{P}(\ell)$ is non-empty if and only if

$$\forall\nu\in\Sigma,\ \ell_\nu \le \frac{1}{2}\sum_{\nu'\in\Sigma}\ell_{\nu'}. \tag{2.4}$$

It should also be clear that if one of inequalities (2.4) is an equality then the index ν_0 for which this happens is unique and the polygons in $\mathcal{P}(\ell)$ are all degenerate, that is all the vertices of such a polygon belongs to the edge indexed by ν_0, of length ℓ_{ν_0}. These degenerate cases are excluded from our discussion and we may suppose from now on that

$$\forall\nu\in\Sigma,\ 0<\ell_\nu<\frac{1}{2}\sum_{\mu\in\Sigma}\ell_\mu. \tag{2.5}$$

A Σ-uple ℓ satisfying this condition may be called an *admissible vector of lengths.* We remark that we seem to exclude from the discussion the case where some edge length ℓ_{ν_0} is null. These cases are in fact covered by considering polygons with less vertices.

[4]Hint: consider ν_0 such that ℓ_{ν_0} is maximal. Consider then $p\in\{1,..,N-1\}$ minimal for the property $\sum_{q=0}^{p}\ell_{\nu_0+q} > \frac{1}{2}\sum_\nu \ell_\nu$, check that $p\le N-2$ and that it is possible to construct a triangle with edges of lengths $\sum_{q=0}^{p-1}\ell_{\nu_0+q}$, ℓ_{ν_0+p} and $\sum_{q=p+1}^{N-1}\ell_{\nu_0+q}$.

We are interested in polygons up to rotations. A first natural normalization, which preserves the natural symetries between indices, leads us to work with the set of *centered polygons*

$$\mathcal{P}_0(\ell) = \mathcal{P}(\ell) \cap \mathbb{C}_0^S = \{z \in \mathcal{P}(\ell),\, \sum_{n\in S} z_n = 0\} \tag{2.6}$$

As we intend to study the critical points of the oriented area of polygons over the space of prescribed edge lengths polygons, the best would be to deal with a ($\mathcal{C}^\infty$-)differential submanifold of a finite dimensional real vector space.

For the time being the sets $\mathcal{P}(\ell)$ et $\mathcal{P}_0(\ell)$ are naturally *algebraic subvarieties* of the finite dimensional real vector space $\mathbb{C}^S \simeq (\mathbb{R}^2)^S \simeq \mathbb{R}^{2N}$, that is the common zero-set of a finite family of real polynomials in $\dim_{\mathbb{R}} \mathbb{C}^S = 2N$ indeterminates.

Indeed, for each $\nu \in \Sigma$, $z \in \mathbb{C}^S$, let us set

$$L_\nu^2(z) = |z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}}|^2. \tag{2.7}$$

We can express L_ν^2 as a function of the real and imaginary parts of $z_{\nu+\frac{1}{2}}$ et $z_{\nu-\frac{1}{2}}$ to make it clear that L_ν^2 is a real quadratic polynomial on $\mathbb{C}^S \simeq (\mathbb{R}^2)^S$.

We get then that $\mathcal{P}(\ell)$ et $\mathcal{P}_0(\ell)$ are the common zero-sets of finite families of real polynomials :

$$\begin{aligned}
\mathcal{P}(\ell) &= \{z \in \mathbb{C}^S,\, \forall \nu \in \Sigma,\, L_\nu^2(z) = \ell_\nu^2\},\\
\mathcal{P}_0(\ell) &= \{z \in \mathbb{C}^S,\, \forall \nu \in \Sigma,\, L_\nu^2(z) = \ell_\nu^2\} \cap \mathbb{C}_0^S\\
&= \{z \in \mathbb{C}^S,\, \forall \nu \in \Sigma,\, L_\nu^2(z) = \ell_\nu^2\} \cap \{z \in \mathbb{C}^S,\, \mathrm{Re}(\sum_{n\in S} z_n) = 0\} \cap \{z \in \mathbb{C}^S,\, \mathrm{Im}(\sum_{n\in S} z_n) = 0\}.
\end{aligned}$$

As far as we are concerned with an extremal problem, the main advantage of $\mathcal{P}_0(\ell)$ over $\mathcal{P}(\ell)$ is its compactness[5]. Let us state it formally.

Proposition 2.2. Let ℓ be an admissible vector of lengths. The set $\mathcal{P}_0(\ell)$ is a non-empty compact subset of $\mathbb{C}^S$.

2.2.2 ...to submanifolds

The real algebraic subvariety $\mathcal{P}_0(\ell)$ of $\mathbb{C}^S$ admits at each point z a tangent vector space $\mathcal{T}_z$ in the *real algebraic sense* defined as the intersection of the kernels of the differentials at z of the defining polynomials. To be precise, as

$$\mathcal{P}_0(\ell) = \{z \in \mathbb{C}^S,\, \forall \nu \in \Sigma,\, L_\nu^2(z) = \ell_\nu^2\} \cap \mathbb{C}_0^S,$$

then the tangent space $\mathcal{T}_z$ to $\mathcal{P}_0(\ell)$ at $z \in \mathcal{P}_0(\ell)$ is defined by

$$\mathcal{T}_z = \{t = (t_n) \in \mathbb{C}^S,\, \forall \nu \in \Sigma,\, (d_z L_\nu^2).t = 0\} \cap \mathbb{C}_0^S. \tag{2.8}$$

We will give the explicit system of equations in (4.3) and solve it in (4.4) to get a parametric representation of $\mathcal{T}_z$.

[5] This is also an elementary exercise. An elementary geometric argument shows that every polygon of $\mathcal{P}_0(\ell)$, being centered at 0, is contained in the planar disk of center 0 and radius twice its perimeter. This shows that the closed set $\mathcal{P}_0(\ell)$ is contained in a closed ball in $\mathbb{C}^S$ centered at 0 of radius depending only on the common perimeter of polygons in $\mathcal{P}_0(\ell)$ and the number of vertices, hence is bounded in $\mathbb{C}^S$, hence compact.

Let us remark that we used the fact that as a linear subspace of $\mathbb{C}^S$, $\mathbb{C}^S_0$ is itself a subvariety of $\mathbb{C}^S$ with itself as its tangent space at every point

$$\begin{aligned}\mathbb{C}^S_0 &= \{z \in \mathbb{C}^S, \operatorname{Re}(\sum_{n\in S} z_n) = 0\} \cap \{z \in \mathbb{C}^S, \operatorname{Im}(\sum_{n\in S} z_n) = 0\} \\ &= \{t \in \mathbb{C}^S, \operatorname{Re}(\sum_{n\in S} t_n) = 0\} \cap \{t \in \mathbb{C}^S, \operatorname{Im}(\sum_{n\in S} t_n) = 0\}\end{aligned}$$

The real dimension of $\mathcal{T}_z$, being the intersection of $N+2$ kernels of linear forms over a space of real dimension $2N$ is greater than $N-2$. If this dimension is equal to $N-2$, that is if the linear forms are independent, we say that the point z is a *regular* point of $\mathcal{P}_0(\ell)$. If the dimension is strictly greater than $N-2$ we say that the point z is a *singular* point of $\mathcal{P}_0(\ell)$.

Proposition 4.1 identifies the singular points of $\mathcal{P}_0(\ell)$ as being the colinear polygons and gives a necessary and sufficient condition on ℓ for $\mathcal{P}_0(\ell)$ (and $\mathcal{P}(\ell)$) to be ($\mathcal{C}^\infty$-)differential submanifolds of $\mathbb{C}^S$, that is to be exempt of these singular points. This condition is fullfilled by a generic admissible ℓ. The computations made to prove this provide us with a parametric description (4.4) of the tangent space $\mathcal{T}_z$.

2.2.3 A quotient.

The *moduli space* $\mathcal{M}(\ell)$ associated with an admissible vector of lengths ℓ is the set $\mathcal{P}(\ell)$ modulo the natural action of direct isometries of the Euclidean plane $\mathbb{C}$. This moduli space $\mathcal{M}(\ell)$ is naturally identified with $\mathcal{P}_0(\ell)$ modulo the diagonal action of $\mathbb{U}$ on $\mathbb{C}^S$, that is the action of SO_2 on centered polygons.

The map D, by restriction, gives a one-to-one map from $\mathcal{P}_0(\ell)$ to

$$\{(\zeta_\nu)_{\nu\in\Sigma}, \forall\nu \in \Sigma, |\zeta_\nu| = \ell_\nu \text{ et } \sum_{\nu\in\Sigma} \zeta_\nu = 0\} \tag{2.9}$$

which is itself in a one-to-one natural normalization correspondence[6] with

$$\{(u_\nu)_{\nu\in\Sigma} \in \mathbb{U}^\Sigma, \sum_{\nu\in\Sigma} \ell_\nu.u_\nu = 0\} \tag{2.10}$$

The space $\mathcal{M}(\ell)$ is studied for example in [FS07] where it is defined as being

$$\{(u_\nu)_{\nu\in\Sigma} \in \mathbb{U}^\Sigma, \sum_{\nu\in\Sigma} \ell_\nu.u_\nu = 0\}/\mathbb{U} \tag{2.11}$$

Defining $\mathcal{M}(\ell)$ this way exhibits clearly the natural correspondence between the spaces $\mathcal{M}(\ell)$ and $\mathcal{M}(\sigma.\ell)$ where we denoted, for a permutation σ of Σ, $\sigma.\ell = (\ell_{\sigma(\nu)})_{\nu\in\Sigma}$.

For a generic admissible ℓ, the submanifold $\mathcal{P}(\ell)$ has (real) dimension $2N-N=N$, the compact submanifold $\mathcal{P}_0(\ell)$ has dimension $N-2$ and the compact manifold $\mathcal{M}(\ell)$ has dimension $N-3$.

To close this presentation, we remark that we voluntarily choose not to identify two polygons that are symetric with respect to some axis.

[6]This correspondence will not be used in this paper.

2.3 The oriented area.

If $\Delta_0 = (0, z_0, z_1)$ is an positively oriented plane triangle, that is if (z_0, z_1) is a direct base of $\mathbb{C} \simeq \mathbb{R}^2$, $z_0 = x_0 + iy_0$, $z_1 = x_1 + iy_1$, its area in the most naive sense can be computed with the determinant which gives the area of the parallelogram $(0, z_0, z_0 + z_1, z_1)$

$$A(\Delta_0) = \frac{1}{2}\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} = \frac{1}{2}(x_0.y_1 - x_1.y_0) = \frac{1}{2i}\mathrm{Im}(\overline{z_0}.z_1) = \frac{1}{4i}(\overline{z_0}.z_1 - z_0.\overline{z_1}).$$

Let $z = (z_k)_{k\in S}$ be a convex polygon, 0 being inside, positively oriented, that is, each triangle $\Delta_k = (0, z_k, z_{k+1})$ is positively oriented, its area is the sum of the areas of the triangles Δ_k, that is

$$A(z) = \sum_{k\in S} \frac{1}{4i}(\overline{z_k}.z_{k+1} - z_k.\overline{z_{k+1}}).$$

Using cyclicity of the indexing graph S, we have also

$$\begin{aligned} A(z) &= \sum_{k\in S} \frac{1}{4i}(\overline{z_k}.z_{k+1} - z_{k-1}.\overline{z_k}) \\ &= \frac{1}{4i}\sum_{k\in S} \overline{z_k}.(z_{k+1} - z_{k-1}) = \frac{1}{2}\sum_{k\in S} \mathrm{Im}((z_{k+1} - z_k)\overline{z}_k). \end{aligned}$$

Let us now remark that this quantity, $A(z)$, defined for any polygon z, is invariant by plane translations, direct plane rotations and that $A(z) = -A(\overline{z})$ where $\overline{z}$ is orthogonally symetric to z with respect to the real axis in $\mathbb{C}$. The absolute value of $A(z)$ is the naive area of z when the polygon z is *simple* and thus admits a triangulation.

