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Comparing minimal simplicial models

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Abstract We compare minimal combinatorial models of homotopy types: arbitrary simplicial complexes, flag complexes and order complexes. Flag complexes are the simplicial complexes which do not have the boundary of a simplex of dimension greater than one as an induced subcomplex. Order complexes are classifying spaces of posets and they correspond to models in the category of finite T_0-spaces. In particular, we prove that stably, that is after a suitably large suspension, the optimal flag complex representing a homotopy type is approximately twice as big as the optimal simplicial complex with that property (in terms of the number of vertices). We also investigate some related questions.

Keywords Triangulation · Simplicial complex · Minimal model · Homotopy type

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1 Introduction

Whenever we have a combinatorial category which models topological spaces we can ask about the minimal size of models. In this short note we study the following numbers defined for a topological space X:

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$$
\begin{aligned}
m_s(X) &= \min\{\#V(K): \ K \text{ is a simplicial complex with } K \simeq X\},\\
m_f(X) &= \min\{\#V(K): \ K \text{ is a flag simplicial complex with } K \simeq X\},\\
m_p(X) &= \min\{\#P: \ P \text{ is a poset with } \Delta(P) \simeq X\}.
\end{aligned} \tag{1}
$$

The symbol $\simeq$ denotes homotopy equivalence. We do not distinguish between an abstract simplicial complex and its geometric realization. The notation $\Delta(P)$ stands for the *order complex* of a poset P, that is the simplicial complex whose faces are the chains of P. Its geometric realization is the classifying space $\mathcal{B}P$ of P. By $\#V(K)$ (resp. $\#P$) we denote the number of vertices in K (resp. the number of elements in P). A simplicial complex is *flag* if its every minimal non-face is of dimension 1 or, equivalently, if it is the maximal simplicial complex with the given 1-skeleton. Moreover, by the results of McCord and Stong (see [12,10]) $m_p(X)$ is equal to the minimal number of points in a finite T_0-space weakly equivalent to X. We say a space X is *of finite type* if $m_s(X) < \infty$. Computing the values of $m_s(X)$ and $m_f(X)$ is related to the rather classical problem of finding minimal triangulations of spaces (here up to homotopy). The properties of $m_p(X)$ were studied in [3] through the perspective of finite T_0-spaces. Note that by definition $m_s(X)$, $m_f(X)$ and $m_p(X)$ are invariants of the homotopy type of X.

We have the following obvious inequalities

$$
m_s(X) \le m_f(X) \le m_p(X) \le 2^{m_s(X)}. \tag{2}
$$

The second one follows since the order complex of a poset is always flag, and the third one is a consequence of the fact that every simplicial complex is homeomorphic to the order complex of its own face poset. One motivation for this work is to see how far $m_f(X)$ can exceed $m_s(X)$.

Define the *homological dimension* of X as

$$
h(X) = \max\{k: \ \widetilde{H}_k(X;\Lambda) \ne 0 \text{ for some group } \Lambda\} \tag{3}
$$

where $\widetilde{H}_k(X;\Lambda)$ denotes the reduced homology groups with coefficients in Λ. If X is acyclic, i.e. all its reduced homology groups vanish, we leave $h(X)$ undefined. Then we have the following proposition.

Proposition 1.1 *For any non-empty, non-acyclic space X we have*

$$
m_s(X) \ge h(X) + 2, \quad m_f(X) \ge 2h(X) + 2, \quad m_p(X) \ge 2h(X) + 2.
$$

Moreover, in each of those inequalities equality holds if and only if $X \simeq S^n$ for some $n \ge 0$.

The statement about $m_s(X)$ is obvious, the one about $m_f(X)$ can be found in [6] and the weaker inequality for $m_p(X)$ follows independently from the results of [3]. For completeness we will provide a short proof in the next section.

The last statement implies, in particular, that $\lim_{k\to\infty} m_f(S^k)/m_s(S^k) = 2$. We will prove that an analogous result holds for suspensions of any space of finite type.