Coefficient inequalities of analytic functions equipped with conic domains involving q-analogue of Noor integral operator

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Abstract

The motivation of this paper is to introduce and study a new class of univalent functions equipped with conic type regions. We also investigate a number of useful properties of this class and coefficient estimates for functions. Several consequences of the results are also pointed out.

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1 Introduction and known results

Let $\mathcal{A}$ indicate the class of functions f analytic in the unit disk $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$ and satisfying the normalization $f(0) = f'(0) - 1 = 0$. Thus, the functions in $\mathcal{A}$ are represented by the following Taylor-Maclaurin series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \Delta). \tag{1.1}$$

Let $\mathcal{S}$ be the subset of $\mathcal{A}$ consisting of the functions that are univalent in Δ. The convolution of functions $f, g \in \mathcal{A}$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad (z \in \Delta),$$

where f is given by (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \qquad (z \in \Delta).$$

For two functions f, $g \in \mathcal{A}$, we say that f is subordinate to g in Δ, denoted by

$$f(z) \prec g(z), \qquad (z \in \Delta),$$

if there exists a function w where

$$w(0) = 0, \ |w(z)| < 1$$

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such that

$$f(z) = g(w(z)), \qquad (z \in \Delta).$$

The quantum (or q-) calculus prove important tools that have been used to study various families of analytic functions due to its applications in mathematics and some related areas. Jackson [9] was among the first few researchers who defined the q-analogue of derivative and integral operators as well as provided some of their applications. However, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [25]). Afterwards, Kanas and Raducanu [12] introduced the q-analogue of Ruscheweyh differential operator and Arif et al. [3] studied its applications for multivalent functions while Khan et al. [15] studied q-calculus by the concept of convolution (see also [1, 7, 22, 23]).

We first give the basic definitions of quantum (or q-) calculus which help us in this study.

The q-derivative of a function $f \in \mathcal{A}$ is defined by

$$\partial_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \qquad (0 < q < 1, \ z \in \Delta).$$

It can easily be seen that for $n \in \mathbb{N} := \{1, 2, \ldots\}$

$$\partial_q \left(\sum_{n=1}^{\infty} a_n z^n \right) = \sum_{n=1}^{\infty} [n,q]\, a_n z^{n-1}, \quad (z \in \Delta),$$

where

$$[n,q] = \frac{1-q^n}{1-q}, \qquad [0,q] = 0.$$

For any nonnegative integer n, the q-number shift factorial is defined by

$$[n,q]! = \begin{cases} 1, & n = 0 \\ [1,q]\,[2,q]\,[3,q]\ldots[n,q], & n \in \mathbb{N} \end{cases}.$$

Also the q-generalized Pochhammer symbol for $x > 0$ is given by

$$[x,q]_n = \begin{cases} 1, & n = 0 \\ [x,q]\,[x+1,q]\ldots[x+n-1,q], & n \in \mathbb{N} \end{cases}.$$

Very recently, in [2], Arif et al. defined the function $F^{-1}_{q,\mu+1}(z)$ $(\mu > -1)$ by

$$F^{-1}_{q,\mu+1}(z) * F_{q,\mu+1}(z) = z \partial_q f(z),$$

where the function $F_{q,\mu+1}(z)$ is given by

$$F_{q,\mu+1}(z) = z + \sum_{n=2}^{\infty} \frac{[\mu+1,q]_{n-1}}{[n-1,q]!} z^n, \qquad (z \in \Delta). \tag{1.2}$$

It is clear that the series defined in (1.2) is convergent absolutely in Δ. By making use of the definition of q-derivative along with the idea of convolution, we now define the integral operator $\zeta_q^\mu : \Delta \to \Delta$ by

$$\zeta_q^\mu f(z) = F_{q,\mu+1}^{-1}(z) * f(z) = z + \sum_{n=2}^{\infty} \Psi_{n-1} a_n z^n, \qquad (z \in \Delta), \tag{1.3}$$

where

$$\Psi_{n-1} = \frac{[n,q]!}{[\mu+1,q]_{n-1}}.$$

From (1.3), we get the identity

$$[\mu+1,q]\, \zeta_q^\mu f(z) = [\mu,q]\, \zeta_q^{\mu+1} f(z) + q^\mu z \partial_q \left(\zeta_q^{\mu+1} f(z)\right). \tag{1.4}$$

We note that $\zeta_q^0 f(z) = z\partial_q f(z),\ \zeta_q^1 f(z) = f(z)$, and

$$\lim_{q \to 1^-} \zeta_q^\mu f(z) = z + \sum_{n=2}^{\infty} \frac{n!}{(\mu+1)_{n-1}} a_n z^n.$$

This shows that, by taking $q \to 1^-$, the operator defined in (1.3) reduces to the familiar Noor integral operator introduced in [17].