We thus define the function *algebraic area* $A : \mathbb{C}^S \to \mathbb{R}$: to each polygon $z = (z_n) \in \mathbb{C}^S$, it associates its *algebraic area*

$$A(z) = \frac{1}{4i}\sum_{k\in S}(z_{k+1} - z_{k-1})\overline{z}_k = \frac{1}{2}\sum_{k\in S} \mathrm{Im}((z_{k+1} - z_k)\overline{z}_k). \tag{2.12}$$

As

$$\forall w \in \mathbb{C},\, \forall\theta \in \mathbb{R},\, A(e^{i\theta}.z + w) = A(z), \tag{2.13}$$

that is, the function A is invariant under the action of translations and the diagonal action of $\mathbb{U}$ on $\mathbb{C}^S$, the function A defines a function on each moduli space $\mathcal{M}(\ell)$.

The function A is clearly a real quadratic polynomial on $\mathbb{C}^S \simeq (\mathbb{R}^2)^S$. Let us compute its differential at $z \in \mathbb{C}^S$ by using direct differentiation rules with respect to the variables z_k, $\overline{z}_k$. Acccording to these rules, we get for example, for each $k \in S$,

$$\begin{aligned} d(z_{k+1}.\overline{z}_k) &= \frac{\partial(z_{k+1}.\overline{z}_k)}{\partial z_{k+1}}.dz_{k+1} + \frac{(\partial z_{k+1}.\overline{z}_k)}{\partial \overline{z}_k}.d\overline{z}_k = \overline{z}_k.dz_{k+1} + z_{k+1}.d\overline{z}_k \\ d(z_{k-1}.\overline{z}_k) &= \overline{z}_k.dz_{k-1} + z_{k-1}.d\overline{z}_k \end{aligned}$$

so that, by summing and shifting the indices in the two last sums

$$\begin{aligned} d_zA &= \frac{1}{4i}\sum_{k\in S}(z_{k+1}-z_{k-1}).d\overline{z}_k + \frac{1}{4i}\sum_{k\in S}\overline{z}_k.dz_{k+1} - \frac{1}{4i}\sum_{k\in S}\overline{z}_k.dz_{k-1} \\ &= \frac{1}{4i}\sum_{k\in S}(z_{k+1}-z_{k-1}).d\overline{z}_k + \frac{1}{4i}\sum_{k\in S}(\overline{z}_{k-1}-\overline{z}_{k+1}).dz_k \\ &= \frac{1}{2}\sum_{k\in S}\mathrm{Im}((z_{k+1}-z_{k-1}).d\overline{z}_k). \end{aligned} \tag{2.14}$$

The differential of A at $z\in\mathbb{C}^S$ is thus the $\mathbb{R}$-linear form on the $\mathbb{R}$-vector space $\mathbb{C}^S$ defined by

$$\forall t\in\mathbb{C}^S,\ d_zA.t = \frac{1}{2}\sum_{k\in S}\mathrm{Im}((z_{k+1}-z_{k-1}).\bar{t}_k). \tag{2.15}$$

Let ℓ, an admissible vector of lengths be given, the function A, being continuous, reaches its minimum and maximum values on the compact space $\mathcal{P}_0(\ell)$. Whenever two polygons z et z' are (orthogonally) symetric with respect to some axis, we have $A(z)=-A(z')$, thus the maximum and minimum values of A over $\mathcal{P}_0(\ell)$ are opposite. FERMAT's principle, gives that, at each extremal point z, regular in $\mathcal{P}_0(\ell)$, d_zA is null over $\mathcal{T}_z$. Let us define

Definition 2.3. Let $z\in\mathbb{C}^S$. We say that z *critical* for A over $\mathcal{P}_0(\ell)$ if

1. the point z is a regular point of $\mathcal{P}_0(\ell)$
2. and d_zA est null over $\mathcal{T}_z$.

and state the main result as

Theorem 2.4. An S-gon is critical for A over $\mathcal{P}_0(\ell)$ if and only if it is cocyclical, non colinear.

3 Some linear and bilinear algebra

We must emphasize an aspect of our computations which, if overlooked, may be misleading.

We will make computations in real plane geometry using the facilities given by computations over the complex field $\mathbb{C}$. We introduced in particular the sets $\mathbb{C}^S$ and $\mathbb{C}^\Sigma$ which are naturally vector spaces over $\mathbb{C}$.

Our problem is nevertheless a real problem. The set of centered polygons $\mathcal{P}_0(\ell)$ is a *real* algebraic subvariety of $\mathbb{C}^S$, its tangent space $\mathcal{T}_z$ at any point is a subspace of $\mathbb{C}^S$ as a real vector space. The functions L^2_ν and the function A although defined by complex formulae are real quadratic polynomials.

While establishing equation (4.4), we will see that we parameterize (non-injectively) $\mathcal{T}_z$, the tangent vector space of $\mathcal{P}_0(\ell)$ at z, by $\mathcal{S}_z$, a $\mathbb{C}$-vector subspace of $\mathbb{C}^S$, by an $\mathbb{R}$-linear map. The computation of the real dimension of $\mathcal{T}_z$, which is our first main goal will reduce to the computation of the complex dimension of $\mathcal{S}_z$.

3.1 Scalar and Hermitian products

We equip $\mathbb{C}^S$ et $\mathbb{C}^\Sigma$ with their usual Hermitian structures, that is, whenever $z, w \in \mathbb{C}^S$, $\zeta, \omega \in \mathbb{C}^\Sigma$, we set the Hermitian products

$$\langle z \mid w \rangle := \sum_{n \in S} z_n \overline{w}_n, \; \langle \zeta \mid \omega \rangle := \sum_{\nu \in \Sigma} \zeta_\nu \overline{\omega}_\nu. \tag{3.1}$$

These Hermitian structures give rise, in the usual way, to Euclidean structures on the $\mathbb{R}$-vector spaces $\mathbb{C}^S$ and $\mathbb{C}^\Sigma$ that is, whenever $z, w \in \mathbb{C}^S$, $\zeta, \omega \in \mathbb{C}^\Sigma$, we set the scalar products

$$\langle\langle z \mid w \rangle\rangle := \operatorname{Re} \langle z \mid w \rangle = \operatorname{Re} \left(\sum_{n \in S} z_n \overline{w}_n \right), \; \langle\langle \zeta \mid \omega \rangle\rangle := \operatorname{Re} \langle \zeta \mid \omega \rangle = \operatorname{Re} \left(\sum_{\nu \in \Sigma} \zeta_\nu \overline{\omega}_\nu \right). \tag{3.2}$$

Let $\mathcal{H}$ be a $\mathbb{C}$-vector space endowed with an Hermitian product $\langle . \mid . \rangle$. For each $u \in \mathcal{H}$, we denote $\|u\|$ the Euclidean norm of u that is, the nonnegative real number $\|u\|$ satisfying

$$\|u\|^2 = \langle\langle u \mid u \rangle\rangle = \langle u \mid u \rangle . \tag{3.3}$$

If each of $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional, Euclidean (resp. Hermitian) vector spaces (with scalar (resp. Hermitian) products denoted by the same symbol $\langle\langle . \mid . \rangle\rangle$ (resp. $\langle . \mid . \rangle$)), we define adjoints of linear maps by

1. if $F : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear, its adjoint $F^* : \mathcal{B} \to \mathcal{A}$ is defined by :
$$\forall a \in \mathcal{A}, b \in \mathcal{B}, \; \langle\langle a \mid F^*.b \rangle\rangle = \langle\langle F.a \mid b \rangle\rangle ; \tag{3.4}$$
2. if $F : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, its adjoint $F^* : \mathcal{B} \to \mathcal{A}$ is defined by :
$$\forall a \in \mathcal{A}, b \in \mathcal{B}, \; \langle a \mid F^*.b \rangle = \langle F.a \mid b \rangle ; \tag{3.5}$$
3. and we notice that in the Hermitian case, a $\mathbb{C}$-linear map is also $\mathbb{R}$-linear and the two preceding definitions of the adjoint coincide.

Let us recall the usual projection formulae on the orthogonal of a line or a plane in an Hermitian space

Lemma 3.1. Let $\mathcal{H}$ be a $\mathbb{C}$-vector space endowed with an Hermitian product $\langle . \mid . \rangle$,

1. Let $u \in \mathcal{H}$ be a non-null vector and consider the vector subspace $u^\perp := E_u = \{v \in \mathcal{H}, \langle v \mid u \rangle = 0\}$ as well as π_u, the orthogonal projection in the Hermitian and Euclidean sense on E_u. We have for each $z \in \mathcal{H}$,
$$\pi_u.z = z - \langle z \mid u \rangle \frac{u}{\|u\|^2}. \tag{3.6}$$
2. Let $u, v \in \mathcal{H}$ be two non-null vectors such that $\langle u \mid v \rangle = 0$ and $\pi_{u,v}$, the orthogonal projection on $E_u \cap E_v$. We have for each $z \in \mathcal{H}$,
$$\pi_{u,v}.z = z - \langle z \mid u \rangle \frac{u}{\|u\|^2} - \langle z \mid v \rangle \frac{v}{\|v\|^2}. \tag{3.7}$$

3.2 Some linear maps

We take, as a convention, the freedom to use by the same letter for operators or objects acting or belonging to different spaces provided they have similar definitions in each of their frameworks. A simple examination of the different spaces at stake (give types to objects) in an expression involving such identifiers removes any ambiguities.

