J. Homotopy Relat. Struct. (2013) 8:1–12
DOI 10.1007/s40062-012-0008-1

Koszul duality in algebraic topology

Dev P. Sinha

Received: 18 October 2011 / Accepted: 20 March 2012 / Published online: 16 May 2012

Abstract The most prevalent examples of Koszul duality of operads are the self-duality of the associative operad and the duality between the Lie and commutative operads. At the level of algebras and coalgebras, the former duality was first noticed as such by Moore, as announced in his ICM talk at Nice (Moore in Actes du Congrès International des Mathématiciens, Tome 1. Gauthier-Villars, Paris, pp. 335–339, 1971). This particular duality has typically been called Moore duality, and some prefer to call the general phenomenon Koszul–Moore duality. The second duality at the level of algebras was realized in the seminal work of Quillen on rational homotopy theory (Quillen in Ann Math 90(2):205–295, 1969). Our aim in these notes based on our talk at the Luminy workshop on Operads in 2009 is to try to provide some historical, topological context for these two classical algebraic dualities. We first review the original cobar and bar constructions used to study loop spaces and classifying spaces, emphasizing the less-familiar geometry of the cobar construction. Then, after some elementary topology, we state duality between bar and cobar complexes in that setting. Before explaining Quillen's work, we also share some other ideas—calculations of Cartan–Serre and Milnor–Moore and philosophy of Eckmann–Hilton—which may have influenced him. After stating Quillen's duality, we share some recent work which relates these constructions to geometry through Hopf invariants and in particular linking phenomena.

Keywords Koszul-Moore duality · Bar and cobar constructions · Operads · Hopf invariants

---

Communicated by Jim Stasheff.

D. P. Sinha (✉)
Department of Mathematics, University of Oregon,
Eugene, OR 97403, USA
e-mail: dps@math.uoregon.edu

1 Bar and cobar constructions

1.1 ΩX and the cobar construction

Studying mapping spaces is one of the central tasks of topology, and loop spaces are the simplest and most fundamental examples (unless one counts maps from finite sets, which yield products). We require a model for loops where the loop sum is associative exactly, not up to homotopy. For us ΩX denotes the Moore loop space which consists of pairs $f : \mathbb{R} \to X$ and a "curfew" $a > 0$ such that $f(x)$ is the basepoint if $x \leq 0$ or if $x \geq a$. Loop sum adds these curfews, which makes multiplication associative.

The cobar construction of Adams and Hilton [2] was informed by the almost concurrent work of James [10] who studied $\Omega\Sigma X$, the loop space on the reduced suspension of X, namely $\Sigma X = X \times \mathbb{I}/(X \times 0 \cup * \times \mathbb{I} \cup 1 \times X)$. There is a canonical inclusion of $J : X \hookrightarrow \Omega\Sigma X$ sending x to $J(x)(t)$, the path which sends t to the image in ΣX of (x, t). Because $\Omega\Sigma X$ is a topological monoid, this map extends to a map from the free moinoid (with unit) on X to $\Omega\Sigma X$ which we call the James map $\hat{J}$. For example, the formal product $y * x * z$ goes to a loop with coordinates (x, t) for $t \in [0, 1]$ then $(y, t - 1)$ for $t \in [1, 2]$, then $(z, t - 2)$ for $t \in [2, 3]$—see Fig. 1 below.

Theorem 1.1 (James [10]) *The James map $\hat{J}$ from the free monoid on X to $\Omega\Sigma X$ is a homotopy equivalence.*

Recall that the homology of any space with an associative multiplication, or even a homotopy associative multiplication, is an associative algebra.

Corollary 1.2 *The homology of $\Omega\Sigma X$ with field coefficient is isomorphic as an algebra to the tensor (that is, free associative) algebra on the homology of X.*

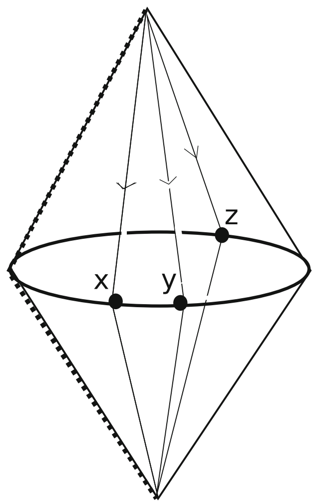

Fig. 1 An illustration of $\hat{J}$ of $y * x * z$ (traversing the path through y first, etc)