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TWISTING COCHAINS AND HIGHER TORSION

KIYOSHI IGUSA

(*communicated by James Stasheff*)

Abstract

This paper gives a short summary of the central role played by Ed Brown's "twisting cochains" in higher Franz-Reidemeister (FR) torsion and higher analytic torsion. Briefly, any fiber bundle gives a twisting cochain which is unique up to fiberwise homotopy equivalence. However, when they are based, the difference between two of them is a higher algebraic K-theory class measured by higher FR torsion. Flat superconnections are also equivalent to twisting cochains.

This paper is dedicated to Edgar Brown.

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Introduction

About 50 years ago Ed Brown [**Bro59**] constructed a small chain complex giving the homology of the total space E of a fiber bundle

$$F \to E \to B$$

whose base B and fiber F are finite cell complexes. It is given by the tensor product of chain complexes for F and B with the usual tensor product boundary map modified by a "twisting cochain." There are many ways to understand the meaning of the twisting cochain.

1. It is the difference between two A_∞ functors.
2. It is a combinatorial flat $\mathbb{Z}$-graded superconnection.
3. It is a family of chain complexes homotopy equivalent to F and parametrized by B.

If $F \to E \to B$ is a smooth bundle with compact manifold fiber and simply connected base then we get another twisting cochain given by fiberwise Morse theory. Comparison of these two twisting cochains gives an algebraic K-theory invariant of the bundle called the *higher Franz-Reidemeister (FR) torsion*. Higher FR-torsion distinguishes different smooth structures on the same topological manifold bundle. Therefore, this construction is a strictly differentiable phenomenon.

The purpose of this paper is to explain some of the basic properties of these constructions and unify them using a simplified version of Ed Brown's construction. A longer exposition can be found in [**Igu05**] which, in turn, gives a summary of the contents of [**Igu02**].

We summarize the contents of this paper. In Section 1 we review the definition of an A_∞ functor. A_∞ structures were first constructed by Stasheff [**Sta63**] and A_∞ categories first appeared in [**Fuk93**]. But here we take A_∞ functors only from ordinary categories to the category of $\mathbb{Z}$-graded projective modules over a ring R. The definition in this restricted case is given by a formula (Equation 2) due to Sugawara[**Sug60**]. We use an old construction of Eilenberg and MacLane[**EM53**] to make homology into an A_∞ functor when it is projective (Equation 3).

In Section 2 we define twisting cochains. We begin with the classical definition of Brown and we also review Brown's construction of the *twisted tensor product*

$$C_*(B) \otimes_\varphi C_*(F)$$

whose homology is equal to the homology of the total space of a fiber bundle $F \to E \to B$. Brown defined this to be the usual tensor product with boundary map ∂_φ twisted by φ (Equation 4). We give a variation of Brown's definitions which arises from certain A_∞ functors. When there is an underlying functor on the category $\mathcal{X}$ whose induced maps are all isomorphisms we get a coefficient sheaf $F(X, Y)$ over $\mathcal{X}$ and the higher homotopies in the A_∞ functor are cochains on $\mathcal{X}$ with values in F. Our twisting cochain is denoted ψ to distinguish it from the classical one of Brown. We use the twisted tensor product to construct a total complex (2.6) which is an actual functor approximating the A_∞ functor.

Section 3 describes Volodin K-theory[**Vol71**] and its relationship to twisting cochains. We construct the Volodin category $\mathcal{V}^b(R)$ and a generalization which we call the *Whitehead category* (3.4). This is a category of acyclic based free chain complexes over a given ring. This category carries a universal twisting cochain of a certain kind and this twisting cochain defines a cohomology class on the classifying space of the Whitehead category:

$$\tau_{2k} \in H^{4k}(\mathcal{W}h^h_\bullet(\mathbb{Q},1);\mathbb{R}).$$

Higher Franz-Reidemeister torsion is in Section 4. We show that, under the right conditions, a smooth fiber bundle with compact manifold fiber $M \to E \to B$ gives two canonical twisting cochains, the topologically defined twisting cochain of Ed Brown and a smoothly defined twisting cochain obtained by fiberwise Morse theory. The fiberwise mapping cone of the comparison map is fiberwise contractible. It gives a mapping of the base B into the Whitehead category provided that we have a basis for the topological twisting cochain. (The Morse theoretic twisting cochain has a basis coming from the critical points.) Such a basis can be chosen in the special case when $\pi_1 B$ acts trivially on the rational homology of M. This based free twisting cochain is classified by a map

$$B \to |\mathcal{W}h^h_\bullet(\mathbb{Q},1)|$$

and we can pull back the universal FR torsion class τ_{2k} to B to obtain the higher FR torsion invariant for the bundle. This invariant has been computed in many cases but here we give only one example: the case when the fiber is a closed oriented even dimensional manifold (Theorem 4.6).

The rest of the paper contains an elementary discussion of flat $\mathbb{Z}$-grade superconnections. The aim is to show that they are equivalent to twisting cochains. Section 5 derives a definition of an *infinitesimal twisting cochain*. This is basically a twisting cochain on very small simplices expressed in terms of differential forms. The prefix "super" refers to a $\mathbb{Z}/2\mathbb{Z}$ grading. However, a superconnection on a $\mathbb{Z}$-graded vector bundle will automatically obtain a $\mathbb{Z}$-grading. The Bismut-Lott definition of such a superconnection [**BL95**] also requires that this $\mathbb{Z}$-graded superconnection have total degree 1.

In Section 6, we view the endomorphism valued differential forms as operators on the vector bundle in the standard way to obtain flat superconnections. In the last section 8 we show that a flat superconnection is the differential in a cochain complex which is dual to Brown's twisted tensor product. Going backwards, the second to last section 7 explains how superconnections can be integrated over 1 and 2 simplices using Chen's iterated integrals to give the beginning of a simplicial twisting cochain. Complete details for integration of superconnections over arbitrary simplices can be found in [**Igu09**].

1. A_∞ functors

In this paper we consider the differential graded category $\mathcal{C}(R)$ of chain complexes of projective R-modules over an associative ring R and $\mathcal{G}r(R)$, the underlying graded category of $\mathbb{Z}$-graded projective R-modules. All R-modules will be right R-modules.

Both categories $\mathcal{C}(R), \mathcal{G}r(R)$ have graded hom sets given by

$$\mathrm{HOM}(C_*, D_*) = \bigoplus \mathrm{HOM}_n(C_*, D_*) = \bigoplus_n \prod_k \mathrm{Hom}_R(C_k, D_{n+k}).$$

But $\mathcal{C}(R)$ has a differential $m_1 : \mathrm{HOM}_n(C_*, D_*) \to \mathrm{HOM}_{n-1}(C_*, D_*)$ given by

$$m_1(f) = df - (-1)^n fd.$$

We will be considering functors from an ordinary category $\mathcal{X}$ into the differential graded category $\mathcal{C}(R)$, but we view these as functors $\mathcal{X} \to \mathcal{G}r(R)$ with additional structure. Usually we assume that the functor takes values in the full subcategory of $\mathcal{G}r(R)$ consisting either of nonnegatively graded R-modules with additional structure given by a degree -1 differential or nonpositively graded modules with degree 1 differential.

To fix a problem which arises in the notation we will use the nerve $\mathcal{N}_\bullet \mathcal{X}^{op}$ of the opposite category. Thus a *p-simplex* in $\mathcal{X}$ (an element of $\mathcal{N}_p \mathcal{X}^{op}$) will be a sequence of morphisms of the form:

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} \cdots \xleftarrow{f_p} X_p \tag{1}$$

To clarify the notation, the composition $X_p \to X_0$ is a morphism in $\mathcal{X}$ which is also a morphism $X_0 \to X_p$ in $\mathcal{X}^{op}$. Note that a 0-simplex consists of one object X_0 with no maps. The main purpose of this is to make the domain X_j of the front j-face of a $j + k$ simplex equal to the range of the back k-face.

The following definition of A_∞ functors, in particular Equation (2), in the restricted case when the domain is an ordinary category is due to Sugawara [**Sug60**].

Definition 1.1. An *A_∞ functor*

$$\Phi = (\Phi, \Phi_0, \Phi_1, \Phi_2, \cdots) : \mathcal{X} \to \mathcal{G}r(R)$$

on an ordinary category $\mathcal{X}$ is an operation which assigns to each object $X \in \mathcal{X}$ a $\mathbb{Z}$-graded projective R-module ΦX and to each sequence of composable morphisms (1) a morphism

$$\Phi_p(f_1, f_2, \cdots, f_p) : \Phi X_p \to \Phi X_0$$

of degree $p - 1$ and satisfies the following cocycle condition for $p \geqslant 0$.

$$\sum_{i=0}^{p} (-1)^i \Phi_i(f_1, \cdots, f_i) \Phi_{p-i}(f_{i+1}, \cdots, f_p) \\ = \sum_{i=1}^{p-1} (-1)^i \Phi_{p-1}(f_1, \cdots, f_i f_{i+1}, \cdots, f_p) \tag{2}$$

For $p \geqslant 1$ this can be written as follows where $m_1(f) = \Phi_0 f - (-1)^{\deg f} f \Phi_0$ and $m_2(f, g) = f \circ g$.

$$m_1(\Phi_p) + \sum_{i=1}^{p-1} (-1)^i m_2(\Phi_i, \Phi_{p-i}) = \sum_{i=1}^{p-1} (-1)^i \Phi_{p-1}(1_{i-1}, m_2, 1_{p-i-1}).$$

For $p = 0, 1, 2, 3$ this equation has the following interpretation.