In this spirit, let us denote by $\mathbb{1}$ the vectors $(1)_{n\in S}\in\mathbb{C}^S$ and $\mathbb{1}=(1)_{\nu\in E}\in\mathbb{C}^\Sigma$, we have then

$$\mathbb{C}_0^S=\mathbb{1}^\perp=\{z\in\mathbb{C}^S,\ \sum_{n\in S}z_n=0\},\ \mathbb{C}_0^\Sigma=\mathbb{1}^\perp=\{\zeta\in\mathbb{C}^\Sigma,\ \sum_{\nu\in\Sigma}\zeta_\nu=0\}. \tag{3.8}$$

If $\pi_0:\mathbb{C}^S\to\mathbb{C}^S$, resp. $\pi_0:\mathbb{C}^\Sigma\to\mathbb{C}^\Sigma$, is the orthogonal projection in $\mathbb{C}^S$ on $\mathbb{C}_0^S$, resp. in $\mathbb{C}^\Sigma$ on $\mathbb{C}_0^\Sigma$, we have, for each $z\in\mathbb{C}^S$, resp. each $\zeta\in\mathbb{C}^\Sigma$,

$$\pi_0.z=z-(\frac{1}{N}\sum_{n\in S}z_n)\mathbb{1},\ \text{resp. } \pi_0.\zeta=\zeta-(\frac{1}{N}\sum_{\nu\in\Sigma}\zeta_\nu)\mathbb{1} \tag{3.9}$$

where $N=\#S=\#\Sigma$. Keeping the same convention,

1. For $a\in\mathbb{C}^S$, resp. $\alpha\in\mathbb{C}^\Sigma$, we define $[\![a]\!]$, resp. $[\![\alpha]\!]$, the diagonal operator whose diagonal is a, resp. α, that is, for each $z\in\mathbb{C}^S$, resp. $\zeta\in\mathbb{C}^\Sigma$, $[\![a]\!].z\in\mathbb{C}^S$, resp. $[\![\alpha]\!].\zeta\in\mathbb{C}^\Sigma$ with

$$\forall n\in S,\ ([\![a]\!].z)_n=a_n.z_n \text{ and } \forall\nu\in\Sigma,\ ([\![\alpha]\!].\zeta)_\nu=\alpha_\nu.\zeta_\nu. \tag{3.10}$$

We have $[\![\alpha]\!]^*=[\![\overline{\alpha}]\!]$ and $[\![a]\!]^*=[\![\overline{a}]\!]$. The identity operator is thus $[\![\mathbb{1}]\!]$ and the operator of multiplication by a complex number λ is $[\![\lambda.\mathbb{1}]\!]=\lambda.[\![\mathbb{1}]\!]$.

2. We define $C:\mathbb{C}^S\to\mathbb{C}^S$ and $C:\mathbb{C}^\Sigma\to\mathbb{C}^\Sigma$ the $\mathbb{R}$-linear maps of conjugacy defined by

$$C.(z_n)_{n\in S}=(\overline{z_n})_{n\in S} \text{ and } C.(\zeta_\nu)_{\nu\in\Sigma}=(\overline{\zeta}_\nu)_{\nu\in\Sigma}. \tag{3.11}$$

We have $C^*=C$.

3.3 The derivation operator D, the midpoint operator M

We define $D:\mathbb{C}^S\to\mathbb{C}^\Sigma$, $D:\mathbb{C}^\Sigma\to\mathbb{C}^S$, $M:\mathbb{C}^S\to\mathbb{C}^\Sigma$, $M:\mathbb{C}^\Sigma\to\mathbb{C}^S$ by the formulae, for each $z\in\mathbb{C}^S$, $\zeta\in\mathbb{C}^\Sigma$

$$\begin{aligned}\forall\nu\in\Sigma,\ (D.z)_\nu &:= z_{\nu+\frac{1}{2}}-z_{\nu-\frac{1}{2}},\ (M.z)_\nu:=\frac{1}{2}(z_{\nu+\frac{1}{2}}+z_{\nu-\frac{1}{2}}), \\ \forall n\in S,\ (D.\zeta)_n &:= \zeta_{n+\frac{1}{2}}-\zeta_{n-\frac{1}{2}},\ (M.\zeta)_n:=\frac{1}{2}(\zeta_{n+\frac{1}{2}}+\zeta_{n-\frac{1}{2}})\end{aligned} \tag{3.12}$$

We record easy facts on these operators, short proofs are either obvious or easy computations.

Lemma 3.2. 1. $D^*=-D$, $M^*=M$,

2. $D.\pi_0=\pi_0.D=D$,

3. $M.\pi_0=\pi_0.M$,

4. $M.D = D.M$ and

$$\forall n \in S,\ (M.D.z)_n = \frac{1}{2}(z_{n+1} - z_{n-1}),\ \forall \nu \in \Sigma,\ (M.D.\zeta)_\nu = \frac{1}{2}(\zeta_{\nu+1} - \zeta_{\nu-1}), \tag{3.13}$$

5. (Leibniz formula) for each $\alpha \in \mathbb{C}^\Sigma$ or $\alpha \in \mathbb{C}^S$,

$$D. [\![\alpha]\!] = [\![M.\alpha]\!] .D + [\![D.\alpha]\!] .M. \tag{3.14}$$

As for a complex number w, $\mathrm{Im}(w) = -\mathrm{Re}(i.w)$, reformulating (2.12) and (2.15) shows that the area $A(z)$ of an S-gon z and its differential can be expressed as

$$A(z) = \frac{i}{2} \langle i.M.D.z \mid z \rangle \tag{3.15}$$

and

$$\forall t \in \mathbb{C}^S,\ d_z A.t = - \langle\!\langle i.M.D.z \mid t \rangle\!\rangle . \tag{3.16}$$

3.4 The integration operators I and K, integration by parts

The map D is clearly an isomorphism from $\mathbb{C}_0^S$ to $\mathbb{C}_0^\Sigma$. Let us call its inverse I and extend it at once linearly on $\mathbb{C}^\Sigma$ by setting $I.\mathbb{1} = 0$.

We may notice the following commutations : $D.M = M.D$, $M.I = I.M$. We also set

$$K = M.I : \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma.$$

As is now customary, we define $I : \mathbb{C}^S \to \mathbb{C}^\Sigma$ and $K : \mathbb{C}^S \to \mathbb{C}^S$ in an analogous way. We will give explicit formulae for I and K in §7.

We record the following facts, whose proofs are obvious given lemma 3.2. Integration by parts (3.18) is obtained in dualizing (3.17) which comes directly from the Leibniz formula (3.14).

Lemma 3.3. 1. $I^* = -I$, $K^* = -K$,

2. $D.I = I.D = \pi_0$,

3. for each $\alpha \in \mathbb{C}^\Sigma$ or $\alpha \in \mathbb{C}^S$,

$$D. [\![\alpha]\!] .I = [\![M.\alpha]\!] .\pi_0 + [\![D.\alpha]\!] K, \tag{3.17}$$

4. (Integration by parts) for each $\alpha \in \mathbb{C}^\Sigma$ or $\alpha \in \mathbb{C}^S$,

$$I. [\![\alpha]\!] .D = \pi_0. [\![M.\alpha]\!] - K. [\![D.\alpha]\!] . \tag{3.18}$$

3.5 Solving linear systems.

We present an easy way to solve linear systems of a particular type. These systems are a conjunction of linear equations whose unknowns are complex numbers and their conjugates. As a first (useful) example, typical from the geometrical problems we are interested in, let us get the formula giving the complex coordinate of the center of the circumcircle of three non aligned points in the plane as a function of their complex coordinates.

Proposition 3.4. Let $u, v, w \in \mathbb{C}$ be three non aligned points. Let us set

$$c = c(u,v,w) = \frac{1}{w-u}\left(\frac{w-v}{\overline{w-v}} - \frac{u-v}{\overline{u-v}}\right).$$

The center of the circumcircle of u, v, w is

$$o = v + \frac{c}{|c|^2}.$$

Proof. The center o of the circumcircle is the intersection of the mediatrices of the segments $[u, v]$ and $[w, v]$. The number $t = o - v$ is solution of the system

$$\left\{\begin{array}{lcl} (t - \frac{u-v}{2}).\overline{(u-v)} + \overline{(t - \frac{u-v}{2})}.(u-v) & = & 0 \\ (t - \frac{w-v}{2}).\overline{(w-v)} + \overline{(t - \frac{w-v}{2})}.(w-v) & = & 0 \end{array}\right. .$$

The first equation expresses the orthogonality between $o - \frac{u+v}{2}$ and $u - v$, the second expresses orthogonality between $o - \frac{w+v}{2}$ and $w - v$.

With little simplification, setting $s = \bar{t}$, we are left to solve a linear system of unknown (t, s),

$$\left\{\begin{array}{lcl} t.\overline{(u-v)} + s.(u-v) & = & |u-v|^2 \\ t.\overline{(w-v)} + s.(w-v) & = & |w-v|^2 \end{array}\right. .$$

This 2×2 system has determinant $\Delta = \overline{(u-v)}.(w-v) - \overline{(w-v)}(u-v)$, which is not null as u, v, w are not aligned. Thus, Cramer's formula gives

$$t = \frac{\overline{(u-v)}.(u-v).(w-v) - \overline{(w-v)}.(w-v)(u-v)}{\Delta} = \frac{(u-v).(w-v).\overline{(u-w)}}{\overline{(u-v)}.(w-v) - \overline{(w-v)}(u-v)} = \frac{1}{\bar{c}}$$

and

$$s = \bar{t} = \frac{1}{c}.$$

Q.E.D.

We elaborate on this by allowing many variables. The following lemma is an elementary linear algebra exercise.

Lemma 3.5 (Solving particular linear systems)**.** Let n, p be positive integers and A a matrix with complex coefficients[7], p lines and n columns.

Let us define the following sets

$$\begin{array}{lcl} \mathcal{T} & = & \{T \in \mathbb{C}^n,\ A.T + \overline{A}.\overline{T} = 0\}, \\ \mathcal{T}_0 & = & \{(T,S) \in \mathbb{C}^n \times \mathbb{C}^n,\ A.T + \overline{A}.S = 0\}, \\ \mathcal{T}_R & = & \{(T,S) \in \mathbb{C}^n \times \mathbb{C}^n,\ A.T + \overline{A}.S = 0, T = \overline{S}\}, \\ \mathcal{T}_I & = & \{(T,S) \in \mathbb{C}^n \times \mathbb{C}^n,\ A.T + \overline{A}.S = 0, T = -\overline{S}\}, \end{array}$$

then

[7] For a matrix or a numerical vector A with complex coefficients, we denote $\overline{A}$ the matrix or vector whose entries are the complex conjugates of the entries of A.

1. $\mathcal{T}_0$ is a $\mathbb{C}$-subspace of $\mathbb{C}^n \times \mathbb{C}^n$, $\mathcal{T}_R$ and $\mathcal{T}_I$ are $\mathbb{R}$-subspaces of $\mathbb{C}^n \times \mathbb{C}^n$, $\mathcal{T}$ is a $\mathbb{R}$-subspace of $\mathbb{C}^n$,
2. $\mathcal{T}_0 = \mathcal{T}_R \oplus \mathcal{T}_I$, in the sense of $\mathbb{R}$-subspaces of $\mathbb{C}^n \times \mathbb{C}^n$, with associated projections[8]
$$\pi_{R//I}(T,S) = \frac{1}{2}(T + \overline{S}, \overline{T} + S),\ \pi_{I//R}(T,S) = \frac{1}{2}(T - \overline{S}, -\overline{T} + S).$$
3. The map $\mathcal{T}_R \to \mathcal{T}_I$, $(T,S) \mapsto (i.T, i.S)$ is an $\mathbb{R}$-isomorphism. The map $\mathcal{T}_R \to \mathcal{T}$, $(T,S) \mapsto T$ is an $\mathbb{R}$-isomorphism.
4. In particular, $\dim_{\mathbb{R}} \mathcal{T} = \dim_{\mathbb{R}} \mathcal{T}_R = \frac{1}{2}\dim_{\mathbb{R}} \mathcal{T}_0 = \dim_{\mathbb{C}} \mathcal{T}_0$.
5. If J and Q are matrices such that
$$\mathcal{T}_0 = \{(T,S) \in \mathbb{C}^n \times \mathbb{C}^n, T = J.S, Q.S = 0\}$$
then
$$\mathcal{T} = \{J.S + \overline{S},\ S \in \mathbb{C}^n, Q.S = 0\} = (J + C)(\mathrm{Ker}\,(Q))$$
and $\dim_{\mathbb{R}} \mathcal{T} = \dim_{\mathbb{C}} \mathrm{Ker}\,(Q)$.