For a function p analytic in Δ,

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}[A,B] \Leftrightarrow p(z) \prec \frac{1+Az}{1+Bz},$$

where $-1 \le B < A \le 1$. This class was established and studied by Janowski [10]. If $A = 1$ and $B = -1$, we have the class $\mathcal{P}$ of functions with a positive real part (see [6]). The classes $\mathcal{P}$ and $\mathcal{P}[A,B]$ are expressed by the relation

$$p(z) \in \mathcal{P} \Leftrightarrow \frac{(A+1)\,p(z) - (A-1)}{(B+1)\,p(z) - (B-1)} \in \mathcal{P}[A,B].$$

For $k \ge 0$, let us consider the classes $k - CV$ and $k - ST$ of k-uniformly convex functions and corresponding k-starlike functions, respectively, introduced by Kanas and Wisniowska. In [11, 13, 14] Kanas and Wisniowska defined the conic domain $\Omega_k, k \ge 0$ by

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

We note that Ω_k represents the conic region bounded successively by the imaginary axis ($k = 0$), the right branch of hyperbola ($0 < k < 1$), a parabola for $k = 1$, and ellipse for $k > 1$. The extremal functions for these conic regions are

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0 \\ 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k = 1 \\ 1 + \frac{2}{1-k^2} \sinh^2 \left\{\frac{2}{\pi}\ (\arccos k) \arctan h\sqrt{z}\right\}, & 0 < k < 1 \\ 1 + \frac{1}{k^2-1} \sin\left(\frac{\pi}{2R(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{\mathrm{d}x}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}\right) + \frac{1}{k^2-1}, & k > 1 \end{cases}, \tag{1.5}$$

where

$$u(z) = \frac{z - \sqrt{t}}{1 - \sqrt{t}z}, \quad (z \in \Delta)$$

and $t \in (0, 1)$ is chosen such that $k = \cosh(\pi R'(t)/(4R(t)))$. Here $R(t)$ is Legender's complete elliptic integral of first kind and $R'(t) = R(\sqrt{1-t^2})$ and $R'(t)$ is the complementary integral of $R(t)$ (see [8, 13, 14]). If $p_k(z) = 1 + Q_1(k) z + Q_2(k) z^2 + ...,\ z \in \Delta$, then it was shown in [14] that for (1.5) one can obtain

$$Q_1 := Q_1(k) = \begin{cases} \frac{2A^2}{1-k^2}, & 0 \le k < 1 \\ \frac{8}{\pi^2}, & k = 1 \\ \frac{\pi^2}{4(k^2-1)\sqrt{t}(1+t)R^2(t)}, & k > 1 \end{cases} \tag{1.6}$$

with $A = \frac{2}{\pi} \arccos t$.

The classes $k - UCV$ and $k - ST$ are defined as follows.
A function $f \in \mathcal{A}$ is said to be in the class $k - UCV$, if and only if

$$\frac{\left(zf'(z)\right)'}{f'(z)} \prec p_k(z), \quad (z \in \Delta,\ k \ge 0).$$

A function $f \in \mathcal{A}$ is said to be in the class $k - ST$, if and only if

$$\frac{zf'(z)}{f(z)} \prec p_k(z), \quad (z \in \Delta,\ k \ge 0).$$

For more study (see [26, 27, 28]. These classes were then generalized to $KD(k, \alpha)$ and $SD(k, \alpha)$ respectively by Shams et al. [21] subject to the conic domain $G(k, \alpha), k \ge 0, 0 \le \alpha < 1$, which is

$$G(k, \alpha) = \{w : \Re(w) > k|w - 1| + \alpha\}.$$

Now using the concepts of Janowski functions and the conic domain, Noor and Malik [18] define the following.

Definition 1.1. A function $p(z)$ is said to be in the class $k - \mathcal{P}[A, B]$, if and only if

$$p(z) \prec \frac{(A + 1)p_k(z) - (A - 1)}{(B + 1)p_k(z) - (B - 1)}, \quad k \ge 0,$$

where $p_k(z)$ is defined in (1.5) and $-1 \le B < A \le 1$.

Geometrically, the function $p \in k - \mathcal{P}[A, B]$ takes all values from the domain $\Omega_k[A, B]$, $1 \le B < A \le 1$, $k \ge 0$ which is defined as:

$$\Omega_k[A, B] = \left\{w : \Re\left(\frac{(B - 1)w - (A - 1)}{(B + 1)w - (A + 1)}\right) > k\left|\frac{(B - 1)w - (A - 1)}{(B + 1)w - (A + 1)} - 1\right|\right\},$$

or equivalently $\Omega_k[A,B]$ is a set of numbers $w=u+iv$ such that

$$\begin{aligned}&\left[\left(B^2-1\right)\left(u^2+v^2\right)-2\left(AB-1\right)u+\left(A^2-1\right)\right]^2\\ > \quad & k^2\left[\left(-2\left(B+1\right)\left(u^2+v^2\right)+2\left(A+B+2\right)u-2\left(A+1\right)\right)^2+4\left(A-B\right)^2v^2\right].\end{aligned}$$

This domain represents the conic type regains for detail see [18, 19] (see also [4, 5, 16, 29]). It can be easily seen that $0-\mathcal{P}\left[A,B\right]=\mathcal{P}\left[A,B\right]$ introduced in [10] and $k-\mathcal{P}\left[1,-1\right]=\mathcal{P}\left(p_k\right)$ introduced in [13].

