J. Homotopy Relat. Struct. (2013) 8:13–34
DOI 10.1007/s40062-012-0011-6

Pre-c-symplectic condition for the product of odd-spheres

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Received: 22 July 2011 / Accepted: 20 June 2012 / Published online: 12 July 2012

Abstract We say that a simply connected space X is *pre-c-symplectic* if it is the fibre of a rational fibration $X \to Y \to \mathbb{C}P^{\infty}$ where Y is cohomologically symplectic in the sense that there is a degree 2 cohomology class which cups to a top class. It is a rational homotopical property but not a cohomological one. By using Sullivan's minimal models (Félix et al. in Rational homotopy theory. Graduate Texts in Mathematics, vol. 205. Springer, Berlin, 2001), we give the necessary and sufficient condition that the product of odd-spheres $X = S^{k_1} \times \cdots \times S^{k_n}$ is pre-c-symplectic and see some related topics. Also we give a charactarization of the Hasse diagram of rational toral ranks for a space X (Yamaguchi in Bull Belg Math Soc Simon Stevin 18:493–508, 2011) as a necessary condition to be pre-c-symplectic and see some examples in the cases of finite-oddly generated rational homotopy groups.

Keywords Symplectic · c-Symplectic · Pre-c-symplectic · Sullivan model · Rational homotopy type · Almost free toral action · Rational toral rank · Hasse diagram of rational toral ranks · KS-model · Elliptic · Formal

Mathematics Subject Classification (2010) 55P62 · 53D05

Communicated by Paul Goerss.

Our definition of *pre-c-symplectic* is completely different from usual one of *presymplectic* (cf. [12,14]).

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1 Introduction

Recall the question:"*If a symplectic manifold is a nilpotent space, what special homotopical properties are apparent? Conversely, what nilpotent spaces have symplectic or c-symplectic structures?*" [9, (4.99)]. Here a rationally Poincaré dual space Y (the graded algebra $H^*(Y;\mathbb{Q})$ is a Poincaré duality algebra [9, Def. 3.1]) with formal dimension

$$fd(Y) := \max\{i \mid H^i(Y;\mathbb{Q}) \neq 0\}$$

$= 2n$ is said to be *c-symplectic (cohomologically symplectic)* if there is a rational cohomology class $\omega \in H^2(Y;\mathbb{Q})$ such that ω^n is a top class for Y [9, Def. 4.87] [22,29], many of which are known to be realized by $2n$-dimensional smooth manifolds [9]. A lot of results on the above problem and related topics are given in rational homotopy theory (cf. [5,6,9,15,16,18–21,29]). For example, Lupton and Oprea [20] study the formalising tendency of certain symplectic manifolds using techniques of D.Sullivan's rational model [28]. Notice that it is known that the connected sum $\mathbb{C}P^2\sharp\mathbb{C}P^2$ is c-symplectic but not symplectic [4] [21, p. 263], for the n-dimensional complex projective space $\mathbb{C}P^n$. In [15,18] [22, Theorem 6.3] [30], we can see conditions that a total space with a degree 2 cohomology class admits a symplectic structure in a certain fibration. But we don't mention anything about symplectic geometry in this paper.

For a simply connected c-symplectic space Y, we have $\omega \in Hom(\pi_2(Y),\mathbb{Q})$ for the class ω of $H^2(Y;\mathbb{Q})$ from Hurewicz isomorphism. In particular, $\pi_2(Y)\otimes\mathbb{Q} \neq 0$. So there is a simply connected space X that is the fibre of a fibration

$$X \to Y \to \mathbb{C}P^\infty \tag{1}$$

where $\mathbb{C}P^\infty = \cup_{n=1}^\infty \mathbb{C}P^n (= K(\mathbb{Z},2))$, $\pi_*(X)\otimes\mathbb{Q}\oplus\mathbb{Q}\cdot t^* = \pi_*(Y)\otimes\mathbb{Q}$ for a cohomology element t with $\deg(t) = 2$ (necessarily we don't need $t = \omega$) and $H^*(\mathbb{C}P^\infty;\mathbb{Q}) = \mathbb{Q}[t]$.

Definition 1.1 We say a simply connected space X to be *pre-c-symplectic* (*pre-cohomologically symplectic*) if X is the fibre of a fibration (1) where Y is c-symplectic.

For example, $\mathbb{C}P^n$ is a symplectic manifold, whose pre-c-symplectic space must be the $2n+1$-dimensional sphere S^{2n+1}. It is induced by the Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ [1, p. 95]. We know that $fd(Y) = 2n$ if and only if $fd(X) = 2n+1$ in (1) from the Gysin exact sequence of of the induced fibration $S^1 \to X \to Y$. When $\dim \pi_2(Y)\otimes\mathbb{Q} > 1$, (1) may not be rational homotopically unique for Y. For example, when Y is $S^2\times\mathbb{C}P^2$, two spaces $S^3\times\mathbb{C}P^2$ and $S^2\times S^5$ are both its pre-c-symplectic spaces (there are three pre-c-symplectic spaces in the case of [20, Example 2.12]). The being c-symplectic and the being pre-c-symplectic are complementary. If a space is c-symplectic, it is not pre-c-symplectic and moreover if a space is pre-c-symplectic, it is not c-symplectic. The being c-symplectic is preserved by product; i.e., $Y_1\times Y_2$ is pre-c-symplectic by the class $\omega_1+\omega_2$ when Y_1 and Y_2 are both c-symplectic by classes ω_1 and ω_2, respectively. But the being pre-c-symplectic can not since then the formal dimension is even.