Proof. 1. It is clear that $\mathcal{T}_0$ is a $\mathbb{C}$-vector subspace of $\mathbb{C}^n \times \mathbb{C}^n$, it is the kernel of the $\mathbb{C}$-linear map
$$\begin{array}{ccc} \mathbb{C}^n \times \mathbb{C}^n & \to & \mathbb{C}^p \\ (T,S) & \mapsto & A.T + \overline{A}.S \end{array}.$$
The maps
$$\begin{array}{ccc} \mathbb{C}^n \times \mathbb{C}^n & \to & \mathbb{C}^n \\ (T,S) & \mapsto & T - \overline{S} \end{array} \text{ and } \begin{array}{ccc} \mathbb{C}^n \times \mathbb{C}^n & \to & \mathbb{C}^n \\ (T,S) & \mapsto & T + \overline{S} \end{array}$$
are $\mathbb{R}$-linear (notice that conjugacy destroys $\mathbb{C}$-linearity). The sets $\mathcal{T}_R$, resp. $\mathcal{T}_I$, are respectively the intersections of the $\mathbb{R}$-subspace $\mathcal{T}_0$ and of the kernel of the first map, resp. the second one.

Eventually, $\mathcal{T}$ is a $\mathbb{R}$-vector subspace of $\mathbb{C}^n$ as it is the kernel of the $\mathbb{R}$-linear map
$$\begin{array}{ccc} \mathbb{C}^n & \to & \mathbb{C}^p \\ (T,S) & \mapsto & A.T + \overline{A}.\overline{T} \end{array}.$$

2. The sets $\mathcal{T}_R$ and $\mathcal{T}_I$ are $\mathbb{R}$-vector subspaces of $\mathcal{T}_0$.

Let us define the $\mathbb{R}$-linear maps $\pi_{R//I}, \pi_{I//R} : \mathcal{T}_0 \to \mathbb{C}^n \times \mathbb{C}^n$ by
$$\forall (T,S) \in \mathcal{T}_0 \subset \mathbb{C}^n \times \mathbb{C}^n,\ \pi_{R//I}(T,S) = \frac{1}{2}(T + \overline{S}, \overline{T} + S),\ \pi_{I//R}(T,S) = \frac{1}{2}(T - \overline{S}, -\overline{T} + S).$$
By adding and substracting the equations $A.T + \overline{A}.S = 0$ and $\overline{A}.\overline{T} + A.\overline{S} = 0$, we get
$$\forall (T,S) \in \mathcal{T}_0,\ \pi_{R//I}(T,S) \in \mathcal{T}_R \text{ and } \pi_{I//R}(T,S) \in \mathcal{T}_I.$$

[8] We denote $\pi_{R//I}$ the projection over $\mathcal{T}_R$ with axis $\mathcal{T}_I$ and $\pi_{R//I}$ the projection on $\mathcal{T}_I$ with axis $\mathcal{T}_R$.

Moreover we clearly know that $\mathcal{T}_R \cap \mathcal{T}_I = \{0\}$ and

$$\forall (T,S) \in \mathcal{T}_0,\ (T,S) = \pi_{R//I}(T,S) + \pi_{I//R}(T,S)$$

so that $\mathcal{T}_0 = \mathcal{T}_R \oplus \mathcal{T}_I$.

3. The maps $\mathcal{T}_R \to \mathcal{T}_I$, $(T,S) \mapsto (i.T, i.S)$ and $\mathcal{T}_R \to \mathcal{T}$, $(T,S) \mapsto T$ are well defined, $\mathbb{R}$-linear and one-to-one : they are thus a isomorphisms of $\mathbb{R}$-vector spaces.

4. The $\mathbb{R}$-vector spaces $\mathcal{T}$, $\mathcal{T}_R$ and $\mathcal{T}_I$ have thus the same (real) dimension and, as $\mathcal{T}_0 = \mathcal{T}_R \oplus \mathcal{T}_I$, we get that $\dim_{\mathbb{R}} \mathcal{T}_0 = \dim_{\mathbb{R}} \mathcal{T}_R + \dim_{\mathbb{R}} \mathcal{T}_I = 2 \dim_{\mathbb{R}} \mathcal{T}$. From the fact that $\mathcal{T}_0$ is also a $\mathbb{C}$-vector space, $\dim_{\mathbb{R}} \mathcal{T}_0 = 2 \dim_{\mathbb{C}} \mathcal{T}_0$, and we get the sought-for equalities.

5. If J and Q are matrices such that

$$\mathcal{T}_0 = \{(T,S) \in \mathbb{C}^n \times \mathbb{C}^n, T = J.S, Q.S = 0\}$$

then

$$\begin{array}{rcl} \mathcal{T}_R & = & \pi_{R//I}(\mathcal{T}_0) \\ & = & \{(T + \overline{S}, \overline{T} + S) \in \mathbb{C}^n \times \mathbb{C}^n,\ (T,S) \in \mathbb{C}^n \times \mathbb{C}^n,\ T = J.S, Q.S = 0\} \\ & = & \{(J.S + \overline{S}, \overline{J.S} + S) \in \mathbb{C}^n \times \mathbb{C}^n,\ S \in \mathbb{C}^n,\ Q.S = 0\} \end{array}$$

and, via the one-to-one correspondence between $\mathcal{T}$ and $\mathcal{T}_R$,

$$\mathcal{T} = \{J.S + \overline{S},\ \exists S \in \mathbb{C}^n, Q.S = 0\} = (J + C)(\mathrm{Ker}\,(Q)).$$

The previous dimension identities give that

$$\dim_{\mathbb{R}} \mathcal{T} = \dim_{\mathbb{R}} \mathcal{T}_R = \dim_{\mathbb{C}} \mathcal{T}_0.$$

The $\mathbb{C}$-linear map $\left\{ \begin{array}{ccc} \mathrm{Ker}\,(Q) & \to & \mathbb{C}^n \times \mathbb{C}^n \\ S & \mapsto & (J.S, S) \end{array} \right.$ is injective with image $\mathcal{T}_0$, thus $\dim_{\mathbb{C}} \mathcal{T}_0 = \dim_{\mathbb{C}} \mathrm{Ker}\,(Q)$ and eventually

$$\dim_{\mathbb{R}} \mathcal{T} = \dim_{\mathbb{C}} \mathrm{Ker}\,(Q)\,.$$

Q.E.D.

3.6 Definition of J_α.

Let us set for each $\alpha \in \mathbb{U}^\Sigma$,

$$J_\alpha = I.\,[\![\alpha]\!]\,.D \tag{3.19}$$

The direct application of lemmas 3.5 and 3.3.(2) readily gives

Lemma 3.6. Let $\alpha = (\alpha_\nu) \in \mathbb{U}^\Sigma$,

1. The set of solutions of the $\mathbb{R}$-linear system

$$\begin{cases} t_{\nu+\frac{1}{2}} - t_{\nu-\frac{1}{2}} + \alpha_\nu(\bar{t}_{\nu+\frac{1}{2}} - \bar{t}_{\nu-\frac{1}{2}}) = 0 & \forall \nu \in \Sigma \\ \sum_{n\in S} t_n = 0 \end{cases}$$

with unknown $t = (t_n) \in \mathbb{C}^S$, is the $\mathbb{R}$-subspace of $\mathbb{C}^S$ given by

$$\begin{aligned} \mathcal{T}_\alpha &= \{t = -J_\alpha.s + \overline{s}, s = (s_n) \in \mathbb{C}^S, \sum_{n\in S} s_n = 0, \sum_{n\in S}(\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}})s_n = 0\} \\ &= (-J_\alpha + C)(\mathbb{1}^\perp \cap (D.\overline{\alpha})^\perp) \end{aligned}$$

2. The dimension of $\mathcal{T}_\alpha$ (over $\mathbb{R}$) is the dimension (over $\mathbb{C}$) of

$$\mathcal{S}_\alpha := \{s \in \mathbb{C}^S, \sum_{n\in S} s_n = 0, \sum_{n\in S}(\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}})s_n = 0\} = \mathbb{1}^\perp \cap (D.\overline{\alpha})^\perp$$

that is

(a) $\dim_\mathbb{R} \mathcal{T}_\alpha = N - 1$ whenever $D.\alpha$ and $\mathbb{1}$ are colinear, that is the α_ν are all equal,

(b) $\dim_\mathbb{R} \mathcal{T}_\alpha = N - 2$ otherwise.

Proof. 1. To prepare the data for lemma 3.5, put

$$\begin{aligned} \mathcal{T}_0 &= \{(t,s) \in \mathbb{C}^S \times \mathbb{C}^S, [\![\beta]\!]\, D.t + [\![\overline{\beta}]\!]\, .D.s = 0, \langle t \mid \mathbb{1}\rangle + \langle s \mid \mathbb{1}\rangle = 0, \langle t \mid \mathbb{1}\rangle - \langle s \mid \mathbb{1}\rangle = 0\} \\ &= \{(t,s) \in \mathbb{C}^S \times \mathbb{C}^S, \begin{cases} [\![\beta]\!]\, D.t & + & [\![\overline{\beta}]\!]\, .D.s & = 0, \\ \langle t \mid \mathbb{1}\rangle & + & \langle s \mid \mathbb{1}\rangle & = 0, \\ \langle t \mid i.\mathbb{1}\rangle & + & \langle s \mid i.\mathbb{1}\rangle & = 0 \end{cases} \} \end{aligned}$$

with β_ν, a square root of $\overline{\alpha}_\nu$ that is, put $n = \#S$, identify S with $\{1, \ldots, n\}$ and $\mathbb{C}^S$ with $\mathbb{C}^n$ and construct the correct $(n+2) \times n$ matrix A so that, up to the renumbering of S,

$$\mathcal{T}_0 = \{(T,S) \in \mathbb{C}^n \times \mathbb{C}^n,\ A.T + \overline{A}.S = 0\}.$$

The set of solutions of our system is then, up to the renumbering of S,

$$\mathcal{T}_\alpha = \{T \in \mathbb{C}^n,\ A.T + \overline{A.T} = 0\}.$$

2. Now observe that by 3.3.(2),

$$\begin{aligned} (t,s) \in \mathcal{T}_0 \quad & \text{if and only if} \quad D.t = -[\![\alpha]\!]\,.D.s \text{ and } \langle s \mid \mathbb{1}\rangle = 0 \text{ and } \langle t \mid \mathbb{1}\rangle = 0 \\ & \text{if and only if} \quad t = -I.[\![\alpha]\!]\,.D.s \text{ and } \langle [\![\alpha]\!]\,.D.s \mid \mathbb{1}\rangle = 0 \text{ and } \langle s \mid \mathbb{1}\rangle = 0 \\ & \text{if and only if} \quad t = -J_\alpha.s \text{ and } \langle s \mid D.\overline{\alpha}\rangle = 0 \text{ and } \langle s \mid \mathbb{1}\rangle = 0. \end{aligned}$$

This allows us to construct the matrices J and Q such that, up to the renumbering of S,

$$\mathcal{T}_0 = \{(T,S) \in \mathbb{C}^n \times \mathbb{C}^n, T = J.S, Q.S = 0\}$$

and get the sought-for conclusions for $\mathcal{T} = \mathcal{T}_\alpha$.