Motivated by the recent work presented by Noor and Malik [18], we introduce a new class of analytic functions associated with conic domains and by using q-analogue of Noor integral operator.

Definition 1.2. A function $f\in\mathcal{A}$ is said to be in the class $k-\mathcal{U}_q^{\mu}[C,D]$, $k\geq 0$, $-1\leq D<C\leq 1$, if and only if

$$\Re\left(\frac{(D-1)T_q^{\mu}\left(z\right)-(C-1)}{(D+1)T_q^{\mu}\left(z\right)-(C+1)}\right)>k\left|\frac{(D-1)T_q^{\mu}\left(z\right)-(C-1)}{(D+1)T_q^{\mu}\left(z\right)-(C+1)}-1\right|,$$

where

$$T_q^{\mu}\left(z\right)=\frac{z\partial_q\left(\zeta_q^{\mu}f(z)\right)}{\zeta_q^{\mu}f(z)},$$

or equivalently

$$T_q^{\mu}\left(z\right)\in k-\mathcal{P}[C,D].$$

Remark 1.3. If we let $q\to 1^-$, and $\mu=1$, then the class $k-\mathcal{U}_q^{\mu}[C,D]$ reduces into the class $k-ST(C,D)$ introduced by Noor and Malik in [18].

Remark 1.4. Setting $q\to 1^-$, $C=1$, $D=-1$ and $\mu=1$, the class $k-\mathcal{U}_q^{\mu}[C,D]$ coincides with the class $k-ST$ introduced by Kanas and Wisniowska in [13, 14].

Remark 1.5. Setting $q\to 1^-$, $C=1-2\alpha$, $D=-1$ and $\mu=1$, the class $k-\mathcal{U}_q^{\mu}[C,D]$ coincides with the class $SD(k,\alpha)$ introduced by Shams et al. in [21].

Remark 1.6. It is easy to check that for $q\to 1^-$, $k=0$ and $\mu=1$, the class $k-\mathcal{U}_q^{\mu}[C,D]$ reduces into the class $S^*(C,D)$ introduced by Janowski [10].

Remark 1.7. If we let $q\to 1^-$, and $\mu=0$, then the class $k-\mathcal{U}_q{}^{\mu}[C,D]$ reduces into the class $k-UCV(C,D)$ introduced by Noor and Malik in [18].

Remark 1.8. Setting $q\to 1^-$, $C=1$, $D=-1$ and $\mu=0$, the class $k-\mathcal{U}_q^{\mu}[C,D]$ coincides with the class $k-UCV$ introduced by Kanas and Wisniowska in [13, 14].

Remark 1.9. Setting $q\to 1^-$, $C=1-2\alpha$, $D=-1$ and $\mu=0$, the class $k-\mathcal{U}_q^{\mu}[C,D]$ coincides with the class $KD(k,\alpha)$ introduced by Shams et al. in [21].

Remark 1.10. It is easy to check that for $q\to 1^-$, $k=0$ and $\mu=0$, the class $k-\mathcal{U}_q^{\mu}[C,D]$ reduces into the class $C(C,D)$ introduced by Janowski [10].

To demonstrate our first theorem, we express the following Lemmas.

Lemma 1.11. [20] Let $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be subordinate to $H(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$. If $H(z)$ is univalent in Δ and $H(\Delta)$ is convex, then

$$|c_n| \leq |C_1|, \qquad n \geq 1.$$

Lemma 1.12. [18] Let $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in k - \mathcal{P}[C, D]$, then

$$|c_n| \leq |Q_1(k, C, D)|, |Q_1(k, C, D)| = \frac{C - D}{2} |Q_1(k)|,$$

where $|Q_1(k)|$ is given by (1.6).

2 Main results

Theorem 2.1. $k - \mathcal{U}_q^{\mu+1}[C, D] \subset k - \mathcal{U}_q^{\mu}[C, D]$ for each $\mu > -1$.

Proof. Let

$$\frac{z\partial_q \left(\zeta_q^{\mu+1} f(z)\right)}{\zeta_q^{\mu+1} f(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) + \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \tag{2.1}$$

where $p(z)$ is analytic in Δ and $p(0) = 1$. Now from (1.4) we get

$$q^{\mu} z\partial_q \left(\zeta_q^{\mu+1} f(z)\right) = [\mu + 1, q] \zeta_q^{\mu} f(z) - [\mu, q] \zeta_q^{\mu+1} f(z).$$

This implies that

$$[\mu + 1, q] \frac{\zeta_q^{\mu} f(z)}{\zeta_q^{\mu+1} f(z)} = q^{\mu} \frac{z\partial_q \left(\zeta_q^{\mu+1} f(z)\right)}{\zeta_q^{\mu+1} f(z)} + [\mu, q]. \tag{2.2}$$

Differentiating logarithmically (2.2) and using (2.1), we obtain

$$\frac{z\partial_q \left(\zeta_q^{\mu} f(z)\right)}{\zeta_q^{\mu} f(z)} = p(z) + \frac{z\partial_q p(z)}{p(z) + \frac{[\mu, q]}{q^{\mu}}}. \tag{2.3}$$

Since $f \in k - \mathcal{U}_q^{\mu+1}[C, D]$, therefore

$$\frac{z\partial_q \left(\zeta_q^{\mu} f(z)\right)}{\zeta_q^{\mu} f(z)} = \left\{ p(z) + \frac{z\partial_q p(z)}{p(z) + \frac{[\mu, q]}{q^{\mu}}} \right\} \in k - \mathcal{P}[C, D].$$

Q.E.D.