($p = 0$) $\Phi_0\Phi_0 = 0$, i.e., $(\Phi X, \Phi_0)$ is a chain complex.

($p = 1$) $\Phi_0\Phi_1(f) = \Phi_1(f)\Phi_0$, i.e.,

$$\Phi_1(f) : (\Phi X_1, \Phi_0) \to (\Phi X_0, \Phi_0)$$

is a chain map.

($p = 2$) $\Phi_0\Phi_2(f_1, f_2) + \Phi_2(f_1, f_2)\Phi_0 = \Phi_1(f_1)\Phi_1(f_2) - \Phi_1(f_1 f_2)$, i.e.,

$$\Phi_2(f_1, f_2) : \Phi_1(f_1 f_2) \simeq \Phi_1(f_1)\Phi_1(f_2)$$

is a chain homotopy.

($p = 3$) $\Phi_0\Phi_3(f_1, f_2, f_3) - \Phi_3(f_1, f_2, f_3)\Phi_0 =$

$$\Phi_2(f_1, f_2 f_3) - \Phi_2(f_1, f_2)\Phi_1(f_3) + \Phi_1(f_1)\Phi_2(f_2, f_3) - \Phi_2(f_1 f_2, f_3).$$

In other words, Φ_3 is a null homotopy of the coboundary of Φ_2.

Proposition 1.2. *Suppose that $\Phi_2 = 0$ and Φ_1 takes isomorphisms to isomorphisms. Then (Φ, Φ_0, Φ_1) is a functor from the category $\mathcal{X}$ to the category of projective R-complexes and degree 0 chain maps and (Φ, Φ_1) is a functor from $\mathcal{X}$ to the category of $\mathbb{Z}$-graded projective R-modules and degree 0 maps.*

Proof. If $\Phi_2 = 0$ then $\Phi_1(f)$ is a chain map $(\Phi X_1, \Phi_0) \to (\Phi X_0, \Phi_0)$ with the property that $\Phi_1(fg) = \Phi_1(f)\Phi_1(g)$. This implies $\Phi_1(id_X)$ is a projection operator. If Φ_1 takes isomorphisms to isomorphisms this must be the identity map on ΦX. □

A *natural transformation* of A_∞ functors is an A_∞ functor on the product category $\mathcal{X} \times I$ where I is the category with two objects $0, 1$ and one nonidentity morphism $0 \to 1$. This is a family of chain maps $(\Phi X, \Phi_0) \to (\Phi' X, \Phi_0)$ which are natural only up to a system of higher homotopies. If these chain maps are homotopy equivalences we say that Φ, Φ' are A_∞ *homotopy equivalent* or *fiber homotopy equivalent*. (We view an A_∞ functor as a family of chain complexes over the nerve of $\mathcal{X}^{op}$.)

One way to construct an A_∞ functor on $\mathcal{X}$ is to start with an actual functor C from $\mathcal{X}$ to the category of projective R-complexes, then replace each $C(X)$ with a homotopy equivalent projective R-complex ΦX with differential Φ_0. Following Eilenberg and MacLane [**EM53**], the higher homotopies are given by

$$\Phi_p(f_1, \cdots, f_p) = q_0 C(f_1)\eta_1 C(f_2)\eta_2 \cdots \eta_{p-1} C(f_p) j_p \tag{3}$$

where $j_i : \Phi X_i \to C(X_i)$ and $q_i : C(X_i) \to \Phi X_i$ are homotopy inverse chain maps and $\eta_i : C(X_i) \to C(X_i)$ is a chain homotopy $id \simeq j_i \circ q_i$.

In the special case when the homology of $C(X)$ is projective (e.g., if R is a field) and $C(X)$ is either nonnegatively or nonpositively graded, the *homology complex*

$$\Phi X = H_*(C(X))$$

(with zero boundary map) gives an example of a homotopy equivalent chain complex. Using the construction above, we obtain the A_∞ *homology functor*.

Suppose that $p : E \to B$ is a fiber bundle where the base B is (the geometric realization of) a simplicial complex. Then for each simplex σ in B the inverse image $E|\sigma = p^{-1}(\sigma)$ is homeomorphic to $F \times \sigma$ and thus homotopy equivalent to the fiber F. Taking either a cellular chain complex or the total singular complex, we get a functor from the category $\operatorname{simp} B$ of simplices in B, with inclusions as morphisms, to the category of augmented chain complexes. When the homology of F is projective, the construction above gives an A_∞ homology functor on $\operatorname{simp} B$.

We will see that this A_∞ functor gives a twisting cochain on B which, by Ed Brown's twisted tensor product construction, gives a chain complex for the homology of E. But first we want to point out that the dual of an A_∞ functor is also an A_∞ functor.

Suppose that $R = K$ is a field. Then we have the degree-wise duality functor on $\mathcal{G}r(K)$ sending $V = \bigoplus V_n$ to V^* where

$$V_n^* = \operatorname{Hom}(V_{-n}, K).$$

Then morphisms of degree q are sent to morphisms of degree q and the order of composition is reversed. So we get a functor $\mathcal{G}r(K) \to \mathcal{G}r(K)^{op}$. Given an A_∞ functor $\Phi : \mathcal{X} \to \mathcal{G}r(K)$ we can compose with this duality functor:

$$\mathcal{X} \to \mathcal{G}r(K) \to \mathcal{G}r(K)^{op}$$

This is the same as a functor $\Phi^* : \mathcal{X}^{op} \to \mathcal{G}r(K)$.

Proposition 1.3. *The composition of an A_∞ functor $\Phi : \mathcal{X} \to \mathcal{G}r(K)$ with the degree-wise duality functor on $\mathcal{G}r(K)$ gives an A_∞ functor Φ^* on $\mathcal{X}^{op}$.*

Proof. This is very straightforward. The interesting point is that there is no change in signs. Apply duality to Equation (2) and reverse the order of the morphisms f_i^* (since they are begin composed in the opposite order in $\mathcal{X}^{op}$). We get:

$$\sum_{i=0}^{p}(-1)^i\Phi_{p-i}^*(f_p^*,\cdots,f_{i+1}^*)\Phi_i^*(f_i^*,\cdots,f_1^*) = \sum_{i=1}^{p-1}(-1)^i\Phi_{p-1}^*(f_p^*,\cdots,f_{i+1}^*f_i^*,\cdots,f_1^*)$$

Now multiply both sides by $(-1)^p$, replace i by $p-i$ and make the notation change: $g_i = f_{p-i+1}^*$ to see that Φ^* satisfies the definition of an A_∞ functor. □

2. Twisting cochains

First we review Ed Brown's original construction [**Bro59**]. Suppose that Λ is a commutative ring, K is a nonnegatively graded Λ-coalgebra and A is a graded Λ-algebra. Then Brown defined a twisting cochain to be a sum $\varphi = \sum_{p\geqslant 0}\varphi_p$ of Λ-linear maps $\varphi_p : K_p \to A_{p-1}$, so that

1. $\varphi_0 = 0$
2. $\partial\varphi_p = \varphi_{p-1}\partial - \sum_{1\leqslant i\leqslant p-1}\varphi_i \cup' \varphi_{p-i}$ where

$$\varphi_i \cup' \varphi_{p-i} = \mu(\varphi_i \otimes \varphi_{p-i})\Delta$$

where μ is the multiplication in A and Δ is the comultiplication of K. When this expression is evaluated, a sign of $(-1)^i$ is produced by the Koszul sign rule. (See Definition 2.1 below.)

Brown also assumed that A has an augmentation, which we do not require.

Given a differential graded A-module M, Brown defined the *twisted tensor product*

$$K \otimes_\varphi M$$

to be the standard graded tensor product over Λ with differential ∂_φ given by

$$\partial_\varphi(x \otimes y) = \partial x \otimes y + (-1)^{\deg x} x \otimes \partial y - \sum_{(x)} (-1)^{\deg x_{(1)}} x_{(1)} \otimes \varphi(x_{(2)}) y \qquad (4)$$

where we use Sweedler notation $\Delta x = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ [**Swe69**].