Q.E.D.

4 The tangent space $\mathcal{T}_z$.

4.1 A parametric representation of $\mathcal{T}_z$

For an S-gon $z \in \mathcal{P}_0(\ell)$, let us set $\zeta = D.z$ and $\alpha \in \mathbb{C}^\Sigma$ defined by

$$\forall \nu \in \Sigma,\ \alpha_\nu = \frac{\zeta_\nu}{\overline{\zeta}_\nu}, \tag{4.1}$$

The vector α is well defined because for each $\nu \in \Sigma$, $|\zeta_\nu| = \ell_\nu > 0$.

The quadratic functions L_ν^2 have been defined by (2.7). For $\nu \in \Sigma$, using rules of differential calculus, as

$$L_\nu^2(z) = |z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}}|^2 = (z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}}).\overline{(z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}})},$$

the differential of L_ν^2 at z is

$$d_z L_\nu^2 = \overline{z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}}}(dz_{\nu+\frac{1}{2}} - dz_{\nu-\frac{1}{2}}) + (z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}})(d\overline{z}_{\nu+\frac{1}{2}} - d\overline{z}_{\nu-\frac{1}{2}}) \tag{4.2}$$

As

$$\mathcal{P}_0(\ell) = \{z \in \mathbb{C}^S, L_\nu^2(z) = \ell_\nu^2, \forall \nu \in \Sigma\} \cap \mathbb{1}^\perp,$$

its tangent space at z, in the real algebraic sense, is

$$\mathcal{T}_z := \{(t_n) \in \mathbb{C}^S,\ \overline{z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}}}(t_{\nu+\frac{1}{2}} - t_{\nu-\frac{1}{2}}) + (z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}})(\overline{t}_{\nu+\frac{1}{2}} - \overline{t}_{\nu-\frac{1}{2}}) = 0, \forall \nu \in \Sigma\} \cap \mathbb{1}^\perp \tag{4.3}$$

Lemma 3.6, gives that, by setting

$$\mathcal{S}_z := \{s = (s_n) \in \mathbb{C}^S, \sum_{n \in S} s_n = 0, \sum_{n \in S} (\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}}) s_n = 0\},$$

we get

$$\mathcal{T}_z = \{-J_\alpha.s + \overline{s}, s \in \mathcal{S}_z\} = (-J_\alpha + C)(\mathcal{S}_z) \tag{4.4}$$

where J_α is defined in §3.6. The space $\mathcal{T}_z$ has real dimension $N-1$ if the α_ν are all equal, $N-2$ otherwise.

4.2 Singular points.

The fact that the α_ν are all equal implies that there exists a complex number of modulus 1, $e^{i\theta_0}$, a family $\varepsilon = (\varepsilon_\nu) \in \{-1, +1\}^\Sigma$ such that, for each $\nu \in \Sigma$, $z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}} = \varepsilon_\nu \ell_\nu . e^{i\theta_0}$. This implies that

$$\sum_{\nu \in \Sigma} \varepsilon_\nu \ell_\nu = 0 \tag{4.5}$$

and the fact that the points z_n are aligned. Reciprocally, if (4.5) is satisfied for some family $\varepsilon = (\varepsilon_\nu) \in \{-1, +1\}^\Sigma$, with each such family, by taking $z = I.(\varepsilon_\nu.\ell_\nu)$, we get an aligned S-gon z in $\mathcal{P}_0(\ell)$ with equal α_ν's. Let us sum up what has just been shown :

Proposition 4.1. In any case, whenever ℓ is an admissible vector of lengths, the colinear S-gons are the singular points of $\mathcal{P}_0(\ell)$.

1. On the one hand, if, for each $\varepsilon \in \{-1, +1\}^\Sigma$, $\sum_{\nu \in \Sigma} \varepsilon_\nu.\ell_\nu \neq 0$, the every point z of $\mathcal{P}_0(\ell)$ is *regular*. The algebraic subvariety $\mathcal{P}_0(\ell)$ is a (real, $\mathcal{C}^\infty$-) differential submanifold of $\mathbb{C}^S$ of dimension $N-2$.

2. On the other hand, if, there exists a Σ-uple $(\varepsilon_\nu) \in \{-1, +1\}^\Sigma$ such that $\sum_{\nu \in \Sigma} \varepsilon_\nu.\ell_\nu = 0$, there is at least one singular point in $\mathcal{P}_0(\ell)$.

4.3 A special vector in $\mathcal{T}_z$.

The diagonal action of $\mathbb{U}$, $(e^{i\theta}, z) \mapsto e^{i\theta}.z$ gives rise to special vector in $\mathcal{T}_z$. Indeed, let us fix some $z \in \mathcal{P}_0(\ell)$. The curve $\theta \mapsto e^{i\theta}.z$, traced over $\mathcal{P}_0(\ell)$, has tangent vector at $\theta = 0$ the vector $i.z$. This vector belongs to the tangent space $\mathcal{T}_z$ and in the parameterization (4.4), this vector is the image of the parameter $s = -\frac{i}{2}\overline{z} \in \mathcal{S}_z$. That is

$$i.z = (-J_\alpha + C).(-\frac{i}{2}\overline{z}). \tag{4.6}$$

The identification $\mathcal{M}(\ell) = \mathcal{P}_0(\ell)/\mathbb{U}$ leads to the identification of the tangent space of $\mathcal{M}(\ell)$ at z with $\mathcal{T}_z/\operatorname{Vect}_{\mathbb{R}}\{i.z\}$.

5 Proof of Theorem 2.4.

We have shown that the singular points of $\mathcal{P}_0(\ell)$, if any, are the colinear S-gons belonging to $\mathcal{P}_0(\ell)$. To get the sought-for conclusion, we consider an S-gon $z \in \mathcal{P}_0(\ell)$, non colinear.

We show that it is critical for the area A on $\mathcal{P}_0(\ell)$ if and only if it is cocyclical. The proof is based on the computation of the operator norm of the restriction of d_zA to the space $\mathcal{T}_z$ once this space has been equipped with an appropriate Euclidean norm[9]. The S-gon z is critical for A on $\mathcal{P}_0(\ell)$ if and only if this norm is null.

Let us define on $\mathcal{S}_z \subset \mathbb{C}^S$ the function

$$\mathcal{A}_z(s) = d_zA.(-J_\alpha.s + \overline{s}) = d_zA.(-J_\alpha + C).s. \tag{5.1}$$

The function $\mathcal{A}_z$ is a real linear form on $\mathcal{S}_z$. Thus there exists a unique vector $a_z \in \mathcal{S}_z$ such that for every $s \in \mathcal{S}_z$,

$$\mathcal{A}_z(s) = \langle\langle a_z \mid s \rangle\rangle\,. \tag{5.2}$$

The Euclidean norm of this vector is

$$\begin{aligned} \|a_z\| &= \sup_{s\in\mathcal{S}_z, \|s\|\le 1} \langle\langle a_z \mid s\rangle\rangle = \sup_{s\in\mathcal{S}_z,\|s\|\le 1} \mathcal{A}_z(s) \\ &= \sup_{w\in\mathcal{T}_z,\|w\|_z\le 1} d_zA.w \end{aligned}$$

where $\|.\|_z$ is the Euclidean norm on $\mathcal{T}_z$ whose unit ball is the ellipsoid $(-J_\alpha + C)(\mathbb{B}_1 \cap \mathcal{S}_z)$ where $\mathbb{B}_1$ is the standard unit ball in $\mathbb{C}^S$. More precisely, we have for each $w \in \mathcal{T}_z = (-J_\alpha + C)(\mathcal{S}_z)$

$$\|w\|_z = \inf\{\|s\|,\ s \in \mathcal{S}_z,\ w = (-J_\alpha + C).s\}. \tag{5.3}$$

The S-gon z is critical for A on $\mathcal{P}_0(\ell)$ if and only if $\|a_z\| = 0$. And we now get an explicit formula for a_z and its norm $\|a_z\|$.

We recall that for a complex number a, $\operatorname{Im}(a) = -\operatorname{Re}(i.a)$ and that we denoted $\langle\langle . \mid . \rangle\rangle$ the standard Euclidean product on $\mathbb{C}^S$. We start with

$$\begin{aligned} \mathcal{A}_z(s) &= \frac{1}{2}\sum_{k\in S} \operatorname{Im}((z_{k+1} - z_{k-1}).\overline{(-J_\alpha.s + \overline{s})_k}) = -\langle\langle i.D.M.z \mid (-J_\alpha + C).s\rangle\rangle \\ &= -\langle\langle(-J_\alpha^* + C)i.D.M.z \mid s\rangle\rangle = \langle\langle i(J_\alpha^* + C)D.M.z \mid s\rangle\rangle =: \langle\langle \tilde{a}_z \mid s\rangle\rangle\,. \end{aligned}$$

[9] In §7, we show that this family of Euclidean norms defines a Riemannian structure on $\mathcal{P}_0(\ell)$ and we compute the gradient of A on $\mathcal{P}_0(\ell)$ relative to this structure.

Let us compute more precisely the vector $\tilde{a}_z = i(J_\alpha^* + C)D.M.z \in \mathbb{C}^S$.

- On one hand, for each $n \in S$,

$$(i.C.M.D.z)_n = \frac{i}{2}\overline{z_{n+1} - z_{n-1}}, \tag{5.4}$$

- on the other hand, using lemmas 3.2 and 3.3, that is computing rules on $I, M, D, [\![\alpha]\!], \pi_0$, we get

$$J_\alpha^*.D.M = D.[\![\overline{\alpha}]\!].I.D.M = D.[\![\overline{\alpha}]\!].M.\pi_0. \tag{5.5}$$

As $z \in \mathbb{C}_0^S$, we have $\pi_0.z = z$ and for each $n \in S$,

$$\begin{aligned}(D.[\![\overline{\alpha}]\!].M.z)_n &= ([\![\overline{\alpha}]\!].M.z)_{n+\frac{1}{2}} - ([\![\overline{\alpha}]\!].M.z)_{n-\frac{1}{2}} = \frac{1}{2}(\overline{\alpha}_{n+\frac{1}{2}}(z_{n+1}+z_n) - \overline{\alpha}_{n-\frac{1}{2}}(z_n + z_{n-1})) \\ &= \frac{1}{2}(\overline{\alpha}_{n+\frac{1}{2}}(z_{n+1}-z_n) - \overline{\alpha}_{n-\frac{1}{2}}(z_{n-1} - z_n)) + z_n(\overline{\alpha}_{n+\frac{1}{2}} - \overline{\alpha}_{n-\frac{1}{2}}) \\ &= \frac{1}{2}\overline{z_{n+1} - z_{n-1}} + z_n(\overline{\alpha}_{n+\frac{1}{2}} - \overline{\alpha}_{n-\frac{1}{2}})\end{aligned}$$

To obtain the last line we have used the fact that

$$\alpha_{n+\frac{1}{2}} = \frac{z_{n+1} - z_n}{\overline{z_{n+1} - z_n}} \text{ et } \alpha_{n-\frac{1}{2}} = \frac{z_{n-1} - z_n}{\overline{z_{n-1} - z_n}}.$$

- Eventually,

$$\forall n \in S,\ (\tilde{a}_z)_n = i\left(\overline{z_{n+1} - z_{n-1}} + z_n\overline{\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}}}\right) \tag{5.6}$$

The vector $C.D.\alpha = (\overline{\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}}})_{n\in S}$ is non-null because the S-gon $z = (z_n)_{n\in\mathbb{S}}$ is not colinear.