Theorem 2.2. A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k - \mathcal{U}_q^{\mu}[C, D]$, if it satisfies the condition

$$\sum_{n=2}^{\infty} \Psi_{n-1} \left\{ 2(k + 1) ([n, q] - 1) + |(D + 1) [n, q] - (C + 1)| \right\} |a_n| \leq C - D, \tag{2.4}$$

where $-1 \leq D < C \leq 1$, $k \geq 0$.

Proof. Assuming that (2.4) holds, then it suffices to show that

$$k\left|\frac{(D-1)T_q^{\mu}(z)-(C-1)}{(D+1)T_q^{\mu}(z)-(C+1)}-1\right|-\Re\left\{\frac{(D-1)T_q^{\mu}(z)-(C-1)}{(D+1)T_q^{\mu}(z)-(C+1)}-1\right\}<1.$$

We have

$$\begin{aligned}
& k\left|\frac{(D-1)T_q^{\mu}(z)-(C-1)}{(D+1)T_q^{\mu}(z)-(C+1)}-1\right|-\Re\left\{\frac{(D-1)T_q^{\mu}(z)-(C-1)}{(D+1)T_q^{\mu}(z)-(C+1)}-1\right\}\\
\leq\ & (k+1)\left|\frac{(D-1)T_q^{\mu}(z)-(C-1)}{(D+1)T_q^{\mu}(z)-(C+1)}-1\right|\\
=\ & (k+1)\left|\frac{(D-1)z\partial_q\left(\zeta_q^{\mu}f(z)\right)-(C-1)\zeta_q^{\mu}f(z)}{(D+1)z\partial_q\left(\zeta_q^{\mu}f(z)\right)-(C+1)\zeta_q^{\mu}f(z)}-1\right|\\
=\ & 2(k+1)\left|\frac{\zeta_q^{\mu}f(z)-z\partial_q\left(\zeta_q^{\mu}f(z)\right)}{(D+1)z\partial_q\left(\zeta_q^{\mu}f(z)\right)-(C+1)\zeta_q^{\mu}f(z)}\right|\\
=\ & 2(k+1)\left|\frac{\sum_{n=2}^{\infty}\Psi_{n-1}\left([n,q]-1\right)a_nz^n}{(D-C)\,z+\sum_{n=2}^{\infty}\Psi_{n-1}\left[(D+1)\,[n,q]-(C+1)\right]a_nz^n}\right|\\
<\ & 2(k+1)\left\{\frac{\sum_{n=2}^{\infty}\Psi_{n-1}\left([n,q]-1\right)|a_n|}{C-D-\sum_{n=2}^{\infty}\Psi_{n-1}\left|(D+1)\,[n,q]-(C+1)\right||a_n|}\right\}.
\end{aligned}$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}\Psi_{n-1}\left\{2(k+1)\left([n,q]-1\right)+\left|(D+1)\,[n,q]-(C+1)\right|\right\}|a_n|\leq C-D.$$

Hence the proof is completed. Q.E.D.

Corollary 2.3. (see [18]) A function $f\in\mathcal{A}$ and of the form (1.1) is in the class $k-ST(C,D)$, if it satisfies the condition

$$\sum_{n=2}^{\infty}\left\{2(k+1)(n-1)+|n(D+1)-(C+1)|\right\}|a_n|\leq C-D.$$

Corollary 2.4. (see [14])If for a function f of the form (1.1) the condition

$$\sum_{n=2}^{\infty}\left\{n(k+1)-k\right\}|a_n|\leq 1$$

holds true for some k, $0\leq k<\infty$, then $f\in k-ST$.

Corollary 2.5. (see [21]) A function $f\in\mathcal{A}$ and of the form (1.1) is in the class $SD(k,\alpha)$, if it satisfies the condition

$$\sum_{n=2}^{\infty}\left[n(k+1)-(k+\alpha)\right]|a_n|\leq 1-\alpha,$$

where $0\leq\alpha<1$ and $k\geq 0$.

Corollary 2.6. (see [24]) A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $S^*(\alpha)$, if it satisfies the condition

$$\sum_{n=2}^{\infty}(n-\alpha)\left|a_n\right| \leq 1-\alpha, \qquad (0 \leq \alpha < 1).$$

Corollary 2.7. (see [18])A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k-UCV(C,D)$, if it satisfies the condition

$$\sum_{n=2}^{\infty} n\left\{2(k+1)(n-1)+|n(D+1)-(C+1)|\right\}\left|a_n\right| \leq C-D.$$

Corollary 2.8. (see [11])If for a function f of the form (1.1) the condition

$$\sum_{n=2}^{\infty} n\left\{n(k+1)-k\right\}\left|a_n\right| \leq C-D$$

holds true for some k, $0 \leq k < \infty$, then $f \in k-UCV$.

