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THE HELP-LEMMA AND ITS CONVERSE IN QUILLEN MODEL CATEGORIES

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Abstract

We show that a map $p : X \to Y$ between fibrant objects in a closed model category is a weak equivalence if and only if it has the right homotopy extension lifting property with respect to all cofibrations. The dual statement holds for maps between cofibrant objects.

The HELP-Lemma states that a homotopy equivalence $p : X \to Y$ of topological spaces has the homotopy extension lifting property, HELP for short, for all closed cofibrations [**2**, Appendix Thm. 3.5]. The lemma, variants of it and their Eckmann-Hilton duals (e.g. see [**1**, II.1.11], [**4**, Thm. 4, Thm. 4*]) have proven to be very useful tools in homotopy theory.

The main purpose of this paper is to make this tool and its Eckmann-Hilton dual available in arbitrary closed model categories in the sense of Quillen [**7**], (see also [**3**]). In addition, we prove a converse.

Surprisingly, this converse has never been explicitely stated in the past with the exception of the very classical case of weak homotopy equivalences of topological spaces (e.g. see [**5**, p. 68]; we refer to it as May's lemma), but it follows from a lemma due to Reedy [**8**, Lemma 2.1]. Our proof is a bit more elementary than Reedy's. Applying the theorem below to the category $\mathcal{T}op$ of topological spaces with the Strøm model structure [**9**] we for example obtain

Proposition 1. *A map of topological spaces $X \to Y$ is a homotopy equivalence if and only if it has the HELP for all closed cofibrations.*

Of course, its Eckmann-Hilton dual also holds.

Our theorem covers Reedy's lemma and May's lemma for weak equivalences of topological spaces, but not for n-equivalences, because n-equivalences in his sense do not satisfy the two-out-of-three axiom for weak equivalences in a model category.

Throughout the paper let $\mathcal{M}$ denote a closed model category in the sense of Quillen.

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Definition 2. Let $i : A \to B$ and $p : X \to Y$ be maps in $\mathcal{M}$.

(1) We say that p has the right HELP with respect to i, if for each not necessarily commutative square

$$\begin{array}{ccc} A & \xrightarrow{f_A} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array} \qquad (*)$$

and each right homotopy $h_A : A \to Y^I$ from $p \circ f_A$ to $g \circ i$, where Y^I is a path object $Y \xrightarrow[\sim]{j} Y^I \xrightarrow{\pi} Y \times Y$ for Y, there is a map $f : B \to X$ and a right homotopy $h : B \to Y^I$ from $p \circ f$ to g such that $f \circ i = f_A$ and $h \circ i = h_A$.

(2) We say that i has the left HELP with respect to p, if for each not necessarily commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g_Y} & Y \end{array}$$

and each left homotopy $h_Y : Z_A \to Y$ from $g_Y \circ i$ to $p \circ f$, where Z_A is a cylinder object $A \sqcup A \xrightarrow{j} Z_A \xrightarrow[\sim]{\sigma} A$ for A, there is a map $g : B \to X$ and a left homotopy $h : Z_A \to X$ from $g \circ i$ to f such that $p \circ g = g_Y$ and $p \circ h = h_Y$.

Theorem 3. *(1) A map $p : X \to Y$ of fibrant objects is a weak equivalence in $\mathcal{M}$ if and only if it has the right HELP with respect to all cofibrations.*

(2) A map $i : A \to B$ of cofibrant objects is a weak equivalence in $\mathcal{M}$ if and only if it has the left HELP with respect to all fibrations.

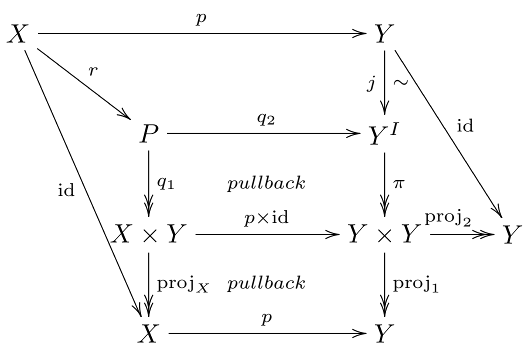
Proof. The two statements are dual so we just prove the first one.

Since X and Y are fibrant the projections

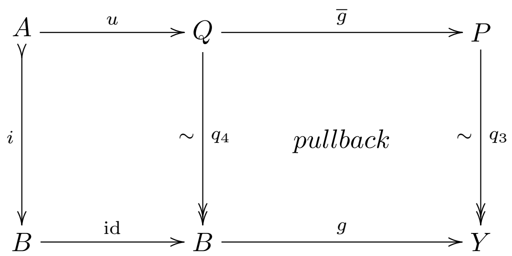
$$X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y \qquad \text{and} \qquad Y \xleftarrow{p_1} Y \times Y \xrightarrow{p_2} Y$$

are fibrations.

Suppose that p is a weak equivalence and that we are given a square $(*)$. Consider the commutative diagram



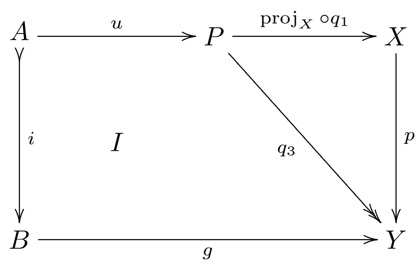
where $r = (\mathrm{id}, j \circ p)$. Since $\mathrm{proj}_1 \circ \pi$ is a weak equivalence, so is $\mathrm{proj}_X \circ q_1$ and hence r. It follows that q_2 and $\mathrm{proj}_2 \circ \pi \circ q_2$ are weak equivalences. Hence $q_3 = \mathrm{proj}_Y \circ q_1 = \mathrm{proj}_2 \circ \pi \circ q_2 : P \twoheadrightarrow Y$ is a weak equivalence. Now consider



where $u = (i, v)$ and $v : A \to P$ is induced by $(f_A, g \circ i) : A \to X \times Y$ and $h_A : A \to Y^I$. Since q_4 is a trivial fibration there is a section $s : B \to Q$ such that $s \circ i = u$. Define

$$f = \mathrm{proj}_X \circ q_1 \circ \overline{g} \circ s : B \to X$$
$$h = q_2 \circ \overline{g} \circ s : B \to Y^I$$

Conversely, suppose that p has the right HELP. Consider the diagram



where square I is supposed to commute. We define

$$f_A = \mathrm{proj}_X \circ q_1 \circ u : A \to X \quad \text{and} \quad h_A = q_2 \circ u : A \to Y^I.$$

Then h_A is a right homotopy from $p \circ f_A$ to $g \circ i$. Hence there exist

$$f : B \to X \quad \text{and} \quad h : B \to Y^I$$

such that h is a right homotopy from $p \circ f$ to g and $f \circ i = f_A$ and $h \circ i = h_A$. Then f and h induce a map

$$k : B \to P$$

such that $k \circ i = u$ and $q_3 \circ k = \mathrm{proj}_2 \circ \pi \circ k = g$.

Hence q_3 has the right lifting property with respect to all cofibration and has to be a trivial fibration. Since $q_2 : P \to Y^I$ is a right homotopy from $p \circ \mathrm{proj}_X \circ q_1$ to q_3 and since a map right homotopic to a weak equivalence is itself a weak equivalence, $p \circ \mathrm{proj}_X \circ q_1$ is a weak equivalence. Since $\mathrm{proj}_X \circ q_1$ is a weak equivalence, so is p. □

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