The vector a_z is the orthogonal projection of $\tilde{a}_z$ on $\mathcal{S}_z = \mathbb{1}^\perp \cap (C.D.\alpha)^\perp$. Lemma 3.1, with the additional remark that $\langle \tilde{a}_z \mid \mathbb{1}\rangle = 0$, gives

$$a_z = \tilde{a}_z - \frac{\left\langle \tilde{a}_z \mid (\overline{\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}}})_{n\in S}\right\rangle}{\sum_{n\in S}|\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}}|^2}(\overline{\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}}})_{n\in S} \tag{5.7}$$

PYTHAGORA's Theorem gives

$$\|a_z\|^2 = \frac{\|\tilde{a}_z\|^2\|(\overline{\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}}})_{n\in S}\|^2 - |\left\langle \tilde{a}_z \mid (\overline{\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}}})_{n\in S}\right\rangle|^2}{\sum_{n\in S}|\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}}|^2}$$

Now, the equality case of CAUCHY–SCHWARZ's inequality gives that, $\|a_z\|^2 = 0$ if and only if the vectors $\tilde{a}_z$ and $C.D.\alpha = (\overline{\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}}})_{n\in S}$ are $\mathbb{C}$-colinear, that is, there exists $o \in \mathbb{C}$ such that

$$\forall n \in S,\ \overline{z_{n+1} - z_{n-1}} + z_n\overline{\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}}} = o.\overline{\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}}}. \tag{5.8}$$

We are now done with the fact that a cocyclical S-gon $z \in \mathcal{P}_0(\ell)$ is critical for A on $\mathcal{P}_0(\ell)$ as we can take for o the center of the circumcircle of z and get the previous identity by Proposition 3.4.

In the opposite direction, if $z \in \mathcal{P}_0(\ell)$ is critical for A on $\mathcal{P}_0(\ell)$, this means that for each $n \in S$,

- either $\alpha_{n+\frac{1}{2}} - \alpha_{n-\frac{1}{2}} = 0$ that is $z_{n+1} = z_{n-1}$,
- either $z_n + \overline{\left(\frac{z_{n+1}-z_{n-1}}{\alpha_{n+\frac{1}{2}}-\alpha_{n-\frac{1}{2}}}\right)} = o$, and thus the point o is, by Proposition 3.4 the center of the circumcircle of the points z_{n-1}, z_n, z_{n+1}.

We can infer from this that every point of z lies on one circle. Let us set

$$Z := \{n \in S,\, \alpha_{n+\frac{1}{2}} = \alpha_{n-\frac{1}{2}}\} = \{n \in S,\, z_{n+1} = z_{n-1}\}.$$

Because the S-gon z is not colinear, there exists a $n_0 \notin Z$. Let us define Γ, the circle with center o and radius $R = |z_{n_0} - o|$, by induction, we get easily that[10] every z_n, $n \in S$ belongs to the circle Γ.

This closes the proof of Theorem 2.4.

6 Lagrange multipliers

6.1 Their value

Let us set, for each $z = (z_n) \in \mathbb{C}^S$, $\lambda = (\lambda_\nu) \in \mathbb{R}^\Sigma$, $\mu \in \mathbb{C}$,

$$\mathcal{L}(z, \lambda, \mu) = 2.A(z) + \frac{1}{2}\sum_{\nu\in\Sigma} \lambda_\nu (L_\nu^2(z) - \ell_\nu^2) + \mathrm{Re}(\mu \sum_{n\in S} \overline{z}_n),$$

the Lagrangian associated to the problem of minimizing $A(z)$ under the constraint $z \in \mathcal{P}_0(\ell)$.

The classical Lagrange Multiplier Theorem states that, with the exception of the singular points of $\mathcal{P}_0(\ell)$, the critical points of $\mathcal{L}$ on $\mathbb{C}^S \times \mathbb{R}^\Sigma \times \mathbb{C}$ corresponds exactly to the critical points of A sur $\mathcal{P}_0(\ell)$.

Let us compute the Lagrange multipliers associated to each critical point of A. We have

$$\begin{aligned} d\mathcal{L} &= \mathrm{Re}\left(\sum_{n\in S} \left\{-i(z_{n+1} - z_{n-1}) + \lambda_{n-\frac{1}{2}}(z_n - z_{n-1}) - \lambda_{n+\frac{1}{2}}(z_{n+1} - z_n) + \mu\right\} d\overline{z}_n\right) \\ &\quad + \frac{1}{2}\sum_{\nu\in\Sigma}(L_\nu^2(z) - \ell_\nu^2) d\lambda_\nu + \mathrm{Re}(\sum_{n\in S} \overline{z}_n d\mu). \end{aligned} \tag{6.1}$$

A triple (z, λ, μ) is critical for $\mathcal{L}$ if and only if

1. $\sum_{n\in S} \overline{z}_n = 0$,
2. $\forall \nu \in \Sigma$, $L_\nu^2(z) - \ell_\nu^2 = 0$,
3. $$\forall n \in S,\, -i(z_{n+1} - z_{n-1}) + \lambda_{n-\frac{1}{2}}(z_n - z_{n-1}) - \lambda_{n+\frac{1}{2}}(z_{n+1} - z_n) + \mu = 0. \tag{6.2}$$

[10] We put as induction hypothesis at rank p : z_{n_0+p}, $z_{n_0+p+1} \in \Gamma$. As $n_0 \notin Z$, this is true at rank $p = 0$. Let us suppose it is true at some rank p and show that it is then true at rank $p+1$. We have to show that $z_{n_0+p+2} \in \Gamma$. There can be two possibilities. Either $p+1 \in Z$ and then $z_{n_0+p+2} = z_{n_0+p} \in \Gamma$, either $p+1 \notin Z$ and then z_{n_0+p}, z_{n_0+p+1} and z_{n_0+p+2} are on a same circle with centre o which can only be Γ.

The first two conditions state that z belongs to $\mathcal{P}_0(\ell)$ while the latter one allows us to compute the multipliers λ, μ.

Let us assume that z is a critical point of A on $\mathcal{P}_0(\ell)$. This S-gon is not colinear and is cocyclical. Thus there exists $o \in \mathbb{C}$, $R > 0$, $\theta = (\theta_n) \in \mathbb{R}^S$ such that $\forall n \in S$, $z_n = o + Re^{i\theta_n}$ and we give in (6.8) the value of λ as a function of the family of angles θ.

Let us rewrite condition (6.2) in matrix terms, inserting the center o, as

$$D. [\![\lambda]\!] .D.z = -2iD.M.(z - o\mathbb{1}) + \mu.\mathbb{1}. \tag{6.3}$$

This shows on one hand that $\mu = 0$ and on the other hand, inverting D, that there exists $c \in \mathbb{C}$ such that

$$[\![\lambda]\!] .D.z = -2iM.(z - o\mathbb{1}) + c\mathbb{1}.$$

Thus there exists $c \in \mathbb{C}$ such that for every $\nu \in \Sigma$,

$$\lambda_\nu(z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}}) = -i(z_{\nu+\frac{1}{2}} - o + z_{\nu-\frac{1}{2}} - o) + c. \tag{6.4}$$

Using the angles $(\theta_n)_{n\in S}$, we can rewrite this as

$$\forall \nu \in \Sigma,\ 2i\lambda_\nu \sin\frac{\theta_{\nu+\frac{1}{2}} - \theta_{\nu-\frac{1}{2}}}{2} = -2i\cos\frac{\theta_{\nu+\frac{1}{2}} - \theta_{\nu-\frac{1}{2}}}{2} + c.e^{-i\frac{\theta_{\nu+\frac{1}{2}}+\theta_{\nu-\frac{1}{2}}}{2}}. \tag{6.5}$$

As $\lambda_\nu \in \mathbb{R}$, if $c \neq 0$, we must have, for every $\nu,\nu' \in \Sigma$,

$$\frac{\theta_{\nu+\frac{1}{2}} + \theta_{\nu-\frac{1}{2}}}{2} \equiv \frac{\theta_{\nu'+\frac{1}{2}} + \theta_{\nu'-\frac{1}{2}}}{2}[\pi].$$

To simplify the analysis, let us suppose we normalized the S-gon z so that for some $\nu_0 \in \Sigma$, $\theta_{\nu_0+\frac{1}{2}} + \theta_{\nu_0-\frac{1}{2}} \equiv 0[2\pi]$. This enforces then that for every $\nu \in \Sigma$,

$$\theta_{\nu+\frac{1}{2}} \equiv -\theta_{\nu-\frac{1}{2}}[2\pi].$$

Let us discuss two disjoint cases :

- if $N = \#S$ is odd, we get then $\theta_{\nu+\frac{1}{2}} \equiv 0[2\pi]$, for every $\nu \in \Sigma$, this is impossible because then z would degenerate to a point ;
- if $N = \#S$ is even, it forces z to be colinear (it reduces to one point or to two alternating points, diametrically opposed on the circle) which is excluded.

Our assumption that z is a critical point for A on $\mathcal{P}_0(\ell)$ thus leads us to $c = 0$ and we can rewrite (6.4) and (6.5) as

$$\lambda_\nu(z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}}) = -i(z_{\nu+\frac{1}{2}} - o + z_{\nu-\frac{1}{2}} - o) \tag{6.6}$$

and

$$\forall \nu \in \Sigma,\ 2i\lambda_\nu \sin\frac{\theta_{\nu+\frac{1}{2}} - \theta_{\nu-\frac{1}{2}}}{2} = -2i\cos\frac{\theta_{\nu+\frac{1}{2}} - \theta_{\nu-\frac{1}{2}}}{2}. \tag{6.7}$$

We remark that we also have for each $\nu \in \Sigma$,

$$L_\nu^2(z) = |z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}}|^2 = \ell_\nu^2 = 4R^2\sin^2\frac{\theta_{\nu+\frac{1}{2}} - \theta_{\nu-\frac{1}{2}}}{2} \neq 0$$

so that

$$\forall \nu \in S,\ \lambda_\nu = -\cot\frac{\theta_{\nu+\frac{1}{2}} - \theta_{\nu-\frac{1}{2}}}{2} = -i.\frac{z_{\nu+\frac{1}{2}} - o + z_{\nu-\frac{1}{2}} - o}{z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}}}. \tag{6.8}$$

6.2 Critical points with a free edge

Let us fix an edge $\nu_0 \in \Sigma$. We may consider the problem of finding the critical points of A under the constraint that every edge length L_ν but L_{ν_0} is fixed to some value $\ell_\nu > 0$.