Corollary 2.9. (see [21]) A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $KD(k,\alpha)$, if it satisfies the condition

$$\sum_{n=2}^{\infty} n\left[n(k+1)-(k+\alpha)\right]\left|a_n\right| \leq 1-\alpha,$$

where $0 \leq \alpha < 1$ and $k \geq 0$.

Corollary 2.10. (see [24]) A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $K(\alpha)$, if it satisfies the condition

$$\sum_{n=2}^{\infty} n(n-\alpha)\left|a_n\right| \leq 1-\alpha, \qquad (0 \leq \alpha < 1).$$

Theorem 2.11. If $f \in k-\mathcal{U}_q^{\mu}[C,D]$ and is of the form (1.1). Then

$$\left|a_n\right| \leq \prod_{j=0}^{n-2}\left(\frac{|Q_1(k)(C-D)-2\Psi_j\left\{[j+1,q]-1\right\}D|}{2\Psi_j\left\{[j+1,q]+1\right\}}\right), \qquad n \geq 2, \tag{2.5}$$

where $|Q_1(k)|$ is defined by (1.6).

Proof. By definition, for $f(z) \in k-\mathcal{U}_q^{\mu}[C,D]$, we have

$$\frac{z\partial_q\left(\zeta_q^{\mu} f(z)\right)}{\zeta_q^{\mu} f(z)} = p(z), \tag{2.6}$$

where

$$p(z) \in k-\mathcal{P}[C,D].$$

Now from (2.6), we have

$$z\partial_q\left(\zeta_q^{\mu} f(z)\right) = \zeta_q^{\mu} f(z) p(z),$$

which implies that

$$
\begin{aligned}
z+\sum_{n=2}^{\infty}\Psi_{n-1}[n,q]\,a_nz^n &= \left(1+\sum_{n=1}^{\infty}c_nz^n\right)\left(z+\sum_{n=2}^{\infty}\Psi_{n-1}a_nz^n\right)\\
z+\sum_{n=2}^{\infty}\Psi_{n-1}[n,q]\,a_nz^n &= \left(1+\sum_{n=1}^{\infty}c_nz^n\right)\left(\sum_{n=1}^{\infty}\Psi_{n-1}a_nz^n\right)\\
z+\sum_{n=2}^{\infty}\Psi_{n-1}[n,q]\,a_nz^n &= \sum_{n=1}^{\infty}\Psi_{n-1}a_nz^n+\left(\sum_{n=1}^{\infty}\Psi_{n-1}a_nz^n\right)\left(\sum_{n=1}^{\infty}c_nz^n\right)\\
\sum_{n=2}^{\infty}\Psi_{n-1}\{[n,q]-1\}\,a_nz^n &= \left(\sum_{n=1}^{\infty}\Psi_{n-1}a_nz^n\right)\left(\sum_{n=1}^{\infty}c_nz^n\right). \qquad (2.7)
\end{aligned}
$$

By using Cauchy product formula on R.H.S of (2.7), we obtain

$$\sum_{n=2}^{\infty}\Psi_{n-1}\{[n,q]-1\}\,a_nz^n=\sum_{n=2}^{\infty}\left[\sum_{j=1}^{n-1}\Psi_{j-1}a_jc_{n-j}\right]z^n. \qquad (2.8)$$

Equating coefficients of z^n on both sides of (2.8), we have

$$\Psi_{n-1}\{[n,q]-1\}\,a_n=\sum_{j=1}^{n-1}\Psi_{j-1}a_jc_{n-j}, \qquad [1]_q^m=1,\ a_1=1.$$

This implies that

$$|a_n|\leq\frac{1}{\Psi_{n-1}\{[n,q]-1\}}\sum_{j=1}^{n-1}\Psi_{j-1}|a_j|\,|c_{n-j}|, \quad \Psi_0=1,\ a_1=1.$$

Using lemma (1.12), we find that

$$|a_n|\leq\frac{|Q_1(k)|\,(C-D)}{2\Psi_{n-1}\{[n,q]-1\}}\sum_{j=1}^{n-1}\Psi_{j-1}|a_j|. \qquad (2.9)$$

Now we prove that

$$
\begin{aligned}
&\frac{|Q_1(k)|\,(C-D)}{2\Psi_{n-1}\{[n,q]-1\}}\sum_{j=1}^{n-1}\Psi_{j-1}|a_j|\\
\leq\ &\prod_{j=1}^{n-1}\left(\frac{|Q_1(k)(C-D)-2\Psi_{j-1}\{[j,q]-1\}D|}{2\Psi_j\{[j+1,q]-1\}}\right),\\
&\frac{|Q_1(k)|\,(C-D)}{2\Psi_{n-1}\{[n,q]-1\}}\sum_{j=1}^{n-1}\Psi_{j-1}|a_j|\\
\leq\ &\prod_{j=0}^{n-2}\left(\frac{|Q_1(k)(C-D)-2\Psi_j\{[j+1,q]-1\}D|}{2\Psi_{j+1}\{[j+2,q]-1\}}\right).
\end{aligned}
$$