Brown then showed that the total singular complex $S_*(E)$ (with coefficients in Λ) of the total space of a fiber bundle $F \to E \to B$ over a path connected space is homotopy equivalent to the twisted tensor product

$$S'_*(B) \otimes_\varphi C_*(F)$$

where $K = S'_*(B)$ is the subcomplex of the total singular complex of B consisting of singular simplices in B with all its vertices at the base point of B, $M = C_*(F)$ is any free Λ-complex homotopy equivalent to the total singular complex of F and A is the differential graded algebra $A = \mathrm{HOM}(M, M)$. (Brown took A to be the total singular complex of the loop space of B which acts on $S_*(F)$.)

The cochains $\varphi_p, p \geqslant 1$ are given as follows. Each 1-simplex σ of B is a loop in B which induces a holonomy $F \to F$. Then $\varphi_1 := \sigma_* - 1$ where $\sigma_* : C_*(F) \to C_*(F)$ is the induced chain map. Any 2-simplex $\sigma : \Delta^2 \to B$ gives a homotopy between one loop and the composition of the other two loops: $\partial_0\sigma\partial_2\sigma \simeq \partial_1\sigma$. We take $\varphi_2(\sigma)$ to be the corresponding chain homotopy $\partial_0\sigma_* \circ \partial_2\sigma_* \simeq \partial_1\sigma_*$. The construction of φ_p for $p \geqslant 3$ is similar and is a special case of the construction given below.

We now consider twisting cochains $\psi = \sum_{p \geqslant 0} \psi_p$ arising from A_∞ functors. In the case of a fiber bundle $p : E \to B$, the idea is that we take simplices in the base B with distinct vertices. For each vertex v, ψ_0 is the boundary map of the complex $C_*(F_v)$. Each 1-simplex gives, not an endomorphism of a single complex $M = C_*(F)$, but morphisms between complexes associated to vertices of the simplex. Thus $\mathrm{HOM}(M, M)$ is replaced by a category of graded R-modules. We also work over an associative ring R. This simply means that the tensor product $S_*(B) \otimes_\varphi M$ is replaced by a direct sum of (shifted) copies of (various) M, one for each free generator of $S_*(B)$. (See Definition 2.6 below).

Suppose that $(\Phi, \Phi_1) : \mathcal{X} \to \mathcal{G}r(R)$ is a functor so that $\Phi_1(f)$ is a degree 0 isomorphism for all $f : X \to Y$. In that case, the graded bifunctor

$$F(X, Y) = \mathrm{HOM}(\Phi X, \Phi Y) \qquad (5)$$

gives a locally trivial coefficient system on the category $\mathcal{X}$.

Since we are using the nerve of the opposite category, a *p-cochain* on $\mathcal{X}$ with coefficients in a bifunctor F is a mapping ψ which assigns to each p-simplex

$$X_* = (X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} \cdots \xleftarrow{f_p} X_p)$$

in $\mathcal{X}$ an element $\psi(X_*) \in F(X_p, X_0)$. The *coboundary* of ψ is the $p+1$ cochain given by

$$\begin{aligned}\delta\psi(X_0, \cdots, X_{p+1}) &= (f_1)_*\psi(X_1, \cdots, X_{p+1}) \\ &+ \sum_{i=1}^{p}(-1)^i\psi(X_0, \cdots, \widehat{X_i}, \cdots, X_{p+1}) + (-1)^p(f_{p+1})^*\psi(X_0, \cdots, X_p). \quad (6)\end{aligned}$$

Definition 2.1. Given a functor $(\Phi, \Phi_1) : \mathcal{X} \to \mathcal{G}r(R)$ as above, a *twisting cochain* ψ on $\mathcal{X}$ with coefficients in (Φ, Φ_1) is a sum of cochains $\psi = \sum_{p\geqslant 0}\psi_p$ where ψ_p is a p-cochain on $\mathcal{X}$ with coefficients in the degree $p-1$ part F_{p-1} of the $\mathrm{HOM}(\Phi, \Phi)$ bifunctor F of (5) so that the following condition is satisfied.

$$\delta\psi = \psi \cup' \psi.$$

Here $\cup'$ is the cup product using the Koszul sign rule:

$$\psi_p \cup' \psi_q(X_0, \cdots, X_{p+q}) = (-1)^p\psi_p(X_0, \cdots, X_p)\psi_q(X_p, \cdots, X_{p+q})$$

since ψ_q has total odd degree.

To obtain a classical twisting cochain φ, we restrict to the case where $\mathcal{X}$ has a single object X, $\Phi_1(f)$ is the identity map on ΦX for all morphisms f and R is a commutative ring. We take K to be the free R-complex of the nerve of $\mathcal{X}^{op}$. This is the differential grade R-coalgebra $K = C_*(\mathcal{N}_\bullet\mathcal{X}^{op})$ which in degree p is freely generated by the set of p simplices $X_0 \leftarrow \cdots \leftarrow X_p$ in $\mathcal{X}$. We let M be the projective R-complex $M = (\Phi X, \psi_0)$. Then $A = \mathrm{HOM}(M, M)$ is a differential graded R-complex. The functor F is trivial and $\delta f = f\partial$. So, the equation $\delta\psi = \psi \cup' \psi$ becomes:

$$\psi_{p-1}\partial = \psi_0 \cup' \psi_p + \psi_p \cup' \psi_0 + \sum_{i=1}^{p-1}\psi_i \cup' \psi_{p-i}$$

Since $\psi_0 = \partial^M$, we have

$$\psi_0 \cup' \psi_p + \psi_p \cup' \psi_0 = \partial^M\psi_p + (-1)^p\psi_p\partial^M = \partial^A\psi_p$$

Therefore, $\varphi_0 = 0$ and $\varphi_p = \psi_p$ for all $p \geqslant 1$ gives a twisting cochain in the sense of Brown. The referee has pointed out that, if $\Phi_1(f)$ is not always the identity map on ΦX, we can still get a classical twisting cochain by letting $\varphi_0 = 0$, $\varphi_p = \psi_p$ for $p \geqslant 2$ and

$$\varphi_1 = \psi_1 + \Phi_1 - id_M.$$

By comparison of definitions we have the following.

Proposition 2.2. *ψ is a twisting cochain on $\mathcal{X}$ with coefficients in (Φ, Φ_1) if and only if*

$$(\Phi, \psi_0, \Phi_1 + \psi_1, \psi_2, \psi_3, \cdots)$$

is an A_∞ functor.

Consider again the A_∞ homology functor

$$\sigma \mapsto H_*(E_\sigma)$$

of a fiber bundle $E \to B$ over a triangulated space B. Suppose that the fiber F has projective homology. In this case ψ_0 and ψ_1 are both zero and the higher homotopies ψ_p for $p \geqslant 2$ are unique up to simplicial homotopy (over $B \times I$). In this case Ed Brown showed that in his twisted tensor product $S'_*(B) \otimes_\psi S_*(F)$, $S_*(F)$ can be replaced with $H_*(F)$ and ψ is given by the A_∞ homology functor constructed above.

An easy spectral sequence comparison argument shows that we may replace $S'_*(B)$ with any homotopy equivalent differential graded coalgebra. We take the cellular complex $C_*(B)$ given by a triangulation of B. Then the twisted tensor product $C_*(B) \otimes_\psi H_*(F)$ is the total complex of the usual bicomplex $C_p(B; H_q(F))$ with boundary map modified by the twisting cochain ψ as follows:

$$\partial_\psi(x \otimes y) = \partial x \otimes y - \sum_{p+q=\deg x} (-1)^p f_p(x) \otimes \psi_q(b_q(x))(y). \tag{7}$$

Here $x = (x_0 \supseteq x_1 \supseteq \cdots \supseteq x_n)$ is a simplex in the first barycentric subdivision of B, $f_p(x) = (x_0 \supseteq \cdots \supseteq x_p)$ is the front p-face of x and $b_q(x) = (x_p \supseteq \cdots \supseteq x_n)$ is the back q-face.

Theorem 2.3 (Brown[**Bro59**])**.** *Assuming that F has projective homology, the twisted tensor product gives the homology of the total space:*

$$H_*(C_*(B) \otimes_\psi H_*(F)) \cong H_*(E).$$

Remark 2.4. The mapping $f_p \otimes \psi_q b_q$ in (7) has bidegree $(-q, q-1)$. It gives the corresponding boundary map in the Serre spectral sequence for E which is given by filtering the twisted tensor product by reverse filtration of $H_*(F)$ (by subcomplexes $H_{*\geqslant n}(F)$) [**Igu02**].

Corollary 2.5. *When $E \to B$ is an oriented $n-1$ sphere bundle, the degree n part ψ_n of the twisting cochain ψ is a cocycle representing the Euler class of E:*

$$[\psi_n] = e^E \in H^n(B; R).$$

Proof. Since $\mathrm{HOM}(H_*(S^{n-1}), H_*(S^{n-1}))$ has elements only in degrees $0, n-1$, $\psi_k = 0$ for $k \neq n$. By definition of a twisting cochain we have

$$\delta\psi_n = (\psi \cup' \psi)_n = 0.$$

Therefore, ψ_n is an $n-1$ cocycle on B. Since it gives the differential in the Serre spectral sequence, it must represent the Euler class. □

In the present setting, Ed Brown's twisted tensor product is equivalent to the following construction.