We are thus led to find the critical points of the Lagrangian

$$\mathcal{L}_{\nu_0}(z,\lambda,\mu) = 2.A(z) + \frac{1}{2}\sum_{\nu\neq\nu_0}\lambda_\nu(L_\nu^2(z) - \ell_\nu^2) + \mathrm{Re}(\mu\sum_{n\in S}\overline{z}_n). \tag{6.9}$$

Let $(z,(\lambda_\nu)_{\nu\neq\nu_0},\mu)$ be a critical point of this Lagrangian such that z is non colinear and[11] $\ell_{\nu_0} := |z_{\nu_0+\frac{1}{2}} - z_{\nu_0-\frac{1}{2}}| > 0$. If we complete the family $(\lambda_\nu)_{\nu\neq\nu_0}$ by setting $\lambda_{\nu_0} = 0$ we get that $(z,(\lambda_\nu)_{\nu\in\Sigma},\mu)$ is a critical point of $\mathcal{L}$ (the one constructed with the family $(\ell_\nu)_{\nu\in\Sigma}$).

To shorten, let us agree to say that z is a critical point of the Lagrangian $\mathcal{L}$ or $\mathcal{L}_{\nu_0}$, if there exists complementary parameters $\lambda,\mu...$ such that (z,λ,μ) is a critical point of $\mathcal{L}$, $\mathcal{L}_{\nu_0}$...

The previous reasoning showed that a critical point z of $\mathcal{L}_{\nu_0}$, non colinear and satisfying $L_{\nu_0}(z) > 0$ is a critical point of an appropriate $\mathcal{L}$ with the additional condition that $\lambda_{\nu_0} = 0$. Such a z, being a critical point of A over the adequate $\mathcal{P}_0(\ell)$ is cocyclical. Let us denote by o the center of the circumcircle of z.

Specializing formula (6.6) for ν_0 to get λ_{ν_0}, it should be clear that we get

$$\lambda_{\nu_0} = 0 \Leftrightarrow \frac{1}{2}(z_{\nu_0+\frac{1}{2}} + z_{\nu_0-\frac{1}{2}}) = o.$$

This is equivalent to the fact that the line segment $[z_{\nu_0-\frac{1}{2}}, z_{\nu_0+\frac{1}{2}}]$ is a diameter of the circumcircle of z.

This result is due to G. Khimshiashvili and D. Siersma, [KS13] ; it is a descendant of [Leg41], Proposition IV, p.132.

6.3 Critical points for prescribed perimeter and other variants

We may now consider the problem of finding the critical points of A under the constraint that the sum of the squares of the edge lengths is prescribed, that is $\sum_{\nu\in\Sigma} L_\nu^2$ is a constant P_2.

This lead us to find the critical points of the Lagrangian

$$\mathcal{L}_{\mathcal{P}_2}(z,\tilde{\lambda},\mu) = 2.A(z) + \frac{1}{2}\tilde{\lambda}.(\sum_\nu L_\nu^2(z) - P_2) + \mathrm{Re}(\mu\sum_{n\in S}\overline{z}_n), \tag{6.10}$$

Let z be a critical point of this Lagrangian such that z is non colinear and[12] for each $\nu \in \Sigma$, $\ell_\nu := |z_{\nu+\frac{1}{2}} - z_{\nu-\frac{1}{2}}| > 0$. We have then that

$$\sum_\nu \ell_\nu^2(z) = P_2$$

and z is a critical point of the Lagrangian

$$\mathcal{L}_{\mathcal{P}_2}(z,\tilde{\lambda},\mu) = 2.A(z) + \frac{1}{2}\tilde{\lambda}.\sum_\nu(L_\nu^2(z) - \ell_\nu^2) + \mathrm{Re}(\mu\sum_{n\in S}\overline{z}_n), \tag{6.11}$$

[11] We exclude the case where $z_{\nu_0+\frac{1}{2}} = z_{\nu_0-\frac{1}{2}}$ from this discussion.

[12] We exclude the case where to consecutive vertices are equal from this discussion.

Such a z is thus a critical point of $\mathcal{L}$ with the additional condition that

$$\forall \nu \in \Sigma, \lambda_\nu = \tilde{\lambda}. \tag{6.12}$$

Thus z is cocyclical, we set as before $z = (z_n)_{n\in S} = (o + R.e^{i\theta_n})_{n\in S}$, and condition (6.12) can be rewritten as

$$\forall \nu, \nu' \in \Sigma,\ \theta_{\nu+\frac{1}{2}} - \theta_{\nu-\frac{1}{2}} \equiv \theta_{\nu'+\frac{1}{2}} - \theta_{\nu'-\frac{1}{2}}[2\pi].$$

For each $n \in S$, we may take $\nu = n + \frac{1}{2}$, $\nu' = (n-1) + \frac{1}{2}$, to get

$$\forall n \in S,\ \theta_{n+1} - \theta_n \equiv \theta_n - \theta_{n-1}[2\pi] \tag{6.13}$$

and this turns out to be equivalent to the existence of angles θ_0 and $\alpha \equiv 0[\frac{2\pi}{N}]$ such that, once a vertex $n_0 \in S$ has been chosen, we have

$$\forall k \in \mathbb{Z}/N.\mathbb{Z},\ z_{n_0+k} = o + R.e^{i.\theta_0}.e^{i.k.\alpha}.$$

This implies that the $N = \#S$ vertices z_n of z are the vertices of a regular N-gon[13]

The problem, in a sense the classical isoperimetric problem–we are then in the conditions of Proposition VII of [Leg41]–, to find the critical points of A under the constraint that the perimeter is prescribed to some value $P_1 > 0$, that is $P(z) = \sum_{\nu\in\Sigma} \sqrt{L_\nu^2(z)}$ is prescribed to be P_1 can be treated in a similar manner.

If we put ψ, the square root function, the method leads us to search such critical points with positive edge lengths as the critical points of some Lagrangian

$$\mathcal{L}_{\mathcal{P}_1}(z, \tilde{\lambda}, \mu) = 2.A(z) + \frac{1}{2}\tilde{\lambda}.\sum_{\nu}(\psi(L_\nu^2(z)) - \psi(\ell_\nu^2)) + \mathrm{Re}(\mu \sum_{n\in S} \overline{z}_n),$$

where the admissible vector length $\ell = (\ell_\nu)_{\nu\in\Sigma}$ satisfies $\sum_\nu \psi(\ell_\nu^2) = P_1$.

These are critical points of the Lagrangian $\mathcal{L}^\psi$ defined by

$$\mathcal{L}^\psi(z, \lambda, \mu) = 2.A(z) + \frac{1}{2}\sum_{\nu\in\Sigma}\lambda_\nu(\psi(L_\nu^2(z)) - \psi(\ell_\nu^2)) + \mathrm{Re}(\mu \sum_{n\in S} \overline{z}_n),$$

with the additional condition that $\forall \nu \in \Sigma, \lambda_\nu = \tilde{\lambda}$.

An S-gon z, non colinear, whose edges have positive lengths, is critical for this Lagrangian if and only if it is cocyclical and a similar computation to the computation led in §6.1 (we keep the same notations) shows that the Lagrange multipliers $(\lambda_\nu)_{\nu\in\Sigma}$ satisfy

$$\forall \nu \in \Sigma,\ \lambda_\nu.\psi'(L_\nu^2(z)) = -\cot\frac{\theta_{\nu+\frac{1}{2}} - \theta_{\nu-\frac{1}{2}}}{2}.$$

As $\psi'(x) = \frac{1}{2\sqrt{x}}$ and $\sqrt{L_\nu^2(z)} = 2R\left|\sin\frac{\theta_{\nu+\frac{1}{2}} - \theta_{\nu-\frac{1}{2}}}{2}\right|$, we get that

$$\forall \nu \in \Sigma,\ \lambda_\nu = -4R\, sign(\sin\frac{\theta_{\nu+\frac{1}{2}} - \theta_{\nu-\frac{1}{2}}}{2})\cos\frac{\theta_{\nu+\frac{1}{2}} - \theta_{\nu-\frac{1}{2}}}{2},$$

[13] This does not mean that z is the usual regular N-gon. Crossed polygons are allowed, for example, for $N = 8$, one may consider $\alpha = \frac{3\pi}{4}$. Other cases are possible such as "twice the square", that is $N = 8$ and $\alpha = \frac{\pi}{2}$. These polygons are called regular stars in [KPS19].

where $sign$ is the sign function. Whenever θ, θ' are angles, we have the equivalence

$$sign(\sin\theta)\cos\theta = sign(\sin\theta')\cos\theta' \Leftrightarrow \theta \equiv \theta'[\pi].$$

Thus, an S-gon $z = (z_n)_{n\in S} = (o + R.e^{i\theta_n})_{n\in S}$ is critical for the Lagrangian with prescribed perimeter if and only if (6.13) holds. We thus obtain the same critical points as for $\mathcal{L}_{\mathcal{P}_2}$, that is regular stars in the terminology of the full list of critical points of the area on the smooth manifold of S-gons with prescribed perimeter obtained in [KPS19].

7 Gradient ascent

7.1 Gradient of the area

To simplify the discussion, we set an admissible length vector ℓ such that $\mathcal{P}_0(\ell)$ is a differential submanifold of $\mathbb{C}^S$, thus it does not contain any colinear S-gons. The same discussion can probably be led in the general case by excluding the singular points, that is the colinear S-gons, but may require some care to avoid these singularities.

In the proof of the main Theorem 2.4 we introduced some kind of gradient of the function A when restricted on the space $\mathcal{P}_0(\ell)$. This gradient is a vector representing the linear form d_zA in the isomorphism between $\mathcal{T}_z$ and its dual $\mathcal{T}_z^*$ given by some scalar product on $\mathcal{T}_z$ which has no reason to be the restriction of the standard scalar product on $\mathbb{C}^S$.

Indeed, looking back at equation (5.3), we define at each $z \in \mathcal{P}_0(\ell)$ a Euclidean norm $\|.\|_z$ on $\mathcal{T}_z$, and thus a scalar product $\langle\!\langle . \mid . \rangle\!\rangle_z$ on $\mathcal{T}_z$ and this defines a Riemannian structure on $\mathcal{P}_0(\ell)$.

Let us be more specific about this. Let us consider some $z \in \mathcal{P}_0(\ell)$ and let us denote, to shorten the notations $H_z = -J_\alpha + C$ where J_α is defined by (3.19) and α is defined by (4.1). The map $H_z : \mathbb{C}^S \to \mathbb{C}^S$ is $\mathbb{R}$-linear and we have $\mathcal{T}_z = H_z(\mathcal{S}_z)$. We denote $H_z^* : \mathbb{C}^S \to \mathbb{C}^S$ its adjoint with respect to the standard Euclidean product on $\mathbb{C}^S$.

Let us denote $\tilde{H}_z : \mathrm{Ker}\,(H_z)^\perp \cap \mathcal{S}_z \to \mathcal{T}_z$, the restriction of H_z. The map $\tilde{H}_z$ is an isomorphism of $\mathbb{R}$-vector spaces.