For this, we use the induction method.
For $n=2$ from (2.9), we have

$$|a_2|\leq\frac{|Q_1(k)|\,(C-D)\,[\mu+1,q]}{2q(1+q)!}.$$

From (2.5), we have

$$|a_2| \leq \frac{|Q_1(k)|\,(C-D)\,[\mu+1,q]}{2q(1+q)!}.$$

For $n=3$ from (2.9), we have

$$\begin{aligned}
|a_3| &\leq \frac{|Q_1(k)|\,(C-D)}{2\Psi_{n-1}\{[n,q]-1\}}\{1+\Psi_1|a_2|\} \\
&\leq \frac{|Q_1(k)|\,(C-D)\,[\mu+1,q]\,[\mu+2,q]}{2q(1+q)(1+q+q^2)!}\left\{1+\frac{|Q_1(k)|\,(C-D)}{2q}\right\}.
\end{aligned}$$

From (2.5), we have

$$\begin{aligned}
|a_3| &\leq \frac{(C-D)\,|Q_1(k)|}{2\Psi_1\{[2,q]-1\}}\left\{\frac{|Q_1(k)(C-D)-2\Psi_{n-1}\{[2,q]-1\}\,D|}{2\Psi_2\{[3,q]-1\}}\right\} \\
&\leq \frac{(C-D)\,|Q_1(k)|}{2\Psi_1\{[2,q]-1\}}\left\{\frac{|Q_1(k)|\,(C-D)+2\Psi_{n-1}\{[2,q]-1\}\,|D|}{2\Psi_2\{[3,q]-1\}}\right\} \\
&\leq \frac{(C-D)\,|Q_1(k)|\,[\mu+1,q]\,[\mu+2,q]}{2q(1+q)(1+q+q^2)!}\left\{1+\frac{|Q_1(k)|\,(C-D)\,[\mu+1,q]}{2q(1+q)!}\right\}.
\end{aligned}$$

Let the hypothesis be true for $n=t$.
From (2.9), we have

$$|a_t| \leq \frac{|Q_1(k)|\,(C-D)}{2\Psi_{t-1}\{[t,q]-1\}}\sum\nolimits_{j=1}^{t-1}\Psi_{j-1}|a_j|.$$

From (2.5), we have

$$\begin{aligned}
|a_t| &\leq \prod\nolimits_{j=0}^{n-2}\left(\frac{|Q_1(k)(C-D)-2\Psi_j\{[j+1,q]-1\}\,D|}{2\Psi_j\{[j+1,q]+1\}}\right) \\
&\leq \prod\nolimits_{j=0}^{t-2}\left(\frac{|Q_1(k)(C-D)+2\Psi_j\{[j+1,q]-1\}|}{2\Psi_j\{[j+1,q]+1\}}\right).
\end{aligned}$$

By the induction hypothesis, we conclude that

$$\begin{aligned}
&\frac{|Q_1(k)|\,(C-D)}{2\Psi_{t-1}\{[t,q]-1\}}\sum\nolimits_{j=1}^{t-1}\Psi_{j-1}|a_j| \\
&\leq \prod\nolimits_{j=0}^{t-2}\left(\frac{|Q_1(k)(C-D)+2\Psi_j\{[j+1,q]-1\}|}{2\Psi_j\{[j+1,q]+1\}}\right). \qquad (2.10)
\end{aligned}$$