Definition 2.6. The *total complex* $E(\psi; \Phi)$ of the twisting cochain ψ with coefficients in the functor (Φ, Φ_1) is given by

$$E(\psi; \Phi) = \bigoplus_{k \geqslant 0} \bigoplus_{(X_0 \leftarrow \cdots \leftarrow X_k)} (X_*) \otimes \Phi X_k$$

with boundary map ∂_ψ given by (7).

Remark 2.7. Note that every simplex $X_* = (X_0 \leftarrow \cdots \leftarrow X_k)$ gives a subcomplex of the total complex $E(\psi;\Phi)$ by:

$$E(X_*) = \bigoplus_{j\geqslant 0} \bigoplus_{a:[j]\to[k]} a^*(X_*) \otimes \Phi X_{a(j)}.$$

This is also the total complex of the A_∞ functor on $[k]$ (considered as a category with objects $0, 1, \cdots, k$ and morphisms $k \to k-1 \to \cdots \to 1 \to 0$) given by pulling back ψ along the functor $X_* : [k] \to \mathcal{X}$.

Note that this gives a functor from the category of simplices in $\mathcal{X}$ to the category of subcomplexes of the total complex with morphisms being inclusion maps. Thus, just as the A_∞ homology functor constructs an A_∞ functor out of an actual functor, the total complex construction gives an actual functor on $\operatorname{simp}\mathcal{N}_\bullet\mathcal{X}^{op}$ from an A_∞ functor on $\mathcal{X}$.

Using the total complex, a twisting cochain on $\mathcal{X}$ can be viewed as a family of chain complexes parametrized by the nerve of $\mathcal{X}^{op}$. With some extra structure, this gives a map from the geometric realization of $\mathcal{X}$ to the Volodin K-theory space of R.

3. Volodin K-theory

Algebraic K-theory is related to twisting cochains in the following way. When two *based, upper triangular twisting cochains* are homotopy equivalent, there is an algebraic K-theory obstruction to deforming one into the other. Formally, we take the pointwise mapping cone. This gives a based free acyclic upper triangular twisting cochain on the category $\mathcal{X}$. This is equivalent to a mapping from the geometric realization $|\mathcal{X}|$ of $\mathcal{X}$ to a fancy version of the Volodin K-theory space of the ring R. To avoid confusion, we assume that R has the property that the rank of a free R-module is well defined, i.e., that $R^n \cong R^m$ implies $n = m$.

When the basis is only well-defined up to permutation and multiplication by elements of a subgroup G of the group of units of R, an acyclic twisting cochain on $\mathcal{X}$ defines a mapping from $|\mathcal{X}|$ into the fiber $\mathcal{W}h^h_\bullet(R, G)$ of the mapping

$$\Omega^\infty\Sigma^\infty(BG_+) \to BGL(R)^+ \times \mathbb{Z}.$$

The well-known basic case is the Whitehead group

$$Wh_1(G) = \pi_0\mathcal{W}h^h_\bullet(\mathbb{Z}[G], G)$$

which is the obstruction to G-collapse of a contractible f.g. based free R-complex. In this section we discuss the different versions of the Volodin construction, show how they are related to twisting cochains and identify the homotopy type of two of them.

The basic definition is sometimes called the "one index" case. It is a space of invertible matrices locally varying by upper triangular column operations. When this definition is expressed as a twisting cochain, the construction seems artificial,

with only one term ψ_1 in the twisting cochain ψ. However, when the missing higher terms ψ_p are inserted we recover the general Volodin space.

Definition 3.1. For every $n \geqslant 2$ the *Volodin category* $\mathcal{V}^n(R)$ is the category whose objects are pairs (A, σ) consisting of an invertible $n \times n$ matrix $A \in GL(n, R)$ and a partial ordering σ of $\{1, 2, \cdots, n\}$. A morphism $(A, \sigma) \to (B, \tau)$ is an $n \times n$ matrix T with coefficients in R so that

1. $\sigma \subseteq \tau$, i.e., τ is a refinement of σ.
2. $AT = B$. (So the morphism $T = A^{-1}B$ is unique if it exists.)
3. $T = (t_{ij})$ is τ-*upper triangular* in the sense that
   (a) $t_{ii} = 1$ for $i = 1, \cdots, n$
   (b) $t_{ij} = 0$ unless $i \leqslant j$ in the partial ordering τ ($i \leqslant j \Leftrightarrow (i, j) \in \tau$).

Note that composition is reverse matrix multiplication:

$$S \circ T = TS.$$

There is a *simplicial Volodin space* $V_\bullet^n(R)$ without explicit partial orderings. A p-simplex $g \in V_p^n(R)$ consists of a $p + 1$ tuple of invertible $n \times n$ matrices

$$g = (g_0, g_1, \cdots, g_p)$$

so that for some partial ordering σ of $\{1, \cdots, n\}$ the matrices $g_i^{-1} g_j$ are all σ-upper triangular. There is a simplicial map

$$\mathcal{N}_\bullet \mathcal{V}^n(R) \to V_\bullet^n(R) \tag{8}$$

from the nerve of the Volodin category $\mathcal{V}^n(R)$ to the simplicial set $V_\bullet^n(R)$ given by forgetting the partial orderings. However, the collection of admissible partial orderings on any $g \in V_p^n(R)$ has a unique minimal element and therefore forms a contractible category. Consequently, (8) induces a homotopy equivalence

$$|\mathcal{V}^n(R)| \simeq |V_\bullet^n(R)|$$

If we stabilize matrices in the usual way by adding a 1 in the lower right corner we get the stable Volodin category

$$\mathcal{V}(R) = \lim_{\to} \mathcal{V}^n(R)$$

and the stable Volodin space $V_\bullet^\infty(R) = \lim_{\to} V_\bullet^n(R)$ which are related to Quillen K-theory by the following well-known theorem due to Vasserstein and Wagoner but best explained by Suslin [**Sus81**].

Theorem 3.2. $|\mathcal{V}(R)| \simeq |V_\bullet^\infty(R)| \simeq \Omega BGL(R)^+$ *where* $\Omega BGL(R)^+$ *is the loop space of the plus construction on the classifying space of* $GL(R) = GL(\infty, R)$.

The Volodin category $\mathcal{V}^n(R)$ has a canonical twisting cochain. It comes from the realization that an invertible matrix is the same as a based contractible chain complex with two terms.

Definition 3.3. The *canonical twisting cochain* on $\mathcal{V}^n(R)$ is given as follows.

1. Let $\Phi(A,\sigma) = C_*$ be the based free graded R-module with $C_0 = C_1 = R^n$ for every object (A,σ) of $\mathcal{V}^n(R)$.
2. $\Phi_1 = (id, id)$ is the identity chain map $C_* \to C_*$ for every morphism.
3. $\psi_0(A,\sigma) = A : R^n \to R^n$.
4. $\psi_1(T) = (0, T - I)$. So $I + \psi_1(T) = (I, T)$ gives a chain isomorphism:

$$\begin{array}{ccc} R^n & \xrightarrow{T} & R^n \\ {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle A} \\ R^n & \xrightarrow{I} & R^n \end{array}$$

The higher homotopies $\psi_p, p \geqslant 2$, are all zero for $\mathcal{V}^n(R)$. However, there is a fancier version of the Volodin category with higher homotopies. We call it the "Whitehead category." This is very similar to the original definition of Volodin [**Vol71**]

Definition 3.4. If G is a subgroup of the group of units of a ring R then the *Whitehead category* $\mathcal{W}h_\bullet(R,G)$ is defined to be the simplicial category whose simplicial set of objects consists of pairs (P,ψ) where ψ is an upper-triangular twisting cochain on the category $[k]$ (as in Remark 2.7) with coefficients in the fixed graded based R-module:

$$R^P := \bigoplus R^{P_i}$$

where $P = \coprod P_i$ is a graded poset (a poset with a grading not necessarily related to the ordering). By *upper-triangular* we mean that $\psi(\sigma)(x)$ is a linear combination of $y < x \in P$ for all simplices σ in $[k]$. As in the Volodin category, Φ_1 is the identity mapping on R^P.

A *morphism* $(P,\psi) \to (Q,\varphi)$ in $\mathcal{W}h_k(R,G)$ consists of a graded order preserving monomorphism going the other way:

$$f : Q \to P$$

so that $S = P - f(Q)$ is a disjoint union of *expansion pairs* which are, by definition, pairs $x^+ > x^-$ otherwise unrelated to every other element of P so that $\psi_0(x^+) = x^- g$ for some $g \in G$, together with a function $\gamma : Q \to G$ so that φ differs from $\psi \circ f$ only by multiplication by γ, i.e., $f_*(\varphi_p(\sigma)(x)) = \psi_p(\sigma)(f(x))\gamma(x)$.

The following theorem, due to J. Klein and the author, is proved in [**Igu02**], Section 5.6.