If $w \in \mathcal{T}_z$, $s \in \mathcal{S}_z$ are such that $H_z.s = w$ and $\tilde{s} \in \mathrm{Ker}\,(H_z)^\perp \cap \mathcal{S}_z$, $\tilde{s} = \tilde{H}_z^{-1}.w$ then $s - \tilde{s} \in \mathrm{Ker}\,(H_z) \cap \mathcal{S}_z$ and, using orthogonality,

$$\|s\| = \sqrt{\|\tilde{s}\|^2 + \|s - \tilde{s}\|^2} \geq \|\tilde{s}\|.$$

To sum up,

$$\|w\|_z = \inf\{\|s\|,\, s \in \mathcal{S}_z,\, w = (-J_\alpha + C).s\} = \|\tilde{s}\| = \|\tilde{H}_z^{-1} w\|.$$

The scalar product $\langle\!\langle . \mid . \rangle\!\rangle_z$ is thus defined on $\mathcal{T}_z$ by the formula (recall that $\langle\!\langle . \mid . \rangle\!\rangle$ denotes the standard scalar product on $\mathbb{C}^S$),

$$\forall w, w' \in \mathcal{T}_z,\ \langle\!\langle w \mid w' \rangle\!\rangle_z = \left\langle\!\!\left\langle \tilde{H}_z^{-1} w \mid \tilde{H}_z^{-1} w' \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle w \mid (\tilde{H}_z.\tilde{H}_z^*)^{-1} w' \right\rangle\!\!\right\rangle$$

where $\tilde{H}_z^*$ is the adjoint of $\tilde{H}_z$ when source and target spaces are endowed with the scalar product inherited from the standard scalar product.

As byproduct of this formula we get the Riemannian metric we put on $\mathcal{P}_0(\ell)$ as well as a formula for the gradient at z of A with respect to this metric. Indeed

- A little bit of thought about adjoints, projections and inclusions gives that

$$\forall w \in \mathcal{T}_z,\ \tilde{H}_z.\tilde{H}_z^*.w = H_z.H_z^*.w,$$

so that the restriction of $H_z.H_z^*$ to $\mathcal{T}_z$ is a symetric automorphism of $\mathcal{T}_z$ and

$$\forall w, w' \in \mathcal{T}_z,\ \langle\!\langle w \mid w' \rangle\!\rangle_z = \left\langle\!\!\left\langle w \mid (H_z.H_z^*)^{-1} w' \right\rangle\!\!\right\rangle.$$

This formula defines a metric tensor on $\mathcal{P}_0(\ell)$ and the formula $H_z = -I.\llbracket \alpha \rrbracket .D + C$, as well as the fact that $z \mapsto \alpha$ is $\mathcal{C}^\infty$ in a neighbourhood of $\mathcal{P}_0(\ell)$, show the $\mathcal{C}^\infty$ regularity of the metric tensor.

- If $w = H_z.s$, $w' = H_z.s'$ with $s \in \mathcal{S}_z$, $s' \in \mathcal{S}_z \cap (\mathrm{Ker}\,(H_z))^\perp$, then, as $\left\langle\!\!\left\langle \tilde{H}_z^{-1}.H_z.s - s \mid s' \right\rangle\!\!\right\rangle = 0$, we get that

$$\langle\!\langle w \mid w' \rangle\!\rangle_z = \langle\!\langle H_z.s \mid H_z.s' \rangle\!\rangle_z = \left\langle\!\!\left\langle \tilde{H}_z^{-1}.H_z.s \mid \tilde{H}_z^{-1}.H_z.s' \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \tilde{H}_z^{-1}.H_z.s \mid s' \right\rangle\!\!\right\rangle = \langle\!\langle s \mid s' \rangle\!\rangle.$$

From this last formula, by setting

$$\nabla_z A = (-J_\alpha + C).a_z = H_z.a_z, \tag{7.1}$$

where a_z is defined by (5.2), we get that $\nabla_z A \in \mathcal{T}_z$ and going back to equations (5.1) and (5.2), we get

$$\forall w \in \mathcal{T}_z,\ \langle\!\langle \nabla_z A \mid w \rangle\!\rangle_z = d_z A.w.$$

This shows that $\nabla_z A$ is the gradient of A on the Riemannian manifold $(\mathcal{P}_0(\ell), \langle\!\langle . \mid . \rangle\!\rangle_z)$.

7.2 Explicit formulae for I and K

Let us set[14] for $t \in]0, N[$ $B(t) = \frac{N-2t}{2N}$, $B(0) = 0$ and extend B to the whole of $\mathbb{R}$ by N-periodicity. The function B is odd. In this way, whenever $k \in \mathbb{Z}/N.\mathbb{Z}$ or $\kappa \in \frac{1}{2} + \mathbb{Z}/N.\mathbb{Z}$, $B(k)$ and $B(\kappa)$ are well defined.

Let us remark that if $t \in]\frac{1}{2}, N - \frac{1}{2}[$, then

$$B(t - \frac{1}{2}) - B(t + \frac{1}{2}) = \frac{1}{N}, B(t - \frac{1}{2}) + B(t + \frac{1}{2}) = B(t),$$

whereas for $t = 0$,

$$B(t - \frac{1}{2}) - B(t + \frac{1}{2}) = \frac{1}{N} - 1, B(t - \frac{1}{2}) + B(t + \frac{1}{2}) = B(t) = 0, .$$

The following lemma is a pure computation that is left to the reader.

[14]The function B such defined is a variant of the first BERNOULLI polynomial. The formulae are in accordance with the following well-known facts. Let $E = \mathcal{C}^\infty(\mathbb{R}) \cap \{f : f \text{ is 1-periodical and } \int_0^1 f(x)\,dx = 0\}$, $D : E \to E$, the derivation operator is an automophism of reciprocal $I : E \to E$ defined by

$$\forall f \in E, \forall x \in \mathbb{R}, I(f)(x) = \int_0^1 B(y) f(x-y)\,dy$$

where B is the 1-periodical function over $\mathbb{R}$ defined by $B(0) = 0$ and $\forall t \in]0, 1[$, $B(t) = \frac{1}{2} - t$.

Lemma 7.1. For every $z \in \mathbb{C}^S$, $\zeta \in \mathbb{C}^\Sigma$, $n \in S$, $\nu \in \Sigma$,

$$(I.z)_\nu = \sum_{\kappa \in \frac{1}{2}+\mathbb{Z}/N.\mathbb{Z}} B(\kappa) z_{\nu-\kappa}, (I.\zeta)_n = \sum_{\kappa \in \frac{1}{2}+\mathbb{Z}/N.\mathbb{Z}} B(\kappa)\zeta_{n-\kappa}. \tag{7.2}$$

$$(K.z)_n = \sum_{k \in \mathbb{Z}/N.\mathbb{Z}} B(k) z_{n-k}, (K.\zeta)_\nu = \sum_{k \in \mathbb{Z}/N.\mathbb{Z}} B(k)\zeta_{\nu-k}. \tag{7.3}$$

7.3 Numerics

As we got an easily computable gradient, one may naturally be tempted to implement numerically a gradient ascent algorithm. The point is to consider the differential equation

$$\frac{dz}{dt} = \nabla_z A \tag{7.4}$$

As $\mathcal{P}_0(\ell)$ is a compact Riemannian manifold, for any initial data $z_0 \in \mathcal{P}_0(\ell)$, the CAUCHY problem associated to (7.4) and this inital data $z(0) = z_0$ admits a unique maximal solution z defined all over $\mathbb{R}$. Limits of z at $\pm\infty$ do exists and are critical points of A on $\mathcal{P}_0(\ell)$ and along a trajectory z, A is nondecreasing.

If we want to naively implement a simple forward EULER scheme in order to maximize A we must take care of the well-known difficulty that a naive discretization does not preserve the fixed edge lengths constraint.

By convexity of each of the sets $\{z \in \mathbb{C}_0^S, L_\nu^2(z) \leq \ell_\nu^2\}$, at each step, every length L_ν increases.

We may adopt the solution to project on $\mathcal{P}_0(\ell)$ at each step of the EULER scheme by a NEWTON's method.

The numerical scheme is thus:

1. Let there be given a time step dt, a threshold $\varepsilon > 0$ and an inital S-gon $Z_0 \in \mathbb{C}_0^S$, $(\ell_\nu^2)_\nu = (L_\nu^2(Z_0))_\nu$. We construct the sequence $(Z_k)_{k\in\mathbb{N}}$ by the following, at each $k \in \mathbb{N}$,

2. Do one EULER step
$$Z'_{k+1} = Z_k + dt.\nabla_{Z_k} A.$$
From a very practical point of view, to compute $\nabla_z A$ for a given S-gon z, we compute first the vector α defined by (4.1), then $\tilde{a_z}$ and a_z using formulae (5.6) and (5.7) and eventually $\nabla_z A = (-J_\alpha + C).a_z$ by using the definition (3.19) of J_α as well as formula (7.2) for I.

3. Follow it by few NEWTON steps, that is a multidimensional variation of the Babylonian method to extract square roots. Put
$$W_0 = Z'_{k+1}, \forall p \in \mathbb{N}, W_{p+1} = \frac{1}{2}.\left(W_p + I.\left[\left[\left(\frac{\ell_\nu^2}{L_\nu^2(W_p)}\right)_\nu\right]\right].D.W_p\right)$$
and iterate on p until $(\ell_\nu^2)_\nu$ and $(L_\nu^2(W_p))_\nu$ are ε-close. Put then $Z_{k+1} = W_p$. If we just want to maximize A there is nothing else to do but if we want to keep a reasonable time evolution in order to solve the ODE (7.4) we need to adjust the time-step at step k by setting dt_k so that $dt_k \simeq \dfrac{\|Z_{k+1} - Z_k\|}{\|\nabla_{Z_k} A\|}$.

We leave to the reader the task to follow the preceding instructions in order to reproduce this numerical experiment. Besides getting a funny animation of the time evolution of a given polygon, interesting quantities are to be looked at and graphed, among them the time evolution of the area is of course the first one but also the perimeter of the developped polygon, that is, the S-gon $(o_n)_{n\in S}$ where o_n is the circumcircle of the points z_{n-1}, z_n, z_{n+1}.

7.4 Towards indices

The numerical experiment of the previous section, given say a 6- or 7-gon with a lot of self intersections, raises an immediate question : area increases with time, as expected, but the graph of the perimeter of the developped polygon exhibits many peaks.

When the developped polygon of z has an infinite length, this means that one of the o_n's is sent to the infinite, that is there is one triplet z_{n-1}, z_n and z_{n+1} of consecutive vertices in z which are aligned. When the developped polygon as a null perimeter, this means that the S-gon z is cocyclical. As we have noticed that the trajectory is contained in $\mathcal{P}_0(\ell)$ and as the trajectory goes eventually to some critical point, it is expected that the perimeter of the developped polygon vanishes at ∞.

A peak in the graph shows an "unfolding" at some vertex, that is, the vertex angle passes through either the null angle or the flat angle. On the contrary, a local minimum in this graph shows the approach of a cocyclical polygon.

This interpretation shows that during the evolution, the polygon got close to some cocyclical polygon and then ran away from this polygon, always in a direction of nondecreasing area.

This is the typical expected behaviour of the trajectories of the differential equation (7.4) when the trajectory gets close to some critical point z_* of A and z_* is a saddle point of A.

The behaviour near a critical point z_* is primarily determined by the eigenvalues of the differential at z_* of $z \mapsto \nabla_z A$, that is the signature of the Hessian matrix of A restricted to $\mathcal{P}_0(\ell)$ at z_*.

These considerations lead us to the question of the non degenerate[15] character of the critical points of A as well as the question on how to geometrically determine the signature of the Hessian of A at some cocyclical S-gon.

Crucial work in this direction has been made by Panina et Zhukova, see [PZ11] and [Zhu13] and the author intends to recover these results within the framework presented in this paper.

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