Multiplying both sides by

$$\frac{|Q_1(k)(C-D)+2\Psi_{t-1}\{[t,q]-1\}|}{2\Psi_j\{[j+1,q]+1\}},$$

we have

$$\begin{aligned}
&\frac{|Q_1(k)(C-D)+2\Psi_{t-1}\{[t,q]-1\}|}{2\Psi_j\{[j+1,q]+1\}}\times\prod_{j=0}^{t-2}\left(\frac{|Q_1(k)(C-D)+2\Psi_j\{[j+1,q]-1\}|}{2\Psi_j\{[j+1,q]+1\}}\right)\\
\geq\ &\left\{\frac{|Q_1(k)(C-D)+2\Psi_{t-1}\{[t,q]-1\}|}{2\Psi_j\{[j+1,q]+1\}}\right\}\times\frac{|Q_1(k)|(C-D)}{2\Psi_{t-1}\{[t,q]-1\}}\sum_{j=1}^{t-1}\Psi_{j-1}|a_j|,\\
&\prod_{j=0}^{t-2}\left(\frac{|Q_1(k)(C-D)+2\Psi_j\{[j+1,q]-1\}|}{2\Psi_j\{[j+1,q]+1\}}\right)\\
\geq\ &\left\{\begin{array}{l}\frac{|Q_1(k)|(C-D)}{2\Psi_j\{[j+1,q]+1\}}\left\{\frac{|Q_1(k)|(C-D)}{2\Psi_{t-1}\{[t,q]-1\}}\sum_{j=1}^{t-1}\Psi_{j-1}|a_j|\right\}\\ +\frac{2\Psi_{t-1}\{[t,q]-1\}}{2\Psi_j\{[j+1,q]+1\}}\left\{\frac{|Q_1(k)|(C-D)}{2\Psi_{t-1}\{[t,q]-1\}}\sum_{j=1}^{t-1}\Psi_{j-1}|a_j|\right\},\end{array}\right.\\
\geq\ &\frac{|Q_1(k)|(C-D)}{2\Psi_j\{[j+1,q]+1\}}\left\{\frac{|Q_1(k)|(C-D)}{2\Psi_{t-1}\{[t,q]-1\}}\sum_{j=1}^{t-1}\Psi_{j-1}|a_j|+\sum_{j=1}^{t-1}\Psi_{j-1}|a_j|\right\},\\
&\prod_{j=0}^{t-2}\left(\frac{|Q_1(k)(C-D)+2\Psi_j\{[j+1,q]-1\}|}{2\Psi_j\{[j+1,q]+1\}}\right)\\
\geq\ &\frac{|Q_1(k)|(C-D)}{2\Psi_j\{[j+1,q]+1\}}\left\{|a_t|+\sum_{j=1}^{t-1}\Psi_{j-1}|a_j|\right\},\\
=\ &\frac{|Q_1(k)|(C-D)}{2\Psi_j\{[j+1,q]+1\}}\sum_{j=1}^{t-1}\Psi_{j-1}|a_j|.
\end{aligned}$$

That is,

$$\begin{aligned}
&\frac{|Q_1(k)|(C-D)}{2\Psi_j\{[j+1,q]+1\}}\sum_{j=1}^{t-1}\Psi_{j-1}|a_j|\\
\leq\ &\prod_{j=0}^{t-2}\left(\frac{|Q_1(k)(C-D)+2\Psi_j\{[j+1,q]-1\}|}{2\Psi_j\{[j+1,q]+1\}}\right)
\end{aligned}$$

which shows that inequality (2.10) is true for $n=t+1$. Hence the required result. Q.E.D.

Corollary 2.12. (see [18]) A function $f\in\mathcal{A}$ and of the form (1.1) is in the class $k-ST(C,D)$, if it satisfies the condition

$$|a_n|\leq\prod_{j=0}^{n-2}\left(\frac{|Q_1(k)(C-D)-2jD|}{2(j+1)}\right).$$

Corollary 2.13. (see [14]) A function $f\in\mathcal{A}$ and of the form (1.1) is in the class $k-ST$, if it satisfies the condition

$$|a_n|\leq\prod_{j=0}^{n-2}\left(\frac{|Q_1(k)+j|}{j+1}\right).$$

Corollary 2.14. (see [21]) A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $SD(k,\alpha)$, if it satisfies the condition

$$|a_n| \leq \prod_{j=0}^{n-2} \left(\frac{|Q_1(k)(1-\alpha)+j|}{j+1} \right), \qquad (-1 \leq D < C \leq 1).$$

Corollary 2.15. (see [6]) A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $S^*(\alpha)$, if it satisfies the condition

$$|a_n| \leq \frac{\prod_{j=2}^{n}(j-2\alpha)}{(n-1)!}, \qquad (0 \leq \alpha < 1).$$

Corollary 2.16. (see [10]) A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $S^*(C,D)$, if it satisfies the condition

$$|a_n| \leq \prod_{j=0}^{n-2} \left(\frac{|(C-D)-jD|}{j+1} \right), \qquad (-1 \leq D < C \leq 1).$$

Corollary 2.17. (see [18]) A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k-UCV(C,D)$, if it satisfies the condition

$$|a_n| \leq \frac{1}{n} \prod_{j=0}^{n-2} \left(\frac{|Q_1(k)(C-D)-2jD|}{2\,(j+1)} \right).$$

Corollary 2.18. (see [11]) A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $k-UCV$, if it satisfies the condition

$$|a_n| \leq \frac{1}{n} \prod_{j=0}^{n-2} \left(\frac{|Q_1(k)+j|}{j+1} \right).$$

Corollary 2.19. (see [21]) A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $KD(k,\alpha)$, if it satisfies the condition

$$|a_n| \leq \frac{1}{n} \prod_{j=0}^{n-2} \left(\frac{|Q_1(k)(1-\alpha)+j|}{j+1} \right), \qquad (-1 \leq D < C \leq 1).$$

Corollary 2.20. (see [6]) A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $C(\alpha)$, if it satisfies the condition

$$|a_n| \leq \frac{\prod_{j=2}^{n}(j-2\alpha)}{n!}, \qquad (0 \leq \alpha < 1).$$

Corollary 2.21. (see [10]) A function $f \in \mathcal{A}$ and of the form (1.1) is in the class $K(C,D)$, if it satisfies the condition

$$|a_n| \leq \frac{1}{n} \prod_{j=0}^{n-2} \left(\frac{|(C-D)-jD|}{j+1} \right), \qquad (-1 \leq D < C \leq 1).$$

References

[1] Ahmad K, Arif M, Liu J-L. *Convolution properties for a family of analytic functions involving q-analogue of Ruscheweyh differential operator*. Turk. J. Math. 2019; 43: 1712-1720.