Theorem 3.5 (Igusa-Klein)**.** *There is a homotopy fiber sequence*

$$|\mathcal{W}h^h_\bullet(R,G)| \to \Omega^\infty\Sigma^\infty(BG_+) \to \mathbb{Z} \times BGL(R)^+$$

where $\mathcal{W}h^h_\bullet(R,G)$ is the simplicial full subcategory of $\mathcal{W}h_\bullet(R,G)$ consisting of (P,ψ) so that each chain complex (R^P,ψ_0) is contractible (i.e., has the homology of the empty set) and $BG_+ = BG \coprod pt$.

Remark 3.6. If we take G to be a finite group then $\Omega^\infty S^\infty(BG_+)$ is rationally trivial above degree 0 so $\mathcal{W}h^h_\bullet(R,G)$ has the rational homotopy type of the Volodin space:

$$|\mathcal{W}h^h_\bullet(R,G)| \simeq_{\mathbb{Q}} |\mathcal{V}(R)| \simeq \Omega BGL(R)^+.$$

In particular, if $R = \mathbb{Q}$, we get [**Bor74**]

$$|\mathcal{W}h^h_\bullet(\mathbb{Q},1)| \simeq_{\mathbb{Q}} \Omega BGL(\mathbb{Q})^+ \simeq_{\mathbb{Q}} BO.$$

Using the Borel regulator maps

$$K_{4k+1}\mathbb{Q} \to \mathbb{R}$$

given by continuous cohomology classes in $H^{2k}(BGL(\mathbb{C});\mathbb{R})$ we get the *universal real higher Franz-Reidemeister torsion* invariants

$$\tau_{2k} \in H^{4k}(\mathcal{W}h^h_\bullet(\mathbb{Q},1);\mathbb{R}).$$

These give characteristic classes for smooth bundles under certain conditions.

4. Higher FR torsion

We will discuss the circumstances under which we obtain well defined algebraic K-theory classes for a fiber bundle. If we have a smooth bundle $p : E \to B$ where E, B and the fiber $M_b = p^{-1}(b)$ are compact connected smooth manifolds and R is a commutative ring so that the fiber homology $H_*(M_b;R)$ is projective then we obtain two canonical twisting cochains on B.

The first is Brown's twisting cochain ψ with coefficients in the fiberwise homology bundle

$$\Phi(b) = H_*(M_b;R).$$

Recall that this requires the fiber homology to be projective.

The second is the fiberwise cellular chain complex $C_*(f_b)$ associated to a fiberwise *generalized Morse function* (GMF) $f : E \to \mathbb{R}$. These are defined to be smooth functions which, on each fiber M_b, have only Morse and birth-death singularities (cubic in one variable plus nondegenerate quadratic in the others). The fiberwise GMF is *not* well-defined up to homotopy. However, there is a canonical choice called a "framed function" ([**Igu87**], [**Igu02**], [**Igu05**]) which exists stably and is unique up to framed fiber homotopy. This gives the following.

Theorem 4.1. *Any compact smooth manifold bundle $E \to B$ gives a mapping*

$$C(f) : B \to |\mathcal{W}h_\bullet(\mathbb{Z}[\pi_1 E], \pi_1 E)|$$

which is well-defined up to homotopy and fiber homotopy equivalent to the fiberwise total singular complex of E with coefficients in $\mathbb{Z}[\pi_1 E]$.

Remark 4.2. The *fiberwise total singular complex* of E is the functor which assigns to each simplex $\sigma : \Delta^k \to B$, the total singular complex of $E|\sigma$. The fiberwise framed function f is defined on a product space $E \times D^N$.

In order to compare the two constructions we need a representation

$$\rho : \pi_1 E \to U(R)$$

of $\pi_1 E$ into the group of units of R with respect to which the fiber homology $H_*(M_b; R)$ is projective over R. By the functorial properties of the Whitehead category we get a mapping

$$B \to |\mathcal{W}h_\bullet(R, G)|$$

where $G \subseteq U(R)$ is the image of ρ. By Theorem 4.1, this will be fiberwise homotopy equivalent to the A_∞ fiberwise homology functor $\Phi_1 + \psi$. The fiberwise mapping cone will be fiberwise contractible but it will not give a mapping to $\mathcal{W}h^h_\bullet(R, G)$ unless the fiberwise homology has a basis. This gives the following.

Corollary 4.3. *If $\pi_1 B$ acts trivially on the fiberwise homology $H_*(M_b; R)$ then a fiberwise mapping cone construction gives a mapping*

$$C(C(f)) : B \to |\mathcal{W}h^h_\bullet(R, G)|$$

which is well-defined up to homotopy.

Remark 4.4. A more precise statement is that we take the direct sum of the fiberwise mapping cone with a fixed contractible projective R-complex P_* with the property that $H_*(M_b; R) \oplus P_*$ is free in every degree.

When $R = \mathbb{Q}$, the construction of higher FR torsion extends to the case then $\pi_1 B$ acts *unipotently* on $H_*(M; \mathbb{Q})$ by which we mean that $H_*(M; \mathbb{Q})$ admits a filtration by $\pi_1 B$ submodules so that the action of $\pi_1 B$ on the successive subquotients is trivial.

Corollary 4.5. *Suppose that $E \to B$ is a compact smooth manifold bundle over a connected space B so that $\pi_1 B$ acts unipotently on the rational homology of the fiber M. Then we have a mapping*

$$B \to |\mathcal{W}h^h_\bullet(\mathbb{Q}, 1)|$$

which is well-defined up to homotopy and we can pull back universal higher torsion invariants to obtain well-defined cohomology classes

$$\tau_{2k}(E) \in H^{4k}(B; \mathbb{R})$$

which are trivial if the bundle is diffeomorphic to a product bundle.

It has been known for many years (by [**FH78**] using the stability theorem [**Igu88**]) that there are smooth bundles which are homeomorphic but not diffeomorphic to product bundles and that these exotic smooth structures are detected by algebraic K-theory. Therefore, when the higher FR torsion was successfully defined, it had already been known to be nonzero in these cases.

However, in these exotic examples the fiber M is either odd dimensional or even dimensional with boundary. We now have the complete calculation of the higher torsion in the case of closed oriented even dimensional fibers.

Theorem 4.6 (6.6 in [**Igu05**])**.** *Suppose that M^{2n} is a closed oriented even dimensional manifold and $M \to E \to B$ is a smooth bundle so that $\pi_1 B$ acts unipotently on the rational homology of M. Then the higher FR torsion invariants $\tau_{2k}(E)$ are well-defined and given by*

$$\tau_{2k}(E) = \frac{1}{2}(-1)^k \zeta(2k+1)\frac{1}{(2k)!}T_{2k}(E) \in H^{4k}(B;\mathbb{R})$$

where $\zeta(s) = \sum \frac{1}{n^s}$ is the Riemann zeta function and

$$T_{2k}(E) = tr_B^E\left(\frac{(2k)!}{2}ch_{2k}(T^vE \otimes \mathbb{C})\right) \in H^{4k}(B;\mathbb{Z})$$

with T^vE being the vertical tangent bundle of E, $ch_{2k}(T^vE \otimes \mathbb{C})$ stands for the degree $4k$ term in the Chern character of $T^vE \otimes \mathbb{C}$ and

$$tr_B^E : H^n(E;\mathbb{Z}) \to H^n(B;\mathbb{Z})$$

is the transfer (with $n = 4k$).

Remark 4.7. Note that $T_{2k}(E)$ is a tangential fiber homotopy invariant. This is in keeping with the belief that there are rationally no stable exotic smooth structures on bundles with closed oriented even dimensional fibers. (*Stable* means stable under product with large dimensional disks D^N. The exotic smooth structure on disk bundles and odd dimensional sphere bundles of [**FH78**] and the explicit examples given by Hatcher ([**Igu02**],[**Goe01**]) are stable.) For more details about this conjecture see [**GI10**]. In that paper we construct virtually all stable exotic smooth structures on bundles with closed odd dimensional fibers and explain why the even dimensional case is so different. See also [**Igu08**] and [**Goe08**] for an outline of those results.

In the special case when $n = 1$, M is an oriented surface and the bundle E is classified by a map of B into the classifying space BT_g of the Torelli group T_g where g is the genus of M. The tangential homotopy invariant T_{2k} is equal to the Miller-Morita-Mumford class in this case ([**Mum83**], [**Mor84**], [**Mil86**]). It is still unknown whether or not any of these classes (tautological classes in degree $4k$) is rationally nontrivial on the Torelli group.

There are several competing versions of higher FR torsion and the version described here is sometimes called *Igusa-Klein (IK) torsion* since the first computation was given in [**IK93**]. Dwyer, Weiss and Williams have defined three kinds of higher Reidemeister which are called *smooth, topological and homotopy DWW torsion* [**DWW03**]. Badzioch, Dorabiala, Klein and Williams have recently shown [**BDKW09**] that smooth DWW torsion is equivalent to IK torsion, making use of the axiomatic characterization of higher torsion given in [**Igu08b**].

Bismut and Lott [**BL95**] have defined higher analytic torsion invariants which have been computed in many cases ([**BL97**], [**BG01**], [**Bun00**], [**Ma97**], [**Goe01**]). In the case of closed oriented even dimensional fibers, the analytic torsion is always zero and Goette has now shown that the expression in Theorem 4.6 gives the difference between BL torsion and IK torsion in all cases. (See the survey article [**Goe08**].)

Higher analytic torsion is defined using flat $\mathbb{Z}$-graded superconnections. It was observed by Goette [**Goe01**] that these are infinitesimal twisting cochains or, as he puts it, that twisting cochains are combinatorial superconnections. We will explain this comment.

5. Flat superconnections

When we review the definition of a flat $\mathbb{Z}$-graded superconnection, we will see that it is the same as an infinitesimal twisting cochain. More precisely, the superconnection is the boundary map of the infinitesimal twisted tensor product. This gives one explanation of the supercommutator rules.

Instead of defining superconnections and showing their relationship to twisting cochains we will take the opposite approach. We ask the question: What is the natural definition of an "infinitesimal twisting cochain?" This question will lead us to the definition of a flat superconnection and we will see that the "superconnection complex" $(\Omega(B,V),D)$ is dual to a twisted tensor product.

Suppose that B is a smooth manifold and $C=\bigoplus_{n\geqslant 0}C_n$ is a nonnegatively graded complex vector bundle over B. Suppose we have a graded flat connection ∇ on C making each C_n into a locally constant coefficient sheaf for the twisting cochain that we want. The example that we keep in mind is when C is the fiberwise homology of a smooth manifold bundle $F\to E\xrightarrow{p}B$. By this we mean the graded vector bundle over B whose fiber over $b\in B$ is the homology of $p^{-1}(b)$. The dual bundle

$$C^*:=\bigoplus_{n\geqslant 0}\operatorname{Hom}(C_n,\mathbb{C})$$

is the fiberwise cohomology bundle $H^*(F)\to C^*\to B$.

Now, imagine that B is subdivided into tiny simplices and we have a twisting cochain on B with coefficients in (C,∇) which satisfies smoothness conditions to be added later. Then, at each vertex v we have a degree -1 endomorphism $\psi_0(v)$ of $C(v)$. This gives a degree 1 endomorphism $A_0=\psi_0^*$ of the dual $C^*(v)$. Suppose we can extend this to a smooth family of such maps

$$A_0\in\Omega^0(B,\operatorname{End}(C^*))=\Omega^0(B,\operatorname{End}(C)^{op})$$

so that $A_0(x)$ has degree 1 and square zero $(A_0(x)^2=0)$ at all $x\in B$.

Next, we take the edges of B. If an edge e goes from v_0 to v_1 the twisting cochain gives us a degree 0 map

$$C(v_0)\xleftarrow{\psi_1(e)}C(v_1)$$

so that $\psi_1(e)$ together with the map (parallel transport) given by the flat connection ∇ is a chain map. This chain map is the parallel transport of a non-flat connection ∇_1 which we now describe.

If we dualize $\psi_1(e)$ and take only the linear term (ignoring Δv^2 terms) we get a degree 0 map $A_1(\Delta v):C^*(v_0)\to C^*(v_1)$ which is linear in Δv. To obtain the smooth version we need to take local coordinates for C so that parallel transport of

∇ is constant, i.e., so that, on C^*, $\nabla^* = d$. Then A_1 becomes a matrix 1-form on B (assuming the twisting cochain is smooth in a suitable sense)

$$A_1 \in \Omega^1(B, \mathrm{End}(C^*))$$

so that parallel transport by the new connection $\nabla_1 = d - A_1$ on C^* keeps A_0 invariant. (The change in sign comes from the fact that parallel transport by $d - A_1$ is given infinitesimally by $I + A_1(\Delta v)$ where $A_1(\Delta v)$ is evaluation of the matrix 1-form A_1 on the vector Δv.) This means that

$$[\nabla_1, A_0] = [d - A_1, A_0] = 0$$

$$dA_0 = [A_1, A_0] = A_1 A_0 + A_0 A_1.$$

We interpret this as an approximately commutative diagram:

$$\begin{array}{ccc} C^*(v_0) & \xrightarrow{I+A_1(\Delta v)} & C^*(v_1) \\ {\scriptstyle A_0}\uparrow & & \uparrow{\scriptstyle A_0+\Delta A_0} \\ C^*(v_0) & \xrightarrow{I+A_1(\Delta v)} & C^*(v_1) \end{array}$$

Higher order terms are needed to make the diagram actually commute. The linear terms give the following approximate equation:

$$\Delta A_0 \cong A_1(\Delta v)A_0 - A_0 A_1(\Delta v)$$

Since A_0 is odd, we get two changes of signs:

$$\Delta A_0 \cong -dA_0(\Delta v) \qquad A_1(\Delta v)A_0 = -(A_1 A_0)(\Delta v)$$

As $\Delta v \to 0$ we get the equation $dA_0 = A_1 A_0 + A_0 A_1$ as claimed.

At the next step, we take two small triangles in B forming a rectangle. The following diagram which commutes up to homotopy by $\psi_2^* = A_2$ indicates what is happening. Here $A_1 = A_1^x dx + A_1^y dy$ where A_1^x, A_1^y are (even) matrix 0-forms and $A_1^x \Delta x$ indicates multiplication by the scalar quantity Δx.

$$\begin{array}{ccc} C^*(v_1') & \xrightarrow{I+A_1^x\Delta x+\Delta A_1^x\Delta x} & C^*(v_2) \\ {\scriptstyle I+A_1^y\Delta y}\uparrow & & \uparrow{\scriptstyle I+A_1^y\Delta y+\Delta A_1^y\Delta y} \\ C^*(v_0) & \xrightarrow{I+A_1^x\Delta x} & C^*(v_1) \end{array}$$

This gives the following approximate equation where $\Delta x, \Delta y$ are scalar quantities and $\Delta v_x, \Delta v_y$ are the corresponding vector quantities giving our rectangle in B.

$$\begin{aligned} (A_0A_2 + A_2A_0)(\Delta v_x, \Delta v_y) &\cong A_1^y\Delta y + \Delta A_1^y \Delta y A_1^x \Delta x - A_1^x \Delta x + \Delta A_1^x \Delta x A_1^y \Delta y \\ &\cong A_1^y A_1^x \Delta x \Delta y + \frac{\partial A_1^y}{\partial x}\Delta x\Delta y - A_1^x A_1^y \Delta x \Delta y - \frac{\partial A_1^x}{\partial y}\Delta x \Delta y \end{aligned}$$

Since A_1^x, A_1^y are even, the right hand side can be written as

$$\left(dA_1 - A_1^2\right)(\Delta v_x, \Delta v_y)$$

In other words, we have

$$A_2 \in \Omega^2(B, \mathrm{End}(C^*))$$

satisfying the equation

$$dA_1 = A_0A_2 + A_1^2 + A_2A_0.$$

In general we will require that

$$dA_{n-1} = \sum_{p+q=n} A_pA_q.$$

(See [**Igu09**] for a full explanation.) This leads to the following definition.

Definition 5.1. An *infinitesimal twisting cochain* on B with coefficients in a graded vector bundle C^* with graded flat connection ∇ ($\nabla = \sum \nabla_k$ where $(-1)^k\nabla_k$ is a flat connection on C^k) is equal to a sequence of $\mathrm{End}(C^*)$-valued forms

$$A_p \in \Omega^p(B, \mathrm{End}_{1-p}(C^*)) = \Omega^0(B, \mathrm{End}_{1-p}(C^*)) \otimes_{\Omega^0(B)} \Omega^p(B)$$

of total degree 1 so that

$$\nabla A_{n-1} = \sum_{p+q=n} A_pA_q. \tag{9}$$

Next we pass to the algebra of operators on $\Omega(B, C^*)$ where we carefully distinguish between differential forms A and the operators $\widetilde{A}$ that they define to arrive at the Bismut-Lott definition of a flat $\mathbb{Z}$-grade superconnection.

6. Forms as operators

If $A \in \Omega(B, \mathrm{End}(C^*))$ is written as $A = \sum \varphi_i \otimes \alpha_i$ with fixed total degree $|A| = |\varphi_i| + |\alpha_i|$, let $\widetilde{A}$ be the linear operator on

$$\Omega(B, C^*) = \Omega^0(B, C^*) \otimes_{\Omega^0(B)} \Omega(B)$$

given by

$$\widetilde{A}(c \otimes \gamma) := \sum_i (-1)^{|c|\cdot|\alpha_i|} \varphi_i(c) \otimes \alpha_i \wedge \gamma \tag{10}$$

Proposition 6.1 (Prop. 1 in [**Qui85**])**.** *If $\omega \in \Omega^k(B)$ then*

$$\widetilde{A} \circ \omega = (-1)^{k|A|} \omega \circ \widetilde{A}.$$

Conversely, any linear operator on $\Omega(B, C^*)$ of fixed total degree having this property is equal to $\widetilde{A}$ for a unique $A \in \Omega(B, \mathrm{End}(C^*))$.