[2] Arif M, Ul Haq M, Liu J-L. *Subfamily of univalent functions associated with q-analogue of Noor integral operator*. Journal of Function Spaces 2018; 2018: 1-5.

[3] Arif M, Srivastava HM, Umar S. *Some applications of a q-analogue of the Ruscheweyh type operator for multivalent functions*. RACSAM 2019; 113: 1211-1221.

[4] Arif M, Mahmood S, Sokol J, Dziok J. *New subclass of analytic functions in conical domain associated with a linear operator*. Acta Mathematica Scientia 2016; 36: 704-7016.

[5] Arif M, Wang ZG, Khan R, Lee SK. *Coefficient inequalities for Janowski-Sakaguchi type functions associated with conic regions*. Hacettepe Journal of Mathematics and Statistics 2018; 47: 261-271.

[6] Goodman AW. *Univalent Functions*. Vol. I. Polygonal Publishing House, Washington, 1983.

[7] Haq M, Raza M, Arif M, Khan Q, Tang H. *q-analogue of differential subordinations*. Mathematics 2019; 7(724): 1-16.

[8] Hussain S, Khan S, Zaighum MA, Darus M. *Certain subclass of analytic functions related with conic domains and associated with Salagean q-differential operator*. AIMS Mathematics 2017; 2 (4): 622-634.

[9] Jackson FH. *On q-functions and a certain difference operator*. Transactions of the Royal Society of Edinburgh 1908; 46: 253-281.

[10] Janowski W. *Some extremal problems for certain families of analytic functions*. Ann. Polon. Math. 1973; 28: 297-326.

[11] Kanas S. *Techniques of the differential subordination for domains bounded by conic sections*. Int. J. Math. Math. Sci. 2003; 38: 2389-2400.

[12] Kanas S, Raducanu D. *Some class of analytic functions related to conic domains*. Math. Slovaca 2014; 64: 1183–1196.

[13] Kanas S, Wisniowska A. *Conic regions and k-uniform convexity.* J. Comput. Appl. Math. 1999; 105: 327-336.

[14] Kanas S, Wisniowska A. *Conic domains and starlike functions*. Rev. Roumaine Math. Pures Appl. 2000; 45: 647-657.

[15] Khan Q, Arif M, Raza M, Srivastava G, Tang H, Rahman S. *Some applications of a new integral operator in q-analog for multivalent functions*. Mathematics 2019; 7: 1-13.

[16] Mahmood S, Arif M, Malik SN. *Janowski type close-to-convex functions associated with conic regions*. Journal of Inequalities and Applications 2017; 259: 1-14.

[17] Noor KI. *On new classes of integral operators*. Journal of Natural Geometry 1999; 16: 71–80.

[18] Noor KI, Malik SN. *On coefficient inequalities of functions associated with conic domains*. Comput. Math. Appl. 2011; 62: 2209-2217.

[19] Noor KI, Sokoł J, Ahmad QZ. *Applications of conic type regions to subclasses of meromorphic univalent functions with respect to symmetric points*. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2016; 111 (4): 947-958.

[20] Rogosinski W. *On the coefficients of subordinate functions*. Proc. Lond. Math. Soc. 1943; 48: 48-82.

[21] Shams S, Kulkarni SR, Jahangiri JM. *Classes of uniformly starlike and convex functions*. Int. J. Math. Math. Sci. 2004; 55: 2959-2961.

[22] Shi L, Khan Q, Srivastava G, Liu J-L, *Arif M. A study of multivalent q-starlike functions connected with circular domain*. Mathematics 2019; 7(670): 1-12.

[23] Shi L, Raza M, Javed K, Hussain S, Arif M. *Class of analytic functions defined by q-integral operator in a symmetric region*. Symmetry 2019; 11(1042): 1-15.

[24] Silverman H. *Univalent functions with negative coefficients*. Proc. Amer. Math. Soc. 1975; 51: 109-116.

[25] Srivastava HM. *Univalent functions, fractional calculus, and associated generalized hypergeometric functions, in Univalent Functions; Fractional Calculus; and Their Applications* (H. M. Srivastava and S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[26] Srivastava HM, Sim YJ, Kwon OS, Cho NE. *Some classes of analytic functions associated with conic regions*. Taiwanese J. Math. 2012; 16: 387-408.

[27] Srivastava HM, Li S-H, Tang H. *Certain classes of k-uniformly close-to-convex functions and other related functions defined by using the Dziok-Srivastava operator*. Bull. Math. Anal. Appl. 2009; 1: 49-63.

[28] Srivastava HM, Ramachandran C, Swaminathan A. *A unified class of k-uniformly convex functions defined by the Salagean derivative operator*. Atti Sem. Mat. Fis. Modena Reggio Emilia 2007; 55: 47-59.

[29] Srivastava HM, Rafiullah M, Arif M. *Some subclasses of close-to-convex mappings associated with conic regions*. Applied Mathematics & Computation 2016; 285; 94-102.