Proof. Since $\widetilde{A}$ acts only on the first tensor factor we get

$$\widetilde{A} \circ \omega = \widetilde{A\omega} = (-1)^{k|A|} \widetilde{\omega A} = (-1)^{k|A|} \omega \circ \widetilde{A}$$

as required. Conversely, any linear operator which is $\Omega(B)$-linear in this sense must be "local" and thus we may restrict to a coordinate chart U over which C^* has a basis of sections. This makes $\Omega(U, C^*|U)$ into a free module over $\Omega(U)$. Thus any

$\Omega(U)$-linear operator is uniquely given by $\widetilde{A}$ where $A \in \Omega(U, \mathrm{End}(C^*|U))$ is given by the value of the operator on the basis of sections of $C^*|U$. We can patch these together on intersections of coordinate charts since, by uniqueness, the differential forms defined using different coordinate charts will agree. □

Here is another straightforward calculation.

Proposition 6.2. $[d, \widetilde{A}] = d \circ \widetilde{A} - (-1)^{|A|} \widetilde{A} \circ d = \widetilde{dA}$.

If A' is another $\mathrm{End}(C^*)$ valued form on B then $\widetilde{AA'} = \widetilde{A} \circ \widetilde{A'}$. So

$$[d, \widetilde{A}_{n-1}] = \widetilde{dA}_{n-1} = \sum_{p+q=n} \widetilde{A}_p \circ \widetilde{A}_q$$

which, in coordinate free notation, is

$$[\nabla, \widetilde{A}] = \widetilde{A} \circ \widetilde{A}$$

Since $|A| = 1$ and $\nabla^2 = 0$, we get

$$(\nabla - \widetilde{A})^2 = (\nabla - \widetilde{A}) \circ (\nabla - \widetilde{A}) = 0.$$

This leads to the following definition due to Bismut and Lott [**BL95**]. (A similar definition appeared in [**Che75**].)

Definition 6.3. Let $V = \bigoplus_{n \geqslant 0} V^n$ be a graded complex vector bundle over a smooth manifold B. Then a *superconnection* on V is defined to be a linear operator D on $\Omega(B, V)$ of total degree 1 so that

$$D\alpha = d\alpha + (-1)^{|\alpha|} \alpha D$$

for all $\alpha \in \Omega(B)$. The superconnection D is called *flat* if

$$D^2 = 0.$$

If D is flat then $(\Omega(B, V), D)$ is a chain complex which we call the *superconnection complex*. We will see later that it is homotopy equivalent to the dual of a twisted tensor product. The superconnection complex is bigraded:

$$\Omega(B, V) = \bigoplus \Omega^p(B, V^q)$$

and the superconnection D has terms of degree $(k, 1-k)$ for $k \geqslant 0$. This gives a spectral sequence in the usual way with $E_1^{p,q} = \Omega^p(B, H^q(V, A_0))$ and

$$E_2^{p,q} = H^p(B; H^q(V, A_0)) \Rightarrow H^{p+q}(\Omega(B, V), D)$$

A flat superconnections on V corresponds to a contravariant A_∞ functor on B. To get a twisting cochain we need an ordinary graded flat connection ∇ on V. Then $D - \nabla$ gives a twisting cochain by reversing the above process.

The first step is to get out of the superalgebra framework by writing D as a sum

$$D = \nabla - \widetilde{A} = \nabla - \widetilde{A}_0 - \widetilde{A}_1 - \widetilde{A}_2 - \cdots$$

where $A_p \in \Omega^p(B, \mathrm{End}_{1-p}(V))$ corresponds to $\widetilde{A}_p$ by (10) and satisfies (9).

Next, we obtain a contravariant twisting cochain on the category of smooth simplices in B with coefficients in the category of cochain complexes by iterated integration of A_*. Then we dualize, relying on Proposition 1.3 to recover the original twisting cochain.

7. Chen's iterated integrals

This section gives a very short discussion and proof of the first two steps in the process of integrating a flat superconnection to obtain a twisting cochain. Details are fully explained in [**Igu09**] although the original idea is contained in Chen's work [**Che73**], [**Che75**], [**Che77**]. A more direct, less computational method of constructing the twisting cochain is explained in the next section.

Since ∇ is a flat connection, we can choose local coordinates so that V is a trivial bundle and $\nabla = d$. Starting with $p = 0$ we note that $A_0(x)$ is a degree 1 endomorphism of V_x with $A_0(x)^2 = 0$ making $C(x) = (V_x, A_0(x))$ into a cochain complex for all $x \in B$. Putting $n = 2$ in (9) we see that the curvature $(d - A_1)^2$ of the connection $d - A_1$ is null homotopic. Also we will see that parallel transport of this connection is a cochain map.

It is well-known that the parallel transport Φ_1 associated to the connection $d-A_1$ on V is given by an iterated integral of the matrix 1-form A_1. Given any piecewise smooth path $\gamma : [0, 1] \to B$, parallel transport is the family of degree zero homomorphisms $\Phi_1(t, s) : C(\gamma(s)) \to C(\gamma(t))$ so that $\Phi_1(s, s) = I = id_V$ and $d - A_1 = 0$, i.e.,

$$\frac{\partial}{\partial t}\Phi_1(t,s) = A_1/t\Phi_1(t,s)$$

$$\frac{\partial}{\partial s}\Phi_1(t,s) = -\Phi_1(t,s)A_1/s$$

where $A_1/t = A_1(\gamma(t))(\gamma') \in \mathrm{End}(C(\gamma(t)))$. The solution is given by Chen's iterated integral [**Che77**]:

$$\Phi_1(s_0,s_1) = I + \int_{s_0\geqslant t\geqslant s_1} dt_1 A_1/t + \int_{s_0\geqslant t_1\geqslant t_2\geqslant s_1} dt_1dt_2(A_1/t_1)(A_1/t_2) + \int_{s_0\geqslant t_1\geqslant t_2\geqslant t_3\geqslant s_1} dt_1dt_2dt_3(A_1/t_1)(A_1/t_2)(A_1/t_3) + \cdots$$

which we abbreviate as:

$$\Phi_1(s_0,s_1) = I + \int_\gamma A_1 + \int_\gamma (A_1)^2 + \int_\gamma (A_1)^3 + \cdots .$$

This can also be written as a limit of products (multiplied right to left)

$$\Phi_1 = \lim_{\Delta t\to 0}\prod(I + (A_1/t_i)\Delta t)$$

In the case when A_1 is constant, parallel transport $C(\gamma(0)) \to C(\gamma(1))$ is given by e^{A_1}. The inverse is given by $\Phi_1(0, 1) = e^{-A_1}$.

Proposition 7.1. $A_0(\gamma(t))\Phi_1(t,s) = \Phi_1(t,s)A_0(\gamma(s))$, *i.e.,* $\Phi_1(t,s)$ *gives a cochain map*

$$C(\gamma(t)) \leftarrow C(\gamma(s)).$$

Proof. By (9) we have:

$$-\frac{d}{dt}A_0(\gamma(t)) = dA_0(\gamma(t))(\gamma') = A_0(\gamma(t))A_1/t - (A_1/t)A_0(\gamma(t))$$

where both negative signs come from the fact that A_0 is odd. So, $X(t) = A_0(\gamma(t))\Phi_1(t,s)$ is the unique solution of the differential equation

$$\frac{\partial}{\partial t}X(t) = (A_1/t)X(t)$$

with initial condition $X(s) = A_0(\gamma(s))$. So, $X(t)$ must also be equal to the other solution of this differential equation which is $X(t) = \Phi_1(t,s)A_0(\gamma(s))$. □

Let

$$\Delta^2 = \{(x,y) \in \mathbb{R}^2 \,|\, 1 \geqslant x \geqslant y \geqslant 0\}$$

and suppose that $\sigma : \Delta^2 \to B$ is a smooth simplex with vertices $v_0 = \sigma(0,0), v_1 = \sigma(1,0), v_2 = \sigma(1,1) \in B$. Then a chain homotopy

$$\Phi_1(v_0,v_2) \simeq \Phi_1(v_0,v_1)\Phi_1(v_1,v_2)$$

can be obtained by an iterated integral of the form

$$\psi_2(\sigma) = \int_\sigma A_2 + \int_\sigma A_2A_1 + \int_\sigma A_1A_2 + \int_\sigma A_2A_1A_1 + \int_\sigma A_1A_2A_1 + \cdots.$$

The integral over σ is the double integral of the pull-back to Δ^2. The factors of A_1 will just give the parallel transport Φ_1 along paths connecting v_0 and v_2 to the point $v = \sigma(x,y)$

$$\psi_2(\sigma) = \int_{1\geqslant x\geqslant y\geqslant 0} \sigma^*(\Phi_1(v_0,v)A_2(v)\Phi_1(v,v_2)) \in \mathrm{Hom}(C(v_2),C(v_0))$$

where $\Phi_1(v_0,v), \Phi_1(v,v_2)$ are given by parallel transport along paths given by two straight lines each as shown in the Figure.

$$\Phi_1(v,v_2) = \Phi_1(v,\sigma(x,x))\Phi_1(\sigma(x,x),v_2)$$

$$\Phi_1(v_0,v) = \Phi_1(v_0,\sigma(x,0))\Phi_1(\sigma(x,0),v).$$

Proposition 7.2. $A_0(v_0)\psi_2(\sigma) + \psi_2(\sigma)A_0(v_2) = \Phi_1(v_0,v_2) - \Phi_1(v_0,v_1)\Phi_1(v_1,v_2)$.

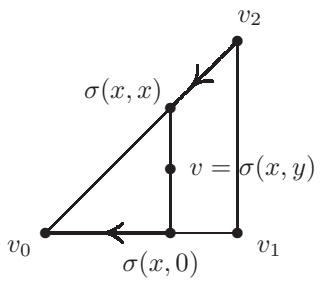
Proof. For $x \in [0,1]$ let $\Phi(x)$ be the parallel transport of $d - A_1$ along the three segment path:

$$\Phi(x) = \Phi_1(v_0,\sigma(x,0))\Phi_1(\sigma(x,0),\sigma(x,x))\Phi_1(\sigma(x,x),v_2).$$

Then

$$\Phi(0) = \Phi_1(v_0,v_2)$$

Figure 1: $\Phi_1(v_0, v), \Phi_1(v, v_2)$ are parallel transport along dark lines.



$$\Phi(1) = \Phi_1(v_0, v_1)\Phi_1(v_1, v_2).$$

So the right hand side of the formula we are proving is

$$\Phi(0) - \Phi(1) = -\int_0^1 d\Phi(x). \tag{11}$$

By Proposition 7.1 the left hand side is equal to

$$\int_\sigma \Phi_1(v_0, v)[A_0(v)A_2(v) + A_2(v)A_0(v)]\Phi_1(v, v_2)$$

(The sign in front of $A_2(v)A_0(v)$ is $(-1)^2 = +1$ since the form degree of A_2 is 2.) By (9) this is equal to

$$= \int_\sigma \Phi_1(v_0, v)[-A_1(v)A_1(v) + dA_1(v)]\Phi_1(v, v_2) \tag{12}$$

In (11), we have

$$-\frac{d\Phi(x)}{dx} = \Phi_1(v_0, \sigma(x,0))X(x)\Phi_1(\sigma_1(x,x), v_2)$$

where

$$X(x) = A_1^x\Phi_1(\sigma(x,0),\sigma(x,x)) - \frac{d}{dx}\Phi_1(\sigma(x,0),\sigma(x,x)) - \Phi_1(\sigma(x,0),\sigma(x,x))A_1^x$$

using the notation $\sigma^*(A_1) = A_1^x dx + A_1^y dy$. (The term $\Phi_1(\sigma(x,0),\sigma(x,x))A_1^y$ which occurred with positive sign in the second term and negative sign in the third term was cancelled.) Comparing this to (12) we are reduced to showing that $X(x) = Y(x)$ where

$$Y(x) = \int_{0\leqslant y\leqslant z} dy\, \Phi_1(\sigma(x,0), v)\left(-A_1^xA_1^y + A_1^yA_1^x + \frac{\partial A_1^y}{\partial x} - \frac{\partial A_1^x}{\partial y}\right)\Phi_1(v, \sigma(x,x)).$$

Expressing $\Phi_1(\sigma(x,0),\sigma(x,x))$ as an iterated integral of $A_1^y dy$ we see that the second term of $X(x)$ is equal to the third term of $Y(x)$ (with $\frac{\partial A_1^y}{\partial x}$). The negative sign comes from the fact that we are going backwards along the y direction ($dt = -dy$). The other three terms of $Y(x)$ form the commutator of A_1^x with each factor

$A_1/dt = -A_1^y$ in the iterated integral representation of $\Phi_1(\sigma(x,0),\sigma(x,x))$. This can be more easily seen in the product limit form:

$$\Phi_1(\sigma(x,0),\sigma(x,x)) = \lim_{\substack{n\to\infty \\ \Delta y = x/n}} \prod_1^n (I - A_1^y \Delta y)$$

The commutator of A_1^x with $I - A_1^y\Delta y$ is:

$$A_1^x(I - A_1^y\Delta y) - (I - A_1^y\Delta y)(A_1^x + \Delta_y A_1^x) = -A_1^x A_1^y \Delta y + A_1^y \Delta y A_1^x - \Delta_y A_1^x + o(\Delta y)$$

So, the commutator of A_1^x with $\Phi_1(\sigma(x,0),\sigma(x,x))$ is

$$A_1^x\Phi_1 - \Phi_1 A_1^x = \lim_{\Delta y\to 0} \sum_i \prod_1^{i-1}(I - A_1^y\Delta y)\,[-A_1^x A_1^y \Delta y + A_1^y \Delta y A_1^x - \Delta_y A_1^x] \prod_{i+1}^{n}(I - A_1^y \Delta y)$$

So, the sum of the remaining two terms of $X(x)$ is equal to the sum of the remaining three terms of $Y(x)$ and we conclude that $X(x) = Y(x)$ proving the proposition. □

The construction that we just explained in detail is a special case of a construction outlined by Chen in [**Che73**], sec. 4.5. Chen constructs mappings

$$\theta_{(n)} : I^{n-1} \to P(\Delta^n, v_n, v_0)$$

from the $n-1$ cube I^{n-1} to the space $P(\Delta^n, v_n, v_0)$ of smooth paths in Δ^n from v_n to v_0 by smoothing a piecewise linear construction very similar to the one we explained. When $n = 2$, this is the 1-parameter family of paths in Δ^2 given in Figure 1 above. The main conceptual difference between Chen's construction and ours is that Chen follows this mapping with a smooth mapping of the n simplex into the space B which sends all vertices of Δ^n to the base point of B. He uses this to obtain a cubical chain complex for ΩB.

A very longwinded description of the higher steps in this process of converting a flat superconnection into a simplicial twisting cochain can be found in [**Igu09**] which uses much of Chen's notation to allow for comparison and to make it easier to understand Chen's work.

In the last section of this paper we will show how the entire process can be done in an easier way.

8. Another method

There is another method for constructing a simplicial twisting cochain from a flat connection. We assume that B is compact and we choose a finite "good cover" for B. (See [**BT82**].) This is a covering of B by contractible open sets U so that all nonempty intersections

$$U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_n}$$

are also contractible.

Lemma 8.1. *If U is a contractible open subset of B and D is a flat superconnection on a graded vector bundle V over B then the cohomology of the superconnection complex over U is isomorphic to the cohomology of V using A_0 as differential:*

$$H^n(\Omega(U, V|U), D) \cong H^n(V, A_0)$$

where the isomorphism is given by restriction to any point in U.

Proof. The spectral sequence collapses since its E_2-term is $H^p(U; H^q(V))$. □

This lemma implies that

$$F : U \mapsto (\Omega(U, V|U), D)$$

is a functor from the nerve $\mathcal{N}_\bullet\mathcal{U}$ of the good cover $\mathcal{U}$ of B to the category of cochain complexes over $\mathbb{C}$ and cochain homotopy equivalences. Applying the A_∞ cohomology functor we get a contravariant A_∞ functor H^*F on $\mathcal{N}_\bullet\mathcal{U}$. By Proposition 1.3 we can dualize to get a covariant A_∞ functor $\Phi_1 + \psi$ on $\mathcal{N}_\bullet\mathcal{U}$ with coefficients in $(\Phi, \Phi_1) = (V^*, \nabla^*)$. Subtracting $\Phi_1 = \nabla^*$ we get the twisting cochain ψ satisfying the following.

Theorem 8.2. *The twisted tensor product $C_*(\mathcal{N}_\bullet\mathcal{U}) \otimes_\psi V^*$ is homotopy equivalent to the dual of the superconnection complex. I.e.,*

$$(\Omega(B, V), D) \simeq \mathrm{Hom}(C_*(\mathcal{N}_\bullet\mathcal{U}) \otimes_\psi V^*, \mathbb{C}).$$

Proof. This holds by induction on the number of open sets in the finite good covering $\mathcal{U}$. When the number is 1 we use Lemma 8.1. To increase the number we use Mayer-Vietoris. □

Remark 8.3. By Ed Brown's Theorem 2.3 this implies that, if the superconnection is constructed correctly, the superconnection complex gives the cohomology of the total space of a smooth manifold bundle. This construction also allows us to compare flat superconnections with twisting cochains, giving a K-theory difference class with a well defined higher torsion. But, this is the subject of another paper.

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Kiyoshi Igusa
`igusa@brandeis.edu`

Department of Mathematics, Brandeis University
Waltham, MA 02